Quasi-hamiltonicity: a series of necessary conditions for a digraph to be hamiltonian

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Abstract

We introduce new necessary conditions, k-quasi-hamiltonicity ($0 \leq k \leq n-1$), for a digraph of order n to be hamiltonian. Every $(k+1)$-quasi-hamiltonian digraph is also k-quasi-hamiltonian; we construct digraphs which are k-quasi-hamiltonian, but not $(k+1)$-quasi-hamiltonian. We design an algorithm that checks k-quasi-hamiltonicity of a given digraph with n vertices and m arcs in time $O(nm^k)$. We prove that $(n-1)$-quasi-hamiltonicity coincides with hamiltonicity and 1-quasi-hamiltonicity is equivalent to pseudo-hamiltonicity introduced (for undirected graphs) by Babel and Woeginger (1997).

1 Introduction

This paper's study was inspired by introduction and investigation of a new necessary condition for hamiltonicity of undirected graphs by Babel and Woeginger [1].

We introduce a series of new necessary conditions, k-quasi-hamiltonicity ($1 \leq k \leq n-1$), for a digraph of order n to be hamiltonian. Every $(k+1)$-quasi-hamiltonian digraph is also k-quasi-hamiltonian; we construct digraphs which are k-quasi-hamiltonian, but not $(k+1)$-quasi-hamiltonian. We prove that $(n-1)$-quasi-hamiltonicity coincides with hamiltonicity and 1-quasi-hamiltonicity is equivalent to pseudo-hamiltonicity introduced (for undirected graphs) by Babel and Woeginger [1]. We also prove a sharp upper bound for the pseudo-hamiltonicity number of a digraph (the problem to compute the pseudo-hamiltonicity number even of an undirected graph is NP-hard [1], though it is polynomial time solvable for special classes of graphs such as trees and cocomparability graphs [1]).

We design an algorithm that checks k-quasi-hamiltonicity of a given digraph with n vertices and m arcs in time $O(nm^k)$. Babel and Woeginger [1] showed that one can verify whether an undirected graph is pseudo-hamiltonian (i.e. 1-quasi-hamiltonian) in polynomial time. Our algorithm restricted to $k = 1$ is of much lower complexity than theirs.

Even 1-quasi-hamiltonicity is more powerful than the well-known and often-used conditions for a digraph to be strong and contain a cycle factor, a collection of vertex-disjoint cycles covering all vertices. As there is an $O(nm^k)$-time algorithm to check k-quasi-hamiltonicity, this condition can be efficiently verified for small values of k. Thus, it can be incorporated in software systems which investigate properties of digraphs (or graphs); one such system is described in [5]. As the existence of a cycle factor and strong connectivity are necessary and sufficient conditions for hamiltonicity in some families of digraphs (for example, in semicomplete bipartite digraphs [6]), k-quasi-hamiltonicity for a constant k may well be a sufficient condition in some wider classes of digraphs.

Our decision to study directed rather than undirected graphs in the framework of this topic is based not only on intention to obtain more general results, but also on the fact that certain properties of digraphs are missing in their undirected counterparts. In particular, a digraph D has a k-cycle ($k \geq 3$) through an arc e if and only if the digraph obtained from D by contracting e contains a cycle through the vertex corresponding to e. This is not true for undirected graphs. One of the key tools in [1] is the notion and properties of a perfect 2-matching in an undirected graph [7], which is a spanning subgraph consisting of vertex-disjoint edges and cycles. A perfect 2-matching becomes simply a cycle factor in the corresponding digraph (replace every edge in an undirected graph by the pair of opposite arcs). Properties of cycle factors in digraphs are well-studied in literature (c.f. [8]).

2 Basic terminology and notation

Digraphs considered in this paper are allowed to have parallel arcs but they do not have loops. The terminology and notation that we use mostly correspond to those in [4]. The set of vertices (arcs, respectively) of a digraph D is denoted by $V(D)$ ($A(D)$, respectively). The number of vertices (arcs, respectively) in the digraph under consideration will be denoted by n (m, respectively). The symbol $\mu_D(x,y)$ stands for the number of arcs from a vertex x to a vertex y in a digraph D. In particular, $\mu_D(x,y) = 0$ means that there is no arc from x to y. A digraph D is *k-regular* if the sum $\sum_{y \in V(D)} \mu_D(x,y) = \sum_{y \in V(D)} \mu_D(y,x) = k$ for every vertex x; a digraph is *regular* if it is k-regular for some k.

In this paper, all paths and cycles are simple and directed. A *k-cycle* is a cycle of length k. A digraph D is said to be *strong* if, for every pair x, y of distinct vertices in D, there is a path from x to y as well as a path from y to x. A *cycle factor* of a digraph D is a collection F of pairwise vertex-disjoint cycles in D such that all vertices of D are in F. In other words, a cycle factor is a spanning 1-regular subdigraph.

Let D be a digraph and let P be a path in D with initial vertex x and terminal vertex y. Then D/P stands for the digraph obtained from D by *contracting* the path P, i.e. $V(D/P) = V(D) \cup \{z\} - V(P)$, where $z \notin V(D)$, and $\mu_{D/P}(uv) = \mu_D(uv)$, $\mu_{D/P}(uz) = \mu_D(ux)$, $\mu_{D/P}(zv) = \mu_D(yv)$ for all distinct $u, v \in V(D) - V(P)$. We will often consider contractions of paths of length one, i.e. arcs e. Clearly, a digraph D has a k-cycle ($k \geq 3$) through an arc e if and only if D/e has a cycle through z.

The *union* of digraphs D and H with the same vertex set is the digraph $R = D \cup H$ with vertex set $V(R) = V(D)(= V(H))$ and arc set defined as follows: $\mu_R(uv) = \mu_D(uv) + \mu_H(uv)$.

3 Quasi-hamiltonicity

Let $QH_1(D)$ be the digraph with vertex set equal to $V(D)$ and arc set

$$\{e \in A(D) : \ e \text{ is contained in a cycle factor of } D\ \}.$$

For $k \geq 2$, $QH_k(D)$ is the digraph with vertex set equal to $V(D)$ and arc set $\{e \in A(D) : QH_{k-1}(D/e) \text{ is strong}\}$. For $k \geq 1$, a digraph D is *k-quasi-hamiltonian*, if $QH_k(D)$ is strong. We assume that every digraph is 0-quasi-hamiltonian. The quasi-hamiltonicity number of a digraph D of order n, $qhn(D)$, is the maximum integer $k(< n)$ such that D is k-quasi-hamiltonian.

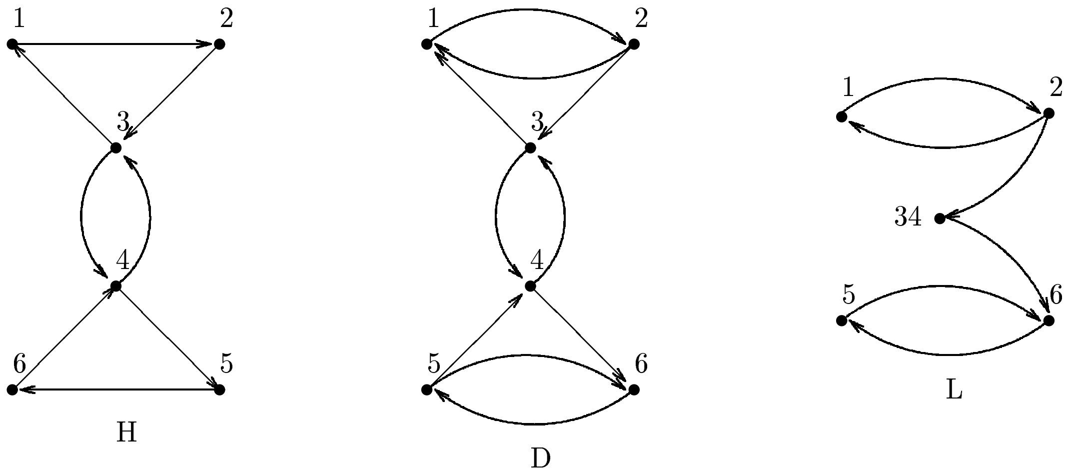

Figure 3.1:

Fig. 3.1 illustrates the notion of quasi-hamiltonicity. The digraph H is 0-quasi-hamiltonian, but not a 1-quasi-hamiltonian ($QH_1(H) = H - \{(3,4),(4,3)\}$ is not strong). Hence, $qhn(H) = 0$. The digraph D is 1-quasi-hamiltonian as $QH_1(D) = D$ is strong (every arc of D belongs to a cycle factor of D). However, D is not 2-quasi-hamiltonian since $QH_2(D)$ is not strong (indeed, $QH_1(D/(3,4)) = QH_1(L)$ is not strong). Thus, $qhn(D) = 1$.

We start with some basic facts on k-quasi-hamiltonicity.

Proposition 3.1 *Let D be a digraph of order $n(\geq 2)$ and let $k \geq 2$. Then $A(QH_k(D)) \subseteq A(QH_{k-1}(D))$. In particular, if D is k-quasi-hamiltonian, it is $(k-1)$-quasi-hamiltonian.*

Proof: We prove the claim by induction on k. Let $e \in A(QH_2(D))$. Thus, $QH_1(D/e)$ is strong which, in particular, means that D/e has a cycle factor. Hence, $e \in A(QH_1(D))$. Let now $k \geq 3$ and let $e \in A(QH_k(D))$. Then, $QH_{k-1}(D/e)$ is strong. By the induction hypothesis, $QH_{k-2}(D/e)$ is also strong. Hence, $e \in A(QH_{k-1}(D))$. □

Theorem 3.2 *A digraph is hamiltonian if and only if it is $(n-1)$-quasi-hamiltonian.*

Proof: Clearly a hamiltonian digraph of order 2 is 1-quasi-hamiltonian. Now assume that all hamiltonian digraphs of order $n-1$ are $(n-2)$-quasi-hamiltonian, and let D be a hamiltonian digraph of order n. Whenever we contract an arc belonging to a hamiltonian cycle we obtain a hamiltonian digraph of order $n-1$, which therefore is $(n-2)$-quasi-hamiltonian. Hence, every arc on a Hamilton cycle lies in $QH_{n-1}(D)$, which implies that $QH_{n-1}(D)$ is strong, i.e. D is $(n-1)$-quasi-hamiltonian. Thus, the "only if" part is proved.

We prove the "if" part. Let D be a digraph, such that $QH_{n-1}(D)$ is strong. Let e_1 be an arc in $QH_{n-1}(D)$. As $QH_{n-2}(D/e_1)$ is strong there exists an arc e_2 in $QH_{n-2}(D/e_1)$. As $QH_{n-3}((D/e_1)/e_2)$ is strong there exists an arc e_3 in $QH_{n-3}((D/e_1)/e_2)$. Continuing this procedure we obtain arcs $e_1, e_2, \ldots, e_{n-2}$, such that $QH_1((((D/e_1)/e_2)\ldots)/e_{n-2})$ is strong. Let

$$D' = (((D/e_1)/e_2)\ldots)/e_{n-2},$$

and observe that since $QH_1(D')$ is strong and D' has order 2, D' must be hamiltonian. By inserting the arcs $e_1, e_2, \ldots, e_{n-2}$ into a Hamilton cycle in D', we obtain a Hamilton cycle in D. □

The proof of the following theorem is given in Section 6.

Theorem 3.3 *For every $k \geq 0$, there exists a digraph D with $qhn(D) = k < n$.*

4 Pseudo-hamiltonicity and 1-quasi-hamiltonicity

A sequence of vertices $Q = v_1 v_2 ... v_{hn} v_1$ in a digraph D of order n is an *h-pseudo-hamiltonian walk* if every vertex of D appears h times in the sequence $v_1 v_2 ... v_{hn}$ and $v_i v_{i+1} \in A(D)$ for every $i = 1, 2, ..., hn$ ($v_{hn+1} = v_1$). A digraph D possessing such a sequence is called *h-pseudo-hamiltonian* and the minimum h for which D is h-pseudo-hamiltonian is the *pseudo-hamiltonicity number* $ph(D)$ of D. If D has no h-pseudo-hamiltonian walk for any positive integer h, then $ph(D) = \infty$. A digraph D is *pseudo-hamiltonian* if $ph(D) < \infty$.

For example, in Fig. 3.1, the digraph D is 2-pseudo-hamiltonian: 1212346565431 is a 2-pseudo-hamiltonian walk of D. This digraph is not 1-pseudo-hamiltonian as D is not hamiltonian. Thus, $ph(D) = 2$. It is not difficult to see that the digraph H in Fig. 3.1 is not pseudo-hamiltonian. We have already seen that D is 1-quasi-hamiltonian, but H is not. The above observations on pseudo-hamiltonicity of D and H actually follow from Theorem 4.2.

Lemma 4.1 follows from the fact that every regular digraph has a cycle factor (see [8]), which implies that every h-regular digraph can be decomposed into h cycle factors.

Lemma 4.1 *Every arc of a regular digraph is included in a cycle factor.*

Theorem 4.2 *A digraph is pseudo-hamiltonian if and only if it is 1-quasi-hamiltonian.*

Proof: Let D be a pseudo-hamiltonian digraph, let Q be an h-pseudo-hamiltonian walk in D, and let $A(Q) = (v_1v_2, v_2v_3, ..., v_{hn-1}v_{hn}, v_{hn}v_1)$ be the sequence of arcs in Q. Construct a new digraph $H(D,Q)$ from D by replacing, for every pair x, y, all arcs from x to y in D by $t(\geq 0)$ parallel arcs from x to y, where t is the number of appearances of xy in $A(Q)$.

By the definition of an h-pseudo-hamiltonian walk, $H(D,Q)$ is an h-regular digraph. Thus, by Lemma 4.1, every arc xy in $H(D,Q)$ is in a cycle factor. Therefore, $\mu_{H(D,Q)}(xy) > 0$ implies $\mu_{QH_1(D)}(xy) > 0$. Since $H(D,Q)$ is strong, we obtain that $QH_1(D)$ is also strong, i.e. D is 1-quasi-hamiltonian.

Now let D be a 1-quasi-hamiltonian digraph, i.e. $QH_1(D)$ is strong. For each arc e in $QH_1(D)$ let F_e be a cycle factor in D including e. Let $D' = \cup_{e \in A(QH_1(D))} F_e$. As a union of cycle factors, D' is regular. Since $QH_1(D)$ is strong, D' is also strong. Therefore, D' has a eulerian walk, which corresponds to a pseudo-hamiltonian walk in D. □

The following theorem provides us by a sharp upper bound for the pseudo-hamiltonicity number of a directed (or undirected) graph. No such bound is given in [1]. The proof of Theorem 4.3 is given in Section 6.

Theorem 4.3 *For a pseudo-hamiltonian digraph D, $ph(D) \leq (n-1)/2$. For every integer $n \geq 3$, there exists a digraph H_n of order n such that $ph(H_n) = \lfloor (n-1)/2 \rfloor$.*

5 Algorithms

An *almost cycle factor* of a digraph D is a collection of vertex-disjoint subdigraphs of D covering $V(D)$ and such that one of the subdigraphs is a path and the rest are cycles. It is well-known (c.f. [6, 8]) that a cycle factor (an almost cycle factor) of a digraph D with vertex set $V(D) = \{x_1, ..., x_n\}$ corresponds to a perfect matching (a matching with $(n-1)$ edges) of a bipartite graph $B(D)$ with partite sets $\{y_1, ..., y_n\}$ and $\{z_1, ..., z_n\}$ and edge set $\{y_i z_j : \mu_D(x_i x_j) > 0\}$. This allows us to reformulate two well-known results on the complexity of finding a perfect matching in a bipartite graph [9].

Lemma 5.1 *In time $O(n^{1/2}m)$ we can check whether a digraph D has a cycle factor and construct one (if it exists). Given an almost cycle factor in a digraph D, in time $O(m)$, we can check whether a digraph D has a cycle factor and construct one (if it exists).*

Theorem 5.2 *We can check whether D is pseudo-hamiltonian in $O(nm)$ time.*

Proof: We first formulate the desired algorithm $\mathcal{B}$.

1. Construct a cycle factor F in D. If no cycle factor exists then the algorithm terminates informing us that D is not pseudo-hamiltonian.

2. Choose a cycle C in F and set $B := \emptyset$.

3. If $V(C) = V(D)$ then the algorithm terminates informing us that D is pseudo-hamiltonian.

4. If $V(C) = B$ then the algorithm terminates informing us that D is not pseudo-hamiltonian.

5. Choose an arbitrary vertex $x \in V(C) - B$. Construct D_x, the digraph obtained from D by deleting all arcs from x to $V(C)$ (we preserve the arcs from $V(C)$ to x, and between x and $D - V(C)$). Consider the following two cases:

5a. If there is a cycle factor, F_x, in D_x, then set $F := F \cup F_x$ and construct C', the strong component in F which includes the vertices from C. Set $C := C'$ and go to Step 3.

5b. If there is no cycle factor in D_x, then set $B := B \cup \{x\}$ and go to Step 3.

We now prove that algorithm $\mathcal{B}$ returns the correct answer.

If $\mathcal{B}$ informs us that D is pseudo-hamiltonian, then it means that F is a strong regular digraph, which therefore contains a eulerian trail, T. Clearly, T is a pseudo-hamiltonian walk in D.

Now suppose that $\mathcal{B}$ informs us D is not pseudo-hamiltonian, but in fact D is. As D is pseudo-hamiltonian, D is 1-quasi-hamiltonian, by Theorem 4.2. Thus, in particular, D has a cycle factor. Hence, the algorithm informs us that D is not pseudo-hamiltonian on Step 4 and terminates. Let B' (C', respectively) be the value of B (C, respectively) when the algorithm terminates, i.e. $B' = V(C')$. Since D is 1-quasi-hamiltonian, $QH_1(D)$ is strong. Thus, there exists a cycle factor in D that contains an arc xy from B' to $V(D) - B'$. Let B'' be the value of B just before x was included in B (on Step 5b). Clearly, $B'' \subseteq B' - \{x\}$. As there is a cycle factor in D through xy, D_x had a cycle factor on Step 5 when x was included in B; a contradiction to the outcome of Step 5.

In order to prove the time complexity, we observe that we can perform Step 5b at most n times as we increase the size of B in each iteration. Furthermore we can perform Step 5a at most n times as we increase the size of $V(C)$ in each iteration. Therefore we perform Steps 3, 4 and 5 at most $2n$ times. As Step 1 takes $O(n^{1/2}m)$ time by the first part of Lemma 5.1, and Steps 3, 4 and 5 take at most $O(m)$ time by the second part of Lemma 5.1 (note that

$F - xx^+$, where x^+ is the successor of x on the cycle in F which contains x, is an almost cycle factor in D_x), we obtain the $O(nm)$ time bound. □

The following theorem implies that one can check k-quasi-hamiltonicity for a constant k in polynomial time.

Theorem 5.3 *In $O(nm^k)$ time, one can check if a digraph is k-quasi-hamiltonian.*

Proof: In this proof, we describe an algorithm $\mathcal{A}$ that, in time $T(k)$, checks whether a digraph D is k-quasi-hamiltonian. We will show that $T(k) = O(nm^k)$.

If $k = 1$, the algorithm $\mathcal{A}$ uses the algorithm $\mathcal{B}$ in Theorem 5.2. Thus, $T(1) = O(nm)$. If $k \geq 2$ then, for each arc e in D, $\mathcal{A}$ verifies whether D/e is $(k-1)$-quasi-hamiltonian. The algorithm $\mathcal{A}$ forms $QH_k(D)$ from all arcs e such that D/e is $(k-1)$-quasi-hamiltonian. Finally, $\mathcal{A}$ checks whether $QH_k(D)$ is strong (in time $O(m)$).

This implies that, for $k \geq 2$,

$$T(k) \leq mT(k-1) + O(m).$$

As $T(1) = O(nm)$, we obtain that $T(k) = O(nm^k)$. □

6 Proofs of Theorems 4.3 and 3.3

Proof of Theorem 4.3:

We first show that, for every pseudo-hamiltonian digraph D, we have $ph(D) \leq (n-1)/2$. Let F be the cycle subgraph found in Step 1 of algorithm $\mathcal{B}$. Every time we add a cycle factor to F in Step 5a of $\mathcal{B}$ we observe that the vertices of some strong component in F are added to C, which implies that we can perform Step 5a at most $n/2 - 1$ times, as every strong component in F has at least two vertices. Therefore $ph(D) \leq n/2$. Assume that $ph(D) = n/2$, which implies that every cycle factor of D consists of $n/2$ cycles of length two and $|V(C)|$ increases by exactly two every time we perform Step 5a. Let F and F' be a pair of different cycle factors in D, and let $x \in V(D)$ be a vertex which does not lie on the same cycle in F and in F'. As $|V(C)|$ increases by exactly two, every time we perform Step 5a, we obtain that the strong component, R, in $F \cup F'$, which includes x has size 4. Let xyx be a cycle in F and xzx be a cycle in F'. Thus, if zwz is a cycle in F then ywy must be a cycle in F'. However this implies that $(F - xyx - zwz) \cup xyzwx$ is a cycle subgraph containing a cycle of length four, a contradiction. Therefore $ph(D) \leq (n-1)/2$.

For every odd integer $n = 2k+1 \geq 3$ the digraph H_n has vertex set $V(H_n) = \{x\} \cup Y \cup Z$, where $Y = \{y_1, y_2, \ldots, y_k\}$ and $Z = \{z_1, z_2, \ldots, z_k\}$, and arc set

$$A(H_n) = \{xy_i, y_ix, xz_i, z_ix, y_iz_i, z_iy_i : \ i = 1, ..., k\}.$$

For every even integer $n \geq 4$, the digraph H_n is obtained from H_{n-1} by appending a new vertex x' together with the arcs $y_1x', x'y_1, x'z_1, z_1x'$.

To reach a vertex in Y from another one, every pseudo-hamiltonian walk in H_n must visit x, thus, we obtain that $ph(H_n) \geq \lfloor (n-1)/2 \rfloor$. For an odd $n = 2k+1$,

$$xy_1^1z_1^1y_1^2z_1^2...y_1^kz_1^kxy_2^1z_2^1y_2^2z_2^2...y_2^kz_2^kx...xy_k^1z_k^1y_k^2z_k^2...y_k^kz_k^kx,$$

where $y_i^j = y_i, z_i^j = z_i$ for every $i = 1, 2, ..., k$, is a k-pseudo-hamiltonian walk in H_n. Similarly, one can construct a k-pseudo-hamiltonian walk in H_n for $n = 2k+2$. Thus, we conclude that $ph(H_n) = \lfloor (n-1)/2 \rfloor$. □

Clearly, for the underlying graph $U(H_n)$ of H_n we obtain that $ph(U(H_n)) = \lfloor (n-1)/2 \rfloor$ as well. Thus, the bound in the previous theorem is sharp for undirected graphs, too.

Proof of Theorem 3.3:

Let B be a collection of arcs in a digraph D such that the arcs of B constitute $s(\geq 1)$ vertex-disjoint paths $P_1, P_2, ..., P_s$. We define $D/B = ((D/P_1)/P_2...)/P_s$. Clearly, the resulting digraph D/B does not depend on the order of the paths.

Let G_k be an undirected graph with vertex set $X \cup Z \cup Y$, where $X = \{x_1, x_2, \ldots, x_k\}$, $Y = \{y_1, y_2, \ldots, y_{k+1}\}$ and $Z = \{z_1, z_2, \ldots, z_{k+1}\}$, and edge set

$$\{xv : \ x \in X, v \in Y \cup Z\} \cup \{y_i z_i : \ i = 1, 2, \ldots, k+1\}.$$

Let D_k be the digraph obtained from G_k by substituting each edge with a pair of mutually opposite arcs with the same end-vertices. We will prove that $qhn(D_k) = k$.

We first prove that D_k is k-quasi-hamiltonian. We say that a collection of arcs B in D_k is *X-path-extendable*, if there exists a set of arcs R, such that the arc set $B \cup R$ induces a path Q starting and terminating in X such that $|V(Q) \cap Y| = |V(Q) \cap Z| = |V(Q) \cap X| - 1 = k-1$. The path Q is called an *X-extension* of B.

We prove that if B is an X-path-extendable set of arcs of cardinality $c \leq k-1$, then $QH_{k-c}(D_k/B)$ is strong. The proof is by induction on $c \in \{k-1, k-2, ..., 0\}$. Assume that $c = k-1$. Let P be an X-extension of B, and let $R^* = P/B$. Clearly there are a pair of distinct indices $i \neq j$, such that $\{z_i, y_i, z_j, y_j\} = V(D_k) - V(P)$. Since both $y_i P z_i y_i \cup z_j y_j z_j$ and $y_j P z_j y_j \cup z_i y_i z_i$ are cycle factors in D_k, $y_i R^* z_i y_i \cup z_j y_j z_j$ and $y_j R^* z_j y_j \cup z_i y_i z_i$ are cycle factors in D_k/B. Thus, $QH_{k-c}(D_k/B) = QH_1(D_k/B)$ includes all arcs in $A(y_i R^* z_i y_i) \cup A(y_j R^* z_j y_j)$, which proves that $QH_1(D_k/B)$ is strong.

Now assume that $QH_{k-(c+1)}(D_k/B)$ is strong when $|B| = c+1$, and let $|B| = c$ $(c \geq 0)$. Let S be the set of arcs in D_k such that $e \in S$ if and only if $B \cup \{e\}$ is X-path-extendable. For an arc $e \in S$, let e^* be the arc in D_k/B corresponding to e. Let $S^* = \{e^* : \ e \in S\}$. For an arc $e \in S$, by the induction hypothesis,

$$QH_{k-(c+1)}(D_k/(B \cup \{e\})) = QH_{k-(c+1)}((D_k/B)/e^*)$$

is strong, so e^* is an arc in $QH_{k-c}(D_k/B)$. Thus, $S^* \subseteq A(QH_{k-c}(D_k/B))$.

Now we prove that the arcs of S^* in D/B induce a strong spanning subdigraph. Let P be an X-extension of B and let $R^* = P/B$. Assume that $\{z_i, y_i, z_j, y_j\} = V(D_k) - V(P)$, Clearly the arcs of R^* lie in S^*. As $c+1 \leq k-1$, we observe that there is an index $k \notin \{i, j\}$, such that no arc in B is incident with either z_k or y_k. As $P y_i z_i$ and $P y_j z_j$ are cycles in D_k, $R^* y_i z_i$ and $R^* y_j z_j$ are cycles in D_k/B. By deleting all arcs in $R^* y_i z_i$ ($R^* y_j z_j$, respectively) incident with z_k and y_k we obtain a path R_1^* (R_2^*, respectively) whose arcs lie in S^*. Clearly the fact that $A(R^*) \cup A(R_1^*) \cup A(R_2^*) \subseteq S^*$ implies that the arcs S^* in D_k/B induce a strong spanning subdigraph.

Using the above result with $B = \emptyset$ we obtain that $QH_k(D_k)$ is strong, so $qhn(D_k) \geq k$.

We now prove that D_k is not $(k+1)$-quasi-hamiltonian. A set of arcs $B = \{u_1 v_1, ..., u_c v_c\}$ in D_k is called *good* if no pair of arcs in B has a common vertex and $\{u_1, ..., u_c\} \subseteq X$, $\{v_1, ..., v_c\} \subseteq Y \cup Z$. We show that, for every $c \in \{k, k-1, ..., 0\}$ and every good set B of cardinality c, the digraph $QH_{k+1-c}(D_k/B)$ is not strong.

If $|B| = k$, then D_k/B is not 1-quasi-hamiltonian. Indeed, let y_i, z_i be a pair of vertices which are not end-vertices in B. Every cycle factor in D_k/B must include the cycle $y_i z_i y_i$. Therefore, $QH_1(D_k/B)$ is not strong as y_i and z_i are not adjacent to any other vertices in $QH_1(D_k/B)$.

Assume that $QH_{k+1-(c+1)}(D_k/B)$ is not strong for some $c+1$ $(1 \leq c+1 \leq k)$ and every B of cardinality $c+1$. Let S be the set of arcs, such that $e \in S$, if and only if $B \cup \{e\}$ is good. For an arc $e \in S$, let e^* be the arc in D_k/B corresponding to e.

Let $e \in S$. As $QH_{k+1-(c+1)}(D_k/(B \cup \{e\}))$ is not strong, $e^* \notin QH_{k+1-c}(D_k/B)$ for every $e \in S$. This means that there is no arc, in $QH_{k+1-c}(D_k/B)$, with initial vertex in $X - V(B)$. Thus, $QH_{k+1-c}(D_k/B)$ is not strong. Using the above result when $B = \emptyset$, we obtain that $QH_{k+1}(D_k)$ is not strong, so $qhn(D_k) \leq k$. □

7 Discussion and open problems

In this paper, we have initiated the study of k-quasi-hamiltonicity, a series of necessary conditions for hamiltonicity of a digraph. As, for every fixed k, k-quasi-hamiltonicity can be checked in polynomial time, these conditions may well be of use for efficient, i.e. polynomial time verifiable, characterizations of hamiltonicity in some special families of digraphs. There are two well-known families of digraphs, semicomplete bipartite digraphs [6] and extended locally semicomplete digraphs [2], where 1-quasi-hamiltonicity is a sufficient condition for hamiltonicity (a digraph from any of these two families is hamiltonian if and only if it is strong and contains a cycle factor).

Possible candidates for efficient characterizations via k-quasi-hamiltonicity (with, perhaps, some additional conditions) may be semicomplete p-partite digraphs [2, 3, 6] (digraphs obtained from complete p-partite graphs by replacing every edge xy with arc xy or arc yx or both xy and yx) and quasi-transitive digraphs [2] (digraphs where the existence of arcs xy and yz $(x \neq z)$ implies the existence of an arc between x and z). It was recently proved that the Hamilton cycle problems in both families are polynomially time solvable (c.f. [2, 3]). However, no efficient theoretical characterizations of hamiltonicity in these families are known so far.

Another interesting problem is to design a faster algorithm for deciding whether a digraph is k-quasi-hamiltonian.

References

[1] L. Babel and G.J. Woeginger, Pseudo-hamiltonian graphs, Proc. of the 23rd International Workshop on Graph-Theoretic Concepts in Computers Science WG'97, Springer Verlag, *LNCS* **1335** (1997), 38–51.

[2] J. Bang-Jensen and G. Gutin, Generalizations of tournaments: A survey, *J. Graph Theory* **28** (1998), 171–202.

[3] J. Bang-Jensen, G. Gutin and A. Yeo, A polynomial algorithm for the Hamiltonian cycle problem in semicomplete multipartite digraphs, *J. Graph Theory* **29** (1998) 111–132.

[4] J. A. Bondy and U. S. R. Murty, *Graph Theory with Applications*, MacMillan Press (1976).

[5] C. Delorme, O. Ordaz and D. Quiroz, Tools for studying paths and cycles in digraphs, *Networks* **31** (1998), 125–148.

[6] G. Gutin, Cycles and paths in semicomplete multipartite digraphs, theorems and algorithms: A survey, *J. Graph Theory* **19** (1995), 481–505.

[7] L. Lovász and M.D. Plummer, Matching Theory, *Annals of Discrete Math.* **29** North-Holland, 1986.

[8] O. Ore, *Theory of Graphs*, Amer. Math. Soc. Coll. Publ., 1962.

[9] C.H. Papadimitriou and K. Steiglitz, *Combinatorial Optimization*, Prentice-Hall, New Jersey, 1982.