Remarks on hamiltonian digraphs

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Abstract

This note is motivated by A. Kemnitz and B. Greger, Congr. Numer. 130 (1998) 127-131. We show that the main result of the paper by Kemnitz and Greger is an easy consequence of the characterization of hamiltonian out-locally semicomplete digraphs by Bang-Jensen, Huang and Prisner, J. Combin. Theory Ser. B 59 (1993) 267-287. We also disprove a conjecture from the paper of Kemnitz and Greger.

There are numerous and various sufficient conditions for hamiltonicity of undirected graphs. For directed graphs not so many are known. One of the main reasons for this situation is the fact that the cycle structure of digraphs is significantly more complicated. As a result, several attempts to generalize sufficient conditions for hamiltonicity of undirected graphs to directed ones have failed. One such example is provided in [3], where the authors describe a family of counterexamples to a 'natural' extension of Fan's sufficient condition [5] for an undirected graph to be hamiltonian. In this note we give another, more striking, example of this kind, which disproves a conjecture from [6]. We also show that the main result of [6] is an easy consequence of the characterization of hamiltonian out-locally semicomplete digraphs by Bang-Jensen, Huang and Prisner [4]. For further information and references on hamiltonian digraphs, see e.g. the chapter on hamiltonicity in [1] as well as recent survey papers [2, 8].

We use the standard terminology and notation on digraphs as described in [1]. A digraph D has vertex set $V(D)$ and arc set $A(D)$. If $xy \in A(D)$, we say that x *dominates* y and that x and y are *adjacent*. The subdigraph of D induced by a set $X \subseteq V(D)$ is denoted by $D\langle X\rangle$. A digraph D is *strong* if either D has only one vertex or there is a path from x to y and a path from y to x for every pair x, y of distinct vertices in D. A digraph D is k-*strong* if deletion of every set of at most $k-1$ vertices from D results in a strong digraph. (The complete symmetric digraph on n vertices is assumed to be $n-1$-strong, but not n-strong.) A digraph D is *out-locally semicomplete* if, for every triple x, y, z of distinct vertices, $zx, zy \in A(D)$ implies that x and y are adjacent. An *oriented graph* is a digraph with no 2-cycle. An *orientation* of a undirected graph G is an oriented graph D such that $V(D) = V(G)$ and vertices $x, y \in V(G)$ are adjacent in G if and only if x, y are adjacent in D. An *orientation* of a digraph D is an oriented graph obtained from D by deleting exactly one arc from every 2-cycle of D. For a vertex v of a digraph D, $N(v)$ stands for the set of vertices in D adjacent with v. The *independence number* of a digraph D is the maximum number of pairwise non-adjacent vertices in D.

Kemnitz and Greger [6] proved the following theorem:

Theorem 1. *If D is a strong oriented graph with at least three vertices and D does not contain, as induced subdigraph, any orientation of the digraphs D_1, D_2, D_3 with vertex sets*

$$V(D_i) = \{u, x, y, z\}, \ i = 1, 2, 3,$$

and arc sets

$$A(D_1) = \{ux, xy, xz\}, \ A(D_2) = A(D_1) \cup \{uy, yu\}, \ A(D_3) = A(D_2) \cup \{uz, zu\},$$

then D is hamiltonian.

Actually, this theorem can be readily obtained from the following characterization of hamiltonian out-locally semicomplete digraphs by Bang-Jensen, Huang and Prisner [4].

Theorem 2. *An out-locally semicomplete digraph D with at least two vertices is hamiltonian if and only if strong.*

Let D be an oriented graph satisfying Theorem 1. We prove that D is out-locally semicomplete. Suppose that D contains a pair y, z of distinct vertices dominated by a common vertex x. Since D is strong and has no 2-cycle, it contains a fourth vertex u which dominates x. Since $D\langle\{u, x, y, z\}\rangle$ is isomorphic to no orientation of the digraphs D_1, D_2, D_3, the vertices y and z must be adjacent. Hence, D is indeed out-locally semicomplete. Since D is strong, by Theorem 2, D is hamiltonian.

Kemnitz and Greger [6] conjectured the following analogue of a result by Oberly and Sumner [7] on hamiltonian undirected graphs: Let D be a strong oriented graph. If $D\langle N(v)\rangle$ is strong for every $v \in V(D)$ and D does not contain, as induced subdigraph, any orientation of $K_{1,3}$, then D is hamiltonian. The following counterexample disproves this conjecture.

Consider the tournament D with $V(D) = \{x_1, x_2, x_3, x_4, x_5\}$ and

$$A(D) = \{x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_5x_1, x_1x_3, x_2x_4, x_3x_5, x_4x_1, x_5x_2\}$$

and any 2-strong tournament T, containing three vertices y_1, y_2, y_3 such that

$$\{y_1y_2, y_2y_3, y_3y_1\} \subseteq A(T).$$

Let us construct an oriented graph T^* with vertex set $V(D) \cup V(T)$ and arc set

$$A(D) \cup A(T) \cup \{y_1x_2, x_4y_1, y_2x_2, x_4y_2, y_3x_4, x_2y_3\}.$$

The oriented graph T^* does not contain, as induced subdigraph, any orientation of $K_{1,3}$ (as the independence number of T^* is 2). Clearly, T^* is a strong oriented graph. It is easy to verify that for every vertex v in T^* we have that $T^*\langle N(v)\rangle$ is strong (check each vertex in $V(D) \cup \{y_1, y_2, y_3\}$ separately and observe that $T - z$ is strong for every $z \in V(T)$). However, T^* is not hamiltonian, as any Hamilton cycle C would have to enter $V(D)$ at x_2 and leave it at x_4 (it could not enter it at x_4 and leave it at x_2 since it could not use y_3 twice). This implies that C would contain the following arcs: x_2x_3, x_3x_5, x_5x_1. However, there is no arc from x_1 to x_4.

We would like to add that we have constructed an infinite family $\mathcal{D}$ of non-hamiltonian oriented graphs such that no digraph of $\mathcal{D}$ has an orientation of $K_{1,3}$ as induced subdigraph and, moreover, for every positive integer k, there is a digraph $D \in \mathcal{D}$, which is k-strong.

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