Solution of a conjecture of Volkmann on the number of vertices in longest paths and cycles of strong semicomplete multipartite digraphs

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Abstract

A digraph obtained by replacing each edge of a complete multipartite graph by an arc or a pair of mutually opposite arcs with the same end vertices is called a semicomplete multipartite digraph. L. Volkmann conjectured that $l \leq 2c-1$, where l (c, respectively) is the number of vertices in a longest path (longest cycle) of a strong semicomplete multipartite digraph. The bound on l is sharp. We settle this conjecture in affirmative.

Running title: Solution of a conjecture of Volkmann

1 Introduction, terminology and notation

L. Volkmann [6] conjectured that $l \leq 2c-1$, where l (c, respectively) is the number of vertices in a longest path (longest cycle) of a strong semicomplete multipartite digraph. An example taken from [2] shows that the bound on l is sharp. (We describe this example in the end of this paper.) We settle Volkmann's conjecture in affirmative.

We will assume that the reader is familiar with the standard terminology on graphs and digraphs and refer the reader to [3]. By a *cycle* and a *path* in a directed graph we mean a directed simple cycle and path, respectively. Let D be a digraph. $V(D)$ ($A(D)$) denotes the vertex (arc) set of D. A digraph D is *strong* if there exists a path from x to y for every choice of distinct vertices x, y of D. A path from x to y is an (x, y)-*path*. The subpath of a cycle C from a vertex v to a vertex w will be denoted by $C[v, w]$. Analogously, $C[v, w[$ stands for the subpath of C from v to the predecessor of w (in the path $C[v, w]$).

If D has an arc $xy \in A(D)$, then we often use the notation $x \rightarrow y$ and say that x *dominates* y and y *is dominated by* x. For disjoint sets X and Y of vertices in D, we say that X *strongly dominates* Y, and use the notation $X \Rightarrow Y$, if there is no arc from Y to X. This means that for every pair $x \in X, y \in Y$ of adjacent vertices x dominates y, but y does not dominate x. For a subset X of $V(D)$, $D\langle X\rangle$ is the subdigraph of D induced by X.

A collection $B_1, B_2, ..., B_m$ of sets is a *partition* of a set B, if and only if, $\cup_{i=1}^{m} B_i = B$ and $B_i \cap B_j = \emptyset$ for all $1 \leq i < j \leq m$. Note that we allow some B_i's to be empty sets. A digraph D is called *semicomplete p-partite* (or, just, *semicomplete multipartite*) if there is a partition $V_1, ..., V_p$ of $V(D)$ such that no set of the partition is empty, no arc has both end vertices in the same V_i, and, for every pair $x_i \in V_i, x_j \in V_j$ $(i \neq j)$ at least one of the arcs $x_i x_j$ and $x_j x_i$ is in D. The sets V_i are the *partite sets* of D. Semicomplete multipartite digraphs are well studied, see, e.g., survey papers [1, 4, 5, 7].

2 Bound

Lemma 2.1 *Let D be a semicomplete multipartite digraph. Let non-empty sets $Q_1, Q_2, ..., Q_l$ form a partition of $V(D)$ such that $Q_i \Rightarrow Q_j$ for every $1 \leq i < j \leq l$. Assume that $|V(D)| > l$ and $D\langle Q_i\rangle$ has a Hamilton path $q_1^i q_2^i ... q_{|Q_i|}^i$ for every $i = 1, 2, ..., l$. Then, D has a $(q_1^1, q_{|Q_l|}^l)$-path with at least $|V(D)| - l + 1$ vertices.*

Proof: We use the induction on l. Clearly the theorem holds when $l = 1$, so assume that $l > 1$.

If $|V(D) - Q_l| > (l - 1)$ then, by the induction hypothesis, there is a $(q_1^1, q_{|Q_{l-1}|}^{l-1})$-path, $p_1 p_2 .. p_k$, in $D - Q_l$ which contains $k \geq |V(D) - Q_l| - (l - 1) + 1 \geq 2$ vertices. As $\{p_{k-1}, p_k\} \Rightarrow q_1^l$ and p_{k-1} and p_k belong to different partite sets, $p_{k-1} \rightarrow q_1^l$ or $p_k \rightarrow q_1^l$. Therefore, the path $p_1 p_2 ... p_s q_1^l q_2^l ... q_{|Q_l|}^l$, where $s = k - 1$ or k, is of the desired type.

If $|V(D) - Q_l| \leq (l - 1)$, then clearly $|Q_l| > 1$. As $q_1^1 \Rightarrow \{q_1^l, q_2^l\}$ and q_1^l and q_2^l belong to different partite sets, $q_1^1 \rightarrow q_1^l$ or $q_1^1 \rightarrow q_2^l$. Therefore, the path $q_1^1 q_s^l q_{s+1}^l ... q_{|Q_l|}^l$, where $s \in \{1, 2\}$, is of the desired type. $\square$

Theorem 2.2 *Let D be a strong semicomplete multipartite digraph and let l be the number of vertices in a longest path in D and let c be the number of vertices in a longest cycle in D. Then $l \leq 2c - 1$.*

Proof: Let $P = p_1p_2..p_l$ be a path in D of maximum length and let $R = V(D) - V(P)$. Let $x_0 = p_l$ and define S_i, x_i and y_i recursively as follows $(i = 1, 2, ...)$.

First let S'_1 be a (p_l, p_k)-path in D, such that k is chosen as small as possible. Let $x_1 = p_k$, let $y_1 = p_l$ and let $S_1 = S'_1 - \{x_1, y_1\}$ (note that $S_1 = \emptyset$, by the maximality of l). Now for $i = 2, 3, 4, \dots$ let S'_i be a (p_t, p_k)-path in $D\langle\{p_t, p_k\} \cup R - (V(S_1) \cup V(S_2) \cup ... \cup V(S_{i-1})\rangle$, such that $p_t \in V(P[x_{i-1}, p_l])$ and $p_k \in V(P[p_1, x_{i-1}[)$, and firstly k is chosen as small as possible, thereafter t is chosen as large as possible. Let also $x_i = p_k$, $y_i = p_t$ and $S_i = S'_i - \{x_i, y_i\}$. (Some paths S_i can be empty)

We continue the above process until $x_i = p_1$. Let the last value of i found above be denoted by m (i.e. $x_m = p_1$). Observe that the paths S'_i always exist as D is strong. Observe also that $y_1 = p_l$ and that T below is a path in D:

$$T = y_1S_1P[x_1, y_2]S_2P[x_2, y_3]S_3...P[x_{m-1}, y_m]S_mx_m$$

Let $U_0 = P[x_m, x_{m-1}] - \{x_m, x_{m-1}\}$ and let $U_i = P[y_{m-i+1}, x_{m-i-1}] - \{y_{m-i+1}, x_{m-i-1}\}$ for $i = 1, 2, .., m-1$. Note that some of the U_i's $(i = 0, 1, 2, .., m-1)$ can be empty. Observe that $V(T), U_0, U_1, ..., U_{m-1}$ partitions the set $V(P) \cup V(S_1) \cup ... \cup V(S_m)$. Let $Z_0, Z_1, ..., Z_{m'-1}$ be the non-empty sets in $U_0, U_1, ..., U_{m-1}$, where the relative ordering has been kept (i.e. if $Z_i = U_{i'}$, $Z_j = U_{j'}$ and $i' < j'$ then $i < j$). Let $B_0 = Z_0 \cup Z_2 \cup ... \cup Z_f$ and $B_1 = Z_1 \cup Z_3 \cup ... \cup Z_g$, where f (g, respectively) is the maximum even integer (odd integer, respectively) not exceeding $m' - 1$.

If $p_l \rightarrow p_1$, then we are done (the cycle Pp_1 is of length l). Thus, we may assume that p_1 is not dominated by p_l. As $x_m = p_1$, we obtain the following:

$$p_1 \Rightarrow Z_1 \cup Z_2 \cup ... \cup Z_{m'-1} \cup \{p_l\}. \tag{1}$$

By the definitions of x_i, y_i and S_i,

$$Z_i \Rightarrow Z_{i+2} \cup Z_{i+3} \cup ... \cup Z_{m'-1} \cup \{p_l\}, \;\; i = 0, 1, .., m' - 2. \tag{2}$$

As $\{x_0, x_1, ..., x_m\} \subseteq V(T)$ and $m \geq m'$, we have

$$|V(T)| \geq m' + 1. \tag{3}$$

As $V(T), B_0, B_1$ partitions the set $V(P) \cup V(S_1) \cup ... \cup V(S_m)$,

$$|V(T)| + |B_0| + |B_1| \geq l. \tag{4}$$

All $Z_i \cup \{x_i\}$ are disjoint sets containing at least two vertices. Thus, there are at most $l/2$ such sets. Hence, we obtain

$$m' \leq \frac{l}{2}. \tag{5}$$

We only consider the case when m' is odd since the case of even m' can be treated similarly.

If $|B_0| + 2 > \frac{m'+1}{2}$ then, by (1), (2) and Lemma 2.1, there is a path W_0 from p_1 to p_l in $D\langle \{p_1, p_l\} \cup B_0 \rangle$ containing at least $|B_0| + 2 - \frac{m'+1}{2} + 1$ vertices (let $V(Q_1) = \{p_1\} \cup Z_0$, $V(Q_2) = Z_2$, ..., $V(Q_{(m'-1)/2}) = Z_{m'-3}$, $V(Q_{(m'+1)/2}) = Z_{m'-1} \cup \{p_l\}$). Analogously if $|B_1| > \frac{m'-1}{2}$, then there is a path W_1 from p_1 to p_l in $D\langle \{p_1, p_l\} \cup B_1 \rangle$ containing at least $|B_1| - \frac{m'-1}{2} + 1$ vertices (let $V(Q_1) = \{p_1\}$, $V(Q_2) = Z_1$, ..., $V(Q_{(m'-1)/2}) = Z_{m'-2}$, $V(Q_{(m'+1)/2}) = \{p_l\}$).

We now consider the cases where none, one or two of the paths W_0 and W_1 exist.

Case 1: Both W_0 and W_1 exist. The cycle $C_0 = W_0T$ contains $|V(T)|+|V(W_0)|-2$ vertices (as p_1 and p_l are counted twice). The cycle $C_1 = W_1T$ contains $|V(T)|+|V(W_1)|-2$ vertices. By (3) and (4), this implies the following:

$$\begin{array}{rcl} |V(C_0)| + |V(C_1)| & = & 2|V(T)| + |V(W_0)| + |V(W_1)| - 4 \\ & \geq & 2|V(T)| + (|B_0| + 2 - \frac{m'+1}{2} + 1) + (|B_1| - \frac{m'-1}{2} + 1) - 4 \\ & = & |V(T)| + (|V(T)| + |B_0| + |B_1|) - m' \\ & \geq & |V(T)| + l - m' \\ & \geq & l + 1. \end{array}$$

This implies that the largest cycle of C_0 and C_1 contains at least $\lceil (l+1)/2 \rceil$ vertices. Thus, we are done.

Case 2: Exactly one of W_0 or W_1 exists. Let $j \in \{0, 1\}$ be defined such that W_j exists, but W_{1-j} does not exist. Using (4) and (5), and observing that either $|B_0| + 2 - \frac{m'+1}{2} + 1 \leq 1$ or $|B_1| - \frac{m'-1}{2} + 1 \leq 1$, we obtain the following ($C_j = W_jT$, as above):

$$\begin{array}{rcl} |V(C_j)| & \geq & |V(T)| - 2 + (|B_0| + 2 - \frac{m'+1}{2} + 1) + (|B_1| - \frac{m'-1}{2} + 1) - 1 \\ & \geq & |V(T)| + |B_0| + |B_1| - m' + 1 \\ & \geq & l - m' + 1 \\ & \geq & l - \frac{l}{2} + 1 \\ & \geq & \frac{l+2}{2}. \end{array}$$

This is the desired result.

Case 3: Neither W_0 nor W_1 exists. This means that $|B_0| + 2 - \frac{m'+1}{2} \leq 0$ and $|B_1| - \frac{m'-1}{2} \leq 0$. Thus,

$$
\begin{array}{rcl}
|V(T)| & \geq & |V(T)| + (|B_0| + 2 - \frac{m'+1}{2}) + (|B_1| - \frac{m'-1}{2}) \\
 & = & |V(T)| + |B_0| + |B_1| + 2 - m' \\
 & \geq & l - m' + 2 \\
 & \geq & \frac{l+2}{2} + 1.
\end{array}
$$

If $p_1 \rightarrow p_l$ then there is a cycle of length at least $\lceil \frac{l+2}{2} + 1 \rceil$ (using all vertices in $V(T)$). By (1), p_l does not dominate p_1, so we may assume that p_1 and p_l are in the same partite set. We have $S_m = \emptyset$ as otherwise $S_m P$ is longer than P which is impossible, hence y_m and p_l are in different partite sets. If $p_l \rightarrow y_m$ then either $P[y_m, p_l]y_m$ or $P[p_1, y_m]p_1$ is a cycle with at least $\lceil \frac{l+1}{2} \rceil$ vertices. Therefore, we may assume that $y_m \rightarrow p_l$. Then the cycle $T[p_l, y_m]p_l$ contains at least $\lceil \frac{l+2}{2} \rceil$ vertices. We are done.

$\square$

3 Bondy's example

This example originates from [2]. Let H be a semicomplete m-partite digraph with partite sets $V_1, V_2, ..., V_m$ ($m \geq 3$) such that $V_1 = \{v\}$ and $|V_i| \geq 2$ for every $2 \leq i \leq n$. Let also $V_2 \Rightarrow v$, $v \Rightarrow V_j$ for $3 \leq j \leq m$, and $V_k \Rightarrow V_i$ for $2 \leq i < k \leq m$. It is easy to verify that a longest path (cycle, respectively) in H contains $2m-1$ (m, respectively) vertices.

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