# MAXIMIZING TRAVELING SALESMAN PROBLEM FOR SPECIAL MATRICES

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**Abstract.** We consider the maximizing traveling salesman problem (MTSP) for two special classes of $n \times n$ matrices with non-negative entries, namely, matrices from $M(n)$ and $M(n,\alpha)$ ($\alpha \geq 3$) defined as follows. An $n \times n$ matrix $W = [w_{ij}] \in M(n)$ if $w_{ij} = 0$ for all $i, j$ such that $|i-j| \neq 1$. An $n \times n$ matrix $W = [w_{ij}] \in M(n,\alpha)$ if $\min_{|i-j|=1} w_{ij} \geq \alpha \max_{|i-j|\neq 1} w_{ij}$ . We describe an $O(n)$-algorithm solving exactly the MTSP for matrices from $M(n)$ and show that this algorithm provides an approximate solution of the MTSP for matrices from $M(n,\alpha)$ for $\alpha \geq 3$ with a relative error of at most $n/(2\alpha(n-1))$. It is proved that the MTSP is NP-hard for matrices from $M(n,\alpha)$ for every fixed positive $\alpha$.

## 1. Introduction

We consider the following problem which is called the maximizing traveling salesman problem (MTSP). Given an $n \times n$ matrix $W = [w_{ij}]$ with non-negative entries, find a cyclic permutation $\pi \in C_n$ such that $\sum_{i=1}^{n} w_{i\pi(i)}$ is as large as possible. It is known that the MTSP is NP-hard [4].

We consider this problem for two special classes of $n \times n$ matrices with non-negative entries, namely, matrices from $M(n)$ and $M(n,\alpha)$ ($\alpha \geq 3$) defined as follows. An $n \times n$

matrix $W = [w_{ij}] \in M(n)$ if $w_{ij} = 0$ for all $i, j$ such that $|i - j| \neq 1$. An $n \times n$ matrix $W = [w_{ij}] \in M(n, \alpha)$ if

$$\min_{|i-j|=1} w_{ij} \geq \alpha \max_{|i-j|\neq 1} w_{ij} \ . \tag{1}$$

We describe an $O(n)$-algorithm solving the MTSP for matrices from $M(n)$ exactly and show that this algorithm provides an approximate solution of the MTSP for matrices from $M(n, \alpha)$ for $\alpha \geq 3$ with a relative error of at most $n/(2\alpha(n-1))$. It is proved that the MTSP is NP-hard for matrices from $M(n, \alpha)$ for every fixed positive $\alpha$.

The MTSP have been investigated in a number of papers where several algorithms have been proposed for approximate solution of the problem for the class of all matrices with nonnegative entries (greedy algorithms) as well as for some special subclasses of this class (cf. [3,5,6]).

Our algorithm has a smaller complexity than that of all previous known algorithms and the smallest upper bound for the relative error of the solution (in the case of matrices from $M(n, \alpha), \alpha \geq 3$).

We describe a simple situation when the MTSP for matrices from $M(n, \alpha)$ can occur. Suppose that the cities are uniformly situated along a road. Since the distance between any pair of neighbours does not exceed that between any pair of non-neighbours, one can assume that the profit obtained by traveling salesman when he moves between neighbours is higher than that when he moves between non-neighbours.

Since the $O(n)$-algorithm above provides a rather good solution for matrices from $M(n, \alpha)$, one can apply the algorithm for approximate solution of the MTSP's for more general classes of matrices (at least as an initial solution for local search algorithms).

A $k$-index matrix $W = [w_{i_1 i_2 \dots i_k}]$ is called a Jacobi matrix if $w_{i_1 i_2 \dots i_k} = 0$ whenever $\max\{|i_s - i_t|\} > 1 (1 \leq s, t \leq k)$. There are linear time algorithms solving the (maximizing) assignment problem [1,2] for two- and three-index Jacobi matrices with non-negative entries. Note that, in general, the three-index assignment problem is NP-hard [4].

## 2. Notation and Definitions

For an $n \times n$ matrix $W = [w_{ij}], G[W]$ denotes the weighted complete symmetric digraph on $n$ vertices $1, 2, \dots, n$ such that the *weight* of an arc $(i, j)$ equals $w_{ij}$. A *tour* of $G[W]$ is a Hamiltonian cycle of $G[W]$, i.e. a simple directed cycle containing all the vertices of $G[W]$. The *weight* $w(H)$ of a tour $H$ of $G[W]$ is the sum of the weights of its arcs. A tour of $G[W]$ is called *maximum* if its weight is maximum.

## 3. Matrices from $M(n)$

**Lemma 1.** *If a matrix $W = [w_{ij}] \in M(n)$, $n \geq 4$, then for any tour $X$ of $G$ there is a tour $Y$ including either the arc $(1,2)$ or the arc $(2,1)$ and $W(X) \leq W(Y)$.*

**Proof:** Suppose that a matrix $W \in M(n)$ and a tour $X$ contains neither $(1,2)$ nor $(2,1)$. W.l.o.g. we assume that $(3,2) \notin X$. Then $w_{i1} = w_{1j} = w_{k2} = 0$, where $(i,1), (1,j), (k,2) \in X$. Put $Y = X \cup \{(i,j), (k,1), (1,2)\} \setminus \{(i,1), (1,j), (k,2)\}$. Obviously, $w(Y) \geq w(X)$ and $Y$ contains $(1,2)$. □

**Theorem 2.** *Given a matrix $W \in M(n)$, then the weight $f(n)$ of a maximum tour of $G[W]$ can be found from the following system of recurrent relations:*

$$f(n) = \max\{w_{12} + f_1(n-1);\ w_{21} + f_2(n-1)\}\ ,$$

$$f_1(i) = \max\{w_{n+1-i,n+2-i} + f_1(i-1);\ f_2(i-1)\}\ ,$$

$$f_2(i) = \max\{w_{n+2-i,n+1-i} + f_2(i-1);\ f_1(i-1)\}, \quad \textit{where} \quad i = 2, 3, \ldots, n-1$$

*with the initial conditions $f_1(1) = f_2(1) = 0$.*

**Proof:** Denote the weight of a maximum path containing vertices $n, n-1, ..., n+1-i$ and starting (terminating, respectively) at $n+1-i$ by $f_1(i)$, ($f_2(i)$, respectively).

We show that for a tour $T$ of $G[W]$, there is a tour $T' = (i_1, i_2, ..., i_n, i_1)$ of at least the same weight as $T$ such that $i_1 < i_2 < ... < i_r = n > i_{r+1} > ... > i_n$. Indeed, let $A = \{j \in \{1, ..., n\} :\ (j-1, j) \in T \vee (j, j+1) \in T\}$ and $B = \{1, ..., n\} - A$. If $A = \{a_1, ..., a_r\}$ where $a_1 < ... < a_r$ and $B = \{b_1, ..., b_s\}$ where $b_1 > ... > b_s$, then $T' = (a_1, ..., a_r, b_1, ..., b_s, a_1)$ is as desired. As the tours constructed in this way have the property that for each $i \in \{1, ..., n\}$ the subgraph induced on the vertices $i, i+1, ..., n$ is a path starting or terminating at $i$, the formulas given in the theorem easily follow. □

Theorem 2 supplies an $O(n)$-algorithm for finding a maximum tour in $G[W]$.

## 4. Matrices from $M(n, \alpha)$

For an $n \times n$ matrix $W = [w_{ij}]$, the elements $w_{ij}$ with $|i-j| = 1$ are called *basic*. Corresponding arcs of $G[W]$ are also called *basic*.

**Theorem 3.** *Given an $\alpha > 0$. The MTSP for matrices from $M(n, \alpha)$ is NP-hard.*

**Proof:** Since $M(n, \alpha_1) \subseteq M(n, \alpha_2)$ for $\alpha_2 \leq \alpha_1$, we can assume that $\alpha \geq 1$. Let $A = [a_{ij}]$ be an $n \times n$ matrix with nonnegative elements. Put $\beta = \alpha \max a_{ij}$, $\gamma = 2n\beta + 1$. Construct the following matrix $B = [b_{ij}] \in M(2n, \alpha)$. Any basic element of $B$ equals $\beta$ or $\gamma$ and a basic element $b_{ij} = \gamma$ if and only if either $j = i + 1 \equiv 2 (\bmod 4)$ or $i = j + 1 \equiv 0 (\bmod 4)$. An arc of $G[B]$ with weight $\gamma$ will be called $\gamma-$ arc. An element $b_{ij} = a_{\lceil i/2 \rceil, \lceil j/2 \rceil}$ if and only if $i$ is the second vertex of a $\gamma$-arc $(k, i)$ and $j$ is the first vertex of another $\gamma$-arc $(j, t)$. All other elements of $B$ equal 0.

Note that a maximum tour of $G[B]$ contains all the $\gamma$-arcs. Hence, the weight of a maximum tour of $G[B]$ equals $n\gamma$ plus the weight of a maximum tour of $G[A]$. □

For $\alpha \geq 3$ the algorithm mentioned in the previous section provides a rather good solution to the MTSP with matrices from $M(n, \alpha)$.

**Theorem 4.** *Let $\alpha \geq 3$. If $W \in M(n, \alpha)$, then a maximum tour $T^*$ in $G[W^*]$, where $w^*_{ij} = w_{ij}$ for the basic elements, and $w^*_{ij} = 0$, otherwise, is an approximate solution of the MTSP for $G[W]$ with the relative error*

$$\frac{w(T) - w(T^*)}{w(T)} \leq \frac{n}{2(n-1)\alpha} \ ,$$

*where $T$ is a maximum tour of $G[W]$.* □

It is easy to see that the last theorem follows from

**Lemma 5.** *Let $\alpha \geq 3$ and $W \in M(n, \alpha)$. Then there is a maximum tour of $G[W]$ having at least $\lceil \frac{n}{2} \rceil$ basic arcs.*

**Proof:** Assume that $X = (i_1, i_2, \ldots, i_n, i_1)$ is a maximum tour of $G[W]$ containing less than $\lceil n/2 \rceil$ basic arcs. Then w.l.o.g. there is a vertex $i_k$ of $G[W]$ such that the arcs $(i_{k-1}, i_k), (i_k, i_{k+1})$ and $(i_{s-1}, i_s)$ are non-basic, where $i_s = i_k + 1$. The tour

$$Y = X \cup \{(i_k, i_s), (i_{s-1}, i_k), (i_{k-1}, i_{k+1})\} \backslash \{(i_{k-1}, i_k), (i_k, i_{k+1}), (i_{s-1}, i_s)\}$$

has more basic arcs than $X$. By (1) $Y$ is also a maximum tour. $\square$


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