Kings in semicomplete multipartite digraphs

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Abstract

A digraph obtained by replacing each edge of a complete p-partite graph by an arc or a pair of mutually opposite arcs with the same end vertices is called a semicomplete p-partite digraph, or just a semicomplete multipartite digraph. A semicomplete multipartite digraph with no cycle of length two is a multipartite tournament. In a digraph D, an r-king is a vertex q such that every vertex in D can be reached from q by a path of length at most r. Strengthening a theorem by K.M. Koh and B.P. Tan (Discrete Math. 147 (1995) 171–183) on the number of 4-kings in multipartite tournaments, we characterize semicomplete multipartite digraphs which have exactly k 4-kings for every $k = 1, 2, 3, 4, 5$.

Keywords: *Kings, semicomplete multipartite digraphs, multipartite tournaments, distances*

1 Introduction, terminology and notation

A digraph obtained by replacing each edge of a complete p-partite graph by an arc or a pair of mutually opposite arcs with the same end vertices is called a *semicomplete p-partite digraph*, or just a *semicomplete multipartite digraph.* A semicomplete multipartite digraph with no mutually opposite arcs is a *multipartite tournament.* In a digraph D, an *r-king* is a vertex q such that every vertex in D can be reached from q by a path of length at most r.

It is well-known that every tournament T has a 2-king (cf. [8], p. 66); in fact, every vertex of maximum out-degree in T is a 2-king. Multipartite tournaments may have two or more vertices of in-degree zero, and, thus, no r-king for any integer r. However, Gutin [2] (and, independently, Petrovic and Thomassen [10]) proved that every multipartite tournament with at most one vertex of in-degree zero contains a 4-king. Moreover, it is easy to construct infinite families of p-partite tournaments (for every $p \geq 2$) which contain 4-kings but have no 3-kings (cf. [2, 5]). Therefore, in study of multipartite tournaments, 4-kings are of special interest. Notice that while in a bipartite tournament every vertex of maximum out-degree is a 4-king, the obvious extension of this result to k-partite tournaments for $k \geq 3$ is not valid [1].

In a number of papers (see, e.g., [3, 5, 6, 7, 9] and the survey papers [4, 11]) the authors investigate the minimum number of 4-kings in multipartite tournaments without vertices of in-degree zero. (If a multipartite tournament has exactly one vertex of in-degree zero, it contains only one 4-king, hence this case is trivial.) To our taste, the most interesting theorem in this direction was obtained by Koh and Tan in [5]. They showed that every k-partite tournament without vertices of in-degree zero contains at least four 4-kings when $k = 2$ and at least three 4-kings when $k \geq 3$. (Petrovic [9], independently, proved the above result in the case of bipartite tournaments.) Moreover, Koh and Tan characterized the cases when a bipartite tournament has exactly four 4-kings (its initial strong component consists of a cycle of length four) and a p-partite ($p \geq 3$) tournament contains exactly three 4-kings (its initial strong component consists of a cycle of length three). Only some sufficient conditions for bipartite tournaments to contain more than five 4-kings and for p-partite ($p \geq 3$) tournaments to possess more than four 4-kings have been obtained in [6, 7].

The above-mentioned theorem by Koh and Tan can be considered as a characterization of bipartite (p-partite, $p \geq 3$) tournaments with exactly k 4-kings for $k \in \{1, 2, 3, 4\}$ ($k \in \{1, 2, 3\}$). Notice that the direct extension of their result to semicomplete multipartite digraphs is not true as the 4-king structure of semicomplete multipartite digraphs is somewhat richer (for example, there are semicomplete multipartite digraphs with exactly two 4-kings).

In this paper, we characterize semicomplete multipartite digraphs which have exactly k 4-kings for every $k \in \{1, 2, 3, 4, 5\}$. The theorem of Koh and Tan immediately follows from our characterization. Observe that our proofs utilize several new ideas. This makes our argumentation relatively short.

We use standard terminology and notation on digraphs as well as some additional notions. For a digraph $D = (V, A)$, and vertices $x, y \in V$, we say that x *dominates* y (denoted by $x \rightarrow y$) if xy is an arc in D, i.e., $xy \in A$. A vertex x is *adjacent* to a vertex a vertex y if either $x \rightarrow y$ or $y \rightarrow x$ (or both hold). Let $D = (V, A)$ be a digraph, let $x \in V$, and let U, W be subsets of V. The subdigraph induced by U is denoted by $D\langle U\rangle$. We define $(U, W) = \{uw \in A : \ u \in U, w \in W\}$, $N^+(x) = \{v \in V : \ xv \in A\}$, $N^+(W) = \cup_{w \in W} N^+(w) - W$.

An (x,y)-*path* is a path from x to y. (All paths in this paper are directed.) The *distance*, $\mathrm{dist}_D(x,y)$, *from* a vertex x *to* a vertex y in D is the least length of a path from x to y, if y is reachable from x, and is equal to ∞, otherwise. We consider a vertex x as an (x,x)-path; thus, $\mathrm{dist}_D(x,x) = 0$. The *distance* $\mathrm{dist}_D(X,Y)$ *from a set* X to *a set* Y of vertices in D is $\max\{\mathrm{dist}_D(x,y) : x \in X, y \in Y\}$. Observe that a vertex v is an r-king if $\mathrm{dist}_D(v,V) \leq r$.

A digraph $D = (V,A)$ is *strong* if $\mathrm{dist}_D(V,V) < \infty$. A maximal strong subdigraph of D is a strong component of D. It is well-known that the vertices of a digraph D can be partitioned into subsets $V_1, ..., V_p$ (i.e., $V_1 \cup ... \cup V_p = V(D)$ and $V_i \cap V_j = \emptyset$ for every $i \neq j$) such that every subdigraph $D_i = D\langle V_i \rangle$ is a strong component of D. Let D_1,D_2,...,D_p be the strong components of D. We call the above ordering an *acyclic ordering* if the existence of an arc from D_i to D_j implies that $i < j$. It is well-known and easy to show that the strong components of a digraph D always have an acyclic ordering. A strong component D_i is *initial* if for no index j there is an arc from D_j to D_i. Notice that if a semicomplete multipartite digraph has two or more initial strong components, each of these components consists of a single vertex.

When it is clear which digraph D is under consideration, we will omit the subscript D in distances between vertices or sets of the vertices of D. We will not distinguish between a singleton $\{x\}$ and its element x, so $\mathrm{dist}(x,W)$ denotes the same as $\mathrm{dist}(\{x\},W)$.

2 Lemmas

Lemma 2.1 *Let* $\{x\}$,$W_0, W_1, ..., W_s$ *be disjoint sets of vertices in a digraph* D. *Let also* $\mathrm{dist}(x,W_0) = t$ *and* $W_{i+1} \subseteq N^+(W_i)$ *for every* $i = 0,1,...,s-1$. *Then* $\mathrm{dist}(x,W_s) \leq t+s$.

Proof: It suffices to show that $\mathrm{dist}(x,w) \leq t+s$ for every $w \in W_s$. Choose an arbitrary vertex $w_s \in W_s$. Clearly, there is $w_{s-1} \in W_{s-1}$ such that $w_{s-1} \rightarrow w_s$. Analogously, there are $w_0, w_1, ..., w_{s-2}$ such that $w_0 \rightarrow w_1 \rightarrow ... \rightarrow w_{s-1} \rightarrow w_s$. Hence, there exists a (w_0,w_s)-path of length at most s. Thus, $\mathrm{dist}(w_0,w_s) \leq s$. Hence, $\mathrm{dist}(x,w_s) \leq \mathrm{dist}(x,w_0) + \mathrm{dist}(w_0,w_s) \leq t+s$. $\square$

Lemma 2.2 *If* $P = p_0 p_1 \dots p_\ell$ *is a shortest path from* p_0 *to* p_ℓ *in a semicomplete multipartite digraph* D, *and* $\ell \geq 3$, *then there is a* (p_ℓ, p_0)*-path of length at most 4 in* $D\langle V(P) \rangle$.

Proof: As $\ell \geq 3$ and P is a shortest path we have $(\{p_0,p_1\},p_\ell) = \emptyset$. If $p_\ell \rightarrow p_0$ we are done, so assume that p_ℓ and p_0 belong to the same partite set of D. This implies that $p_\ell \rightarrow p_1$. Analogously, $(p_0,\{p_2,p_3\}) = \emptyset$, which implies that either $p_\ell p_1 p_2 p_3 p_0$ or $p_\ell p_1 p_2 p_0$ is a (p_ℓ,p_0)-path of length at most 4 in $D\langle V(P) \rangle$. $\square$

3 Main results

The following theorem is our main result on strong semicomplete digraphs.

Theorem 3.1 *If $D = (V, A)$ is a strong semicomplete digraph, containing at least six vertices, then there are at least five 4-kings in D. Furthermore, if there are exactly five 4-kings in D, then there is a path $P = p_0p_1p_2p_3p_4$, such that $\mathrm{dist}(p_0, p_4) = 4$, and $(V - V(P), \{p_1, p_2, p_3, p_4\}) = \emptyset$.*

Proof: Let X denote the set of all 4-kings in D and let $Y = V - X$. Clearly our theorem is true if $Y = \emptyset$, so we may assume that Y is not empty; let $w \in Y$ be arbitrary. Now define W_i as follows: $W_i = \{v \in V : \mathrm{dist}(w, v) = i\}$, for all $i = 0, 1, \ldots, m$, where $m = \mathrm{dist}(w, V)$. As $w \in Y$, $m \geq 5$.

By the definition of the sets W_i we observe that

$$(W_0 \cup W_1 \cup ... \cup W_{i-2}, W_i) = \emptyset, \tag{1}$$

for all $2 \leq i \leq m$.

We prove the following two claims:

Claim A: For every $z \in W_i$, with $i \geq 3$, we have $\mathrm{dist}(z, W_0 \cup W_1 \cup ... \cup W_i) \leq 4$.

As a shortest path from w to z is of length at least 3, by Lemma 2.2, $\mathrm{dist}(z, w) \leq 4$. Let $q \in W_1 \cup ... \cup W_i - z$ and let $r_0r_1...r_j$ be a shortest (w, q)-path in D. If $1 \leq j \leq 3$, then, by (1) and the fact that either r_0 or r_1 is adjacent to z (as D is semicomplete multipartite), we conclude that z dominates at least one of the vertices r_0, r_1. Thus, either $zr_0r_1...r_j$ or $zr_1...r_j$ is a (z, q)-path in D of length at most 4. If $j \geq 4$ then, since z dominates at least one of the vertices r_{j-3}, r_{j-2} (by (1)), either $zr_{j-3}r_{j-2}r_{j-1}r_j$ or $zr_{j-2}r_{j-1}r_j$ is a (z, q)-path in D of length at most 4.

Claim B: If $y \in Y \cap W_i$, $0 \leq i \leq m - 1$, then either (i) or (ii) below holds :

(i) for every $z \in W_{i+1}$ there is a (w, z)-path in $D - y$.

(ii) $\mathrm{dist}(y, W_{i+1}) \leq 3$.

Assume that neither (i) nor (ii) holds. This implies that there exist vertices z_1 and z_2 in W_{i+1} such that there is no (w, z_1)-path in $D - y$ and there is no (y, z_2)-path of length at most 3 in D. Let $P = p_0p_1...p_{i+1}$ be a shortest path from w to z_1 in D and let $R = r_0r_1...r_{i+1}$ be a shortest path from w to z_2 in D. Clearly $(y = p_i) \rightarrow z_1$ and $y \not\rightarrow z_2$, which implies that $z_1 \neq z_2$.

If z_1 and z_2 belong to the same partite set, then r_i is adjacent to z_1. If $r_i \rightarrow z_1$, then $r_0r_1...r_iz_1$ is a (w, z_1)-path in $D - y$, a contradiction. If $z_1 \rightarrow r_i$, then $yz_1r_iz_2$ is a

(y, z_2)-path of length 3 in D, a contradiction. Therefore z_1 and z_2 belong to different partite sets. If $z_1 \rightarrow z_2$, then yz_1z_2 is a (y, z_2)-path of length 2 in D and if $z_2 \rightarrow z_1$, then $r_0r_1...r_{i+1}z_1$ is a (w, z_1)-path in $D - y$, a contradiction.

We now prove the theorem by induction. If $|V| = 6$ then either all the vertices of D are 4-kings, or there is a path $p_0p_1p_2p_3p_4p_5$ in D, such that $\mathrm{dist}(p_0, p_5) = 5$. In both cases the theorem holds. Indeed, in the second case, observe that $p_0 \in Y$ and, thus, $\{p_3, p_4, p_5\} \subseteq X$, by Claim A. Clearly $\{p_1, p_2\} \subseteq X$, as $p_0 \in N^+(\{p_2, p_3\})$.

Now assume that $|V| \geq 7$ and that the theorem holds for all smaller strong semicomplete multipartite digraphs (with at least six vertices). We consider the following two cases.

Case 1. There is a vertex $y \in Y \cap W_i$, $0 \leq i \leq m - 1$, such that (i) of Claim B holds.

We first prove that, for every $q \in V - y$, there is a (w, q)-path in $D - y$. Let $q \in W_j$ be arbitrary and let $P = p_0p_1..p_j$ be a shortest path from w to q in D. If $j \leq i$ then clearly $y \notin V(P)$, so we are done. If $j \geq i + 1$ then (as (i) holds) there is a (w, p_{i+1})-path in $D - y$, which together with $p_{i+2}...p_j$ forms a (w, p_j)-path in $D - y$.

Let $u \in W_m$ be arbitrary and let $R = r_0r_1...r_\ell$ be a shortest path from w to u in $D - y$. Clearly $\ell \geq m \geq 5$.Thus, by Lemma 2.2, we conclude that $D\langle V(R)\rangle$ is strong.

Let Q_1, Q_2,..., Q_s $(s \geq 1)$ be an acyclic ordering of the strong components of $D - y$. As $\mathrm{dist}_{D-y}(w, V - y) < \infty$, the vertex w belongs to an initial strong component of $D - y$. As $D\langle V(R)\rangle$ is strong, this component has more than one vertex. Since $D - y$ is semicomplete multipartite, we conclude that Q_1 is the unique initial strong component of $D - y$. As $V(R) \subseteq V(Q_1)$, Q_1 has at least six vertices.

Let q be a 4-king in Q_1, and let $r \in V - q$ be arbitrary. If $r \in Q_1$ then clearly there is a (q, r)-path of length at most 4 in D. If $r \in Q_t$, $t \geq 2$, then either $q \rightarrow r$ or there is some vertex $u \in Q_1$ such that $q \rightarrow u \rightarrow r$. This implies that there is a (q, r)-path of length at most 2 in D. Suppose that $r = y$ and $\mathrm{dist}(q, y) > 4$. It follows from the above arguments that $\mathrm{dist}(q, V - y) \leq 4$. Therefore, $\mathrm{dist}(q, y) = 5$. Let $W_i^{(q)} = \{v \in V :\ \mathrm{dist}(q, v) = i\}$, $i = 0, 1, ..., 5$. As $W_5^{(q)} = y$, by Claim A, y is a 4-king, a contradiction. Therefore every 4-king in $D\langle Q_1\rangle$ is also a 4-king in D.

Using the induction hypothesis for $D\langle Q_1\rangle$, we obtain that there are at least five 4-kings in D. If there are precisely five 4-kings in D, then there are precisely five 4-kings in $D\langle Q_1\rangle$, and, thus, there is a path $P = p_0p_1p_2p_3p_4$ in $D\langle Q_1\rangle$, which is a shortest possible (p_0, p_4)-path in $D\langle Q_1\rangle$ and $(V(Q_1) - V(P), \{p_1, p_2, p_3, p_4\}) = \emptyset$. If there is no arc from y to a vertex in $\{p_1, p_2, p_3, p_4\}$, then clearly $(V - V(P), \{p_1, p_2, p_3, p_4\}) = \emptyset$ and we are done, so assume that there is an arc yp_k, where $1 \leq k \leq 4$. We will show that y is another 4-king in D and, thus, obtain a contradiction to our assumption on the existence of the arc yp_k.

If $D-y$ is not strong, by $\mathrm{dist}(p_k, V-(V(Q_1)\cup y)) \leq 2$, we have $\mathrm{dist}(y, V-V(Q_1)) \leq 3$. Let $x \in V(Q_1)-V(P)$. If $k<4$, then either $p_k \to x$ or $p_{k+1} \to x$ (or both). Hence, $\mathrm{dist}(y,x) \leq 3$. Let $k=4$. If x and p_4 are adjacent, then $y \to p_4 \to x$. If x and p_4 are not adjacent, then either $y \to p_4 \to p_1 \to x$ or $y \to p_4 \to p_2 \to x$ (or both). Hence, $\mathrm{dist}(y,x) \leq 3$.

To demonstrate that y is a 4-king, it is now sufficient to prove that $\mathrm{dist}(y,p_j) \leq 4$ for every $j \in \{0,1,2,3,4\}$. For $j>k$, $\mathrm{dist}(y,p_j) \leq 4$. For $0 \leq j < k < 4$, p_j is dominated by either p_k or p_{k+1}; thus, $\mathrm{dist}(y,p_j) \leq 3$. Let $k=4$. If $p_4 \to p_0$, then $\mathrm{dist}(y,\{p_0,p_1,p_2,p_4\}) \leq 4$. As p_4 dominates either p_1 or p_2 (or both), $\mathrm{dist}(y,p_3) \leq 4$. Assume that p_4 and p_0 are not adjacent. Then, $p_4 \to p_1$. Hence, $\mathrm{dist}(y,\{p_1,p_2,p_3,p_4\}) \leq 4$. As Q_1 is strong and $|V(Q_1)| \geq 6$, there is an arc vp_0 in D, where $v \in V(Q_1)-V(P)$. As p_4 and p_0 are not adjacent, $p_4 \to v$. Thus, $\mathrm{dist}(y,p_0) \leq 2$.

Case 2. For every $i=0,1,...,m-1$ and every $y \in Y \cap W_i$, (ii) of Claim B holds.

By Claim A, we have $W_m \subseteq X$. By Claim A and (ii) of Claim B, we obtain that $W_{m-1} \subseteq X$. By Claim A, $\mathrm{dist}(W_{m-2}, W_0 \cup W_1 \cup ... \cup W_{m-2}) \leq 4$. By (ii) of Claim B, $\mathrm{dist}(W_{m-2}, W_{m-1}) \leq 3$. As $W_m \subseteq N^+(W_{m-1})$, by Lemma 2.1, $\mathrm{dist}(W_{m-2}, W_m) \leq 4$. Thus, $W_{m-2} \subseteq X$.

If $|X| \geq 6$, we are done. Hence, assume that $|X| \leq 5$. However this implies that at least one of the sets W_{m-2}, W_{m-1}, W_m is a singleton.

Let $P = p_0p_1...p_m$ be a path from w to a vertex $p_m \in W_m$. Clearly p_{m-3} has a path of length 3 to every vertex in W_m, as $|W_{m-2}|=1$, $|W_{m-1}|=1$ or $|W_m|=1$. As above we conclude, by (ii) of Claim B and Lemma 2.1, that p_{m-3} has a path of length at most 3 to every vertex of W_{m-2} and a path of length at most 4 to every vertex of W_{m-1}. By Claim A, $\mathrm{dist}(p_{m-3}, W_0 \cup W_1 \cup ... \cup W_{m-3}) \leq 4$. Therefore $p_{m-3} \in X$.

Hence, at least two of the sets W_{m-2}, W_{m-1} and W_m are singletons. As above we observe that p_{m-4} has a path of length 3 to every vertex of W_{m-1} and a path of length 4 to every vertex of W_m. It furthermore has a path of length at most 3 to every vertex of W_{m-3} and a path of length at most 4 to every vertex of W_{m-2} (by (ii) of Claim B and Lemma 2.1). By Claim A, $\mathrm{dist}(p_{m-4}, W_0 \cup W_1 \cup ... \cup W_{m-4}) \leq 4$. Thus, $p_{m-4} \in X$.

Therefore $X = \{p_{m-4}, p_{m-3}, p_{m-2}, p_{m-1}, p_m\}$ (as we have assumed that $|X| \leq 5$), and $W_m = \{p_m\}$, $W_{m-1} = \{p_{m-1}\}$, and $W_{m-2} = \{p_{m-2}\}$. Now it suffices to prove that $(V-X, \{p_{m-3}, p_{m-2}, p_{m-1}, p_m\}) = \emptyset$. Assume that this is not true. Thus, there is a vertex $q \in V-X$ which dominates a vertex in the set $\{p_{m-3}, p_{m-2}, p_{m-1}, p_m\}$. Clearly there are only the following three possibilities: (a) $q \in W_{m-3} - p_{m-3}$ and $q \to p_{m-2}$; (b) $q \in W_{m-3} - p_{m-3}$ and $q \to p_{m-3}$; (c) $q \in W_{m-4} - p_{m-4}$ and $q \to p_{m-3}$. In all these cases, $W_0 \cup W_1 \cup ... \cup W_{m-3} \subseteq N^+(\{p_{m-1}, p_m\})$, and $\mathrm{dist}(q, \{p_{m-1}, p_m\}) \leq 3$. Thus, by Lemma 2.1, $\mathrm{dist}(q, W_0 \cup W_1 \cup ... \cup W_{m-3}) \leq 4$. Observe that $\mathrm{dist}(q, p_{m-2}) \leq 2$. Thus, q is a 4-king, a contradiction. $\square$

This theorem implies the following:

Theorem 3.2 *Let $D = (V, A)$ be a semicomplete multipartite digraph and let k be the number of 4-kings in D. Then (a) $k = 1$ if and only if D has exactly one vertex of in-degree zero, (b) $k = 2, 3$ or 4 if and only if the initial strong component of D has k vertices, (c) $k = 5$ if and only if either the initial strong component Q of D has five vertices or Q contains at least six vertices and possesses a path $P = p_0p_1p_2p_3p_4$ such that* $\mathrm{dist}(p_0, p_4) = 4$ *and $(V - V(P), \{p_1, p_2, p_3, p_4\}) = \emptyset$.*

Proof: Assume that $|V| \geq 2$ (the case of $|V| = 1$ is trivial). Observe that if the initial strong component Q of D has at most five vertices, then every vertex of Q is a 4-king and, clearly, no other vertex in D is a 4-king. If $|V(Q)| \geq 6$, then the 4-kings of Q are 4-kings of D and no other vertex in D is a 4-king. □

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References

[1] W.D. Goddard, G. Kubicki, O. Oellermann and S.L. Tian, On multipartite tournaments. *J. Combin. Theory Ser. B* **52** (1991) 284–300.

[2] G. Gutin, The radii of n-partite tournaments. *Math. Notes* **40** (1986) 743–744.

[3] G. Gutin, m-sources in complete multipartite digraphs. *Vestsī Akad. Navuk BSSR Ser. Fīz.-Mat. Navuk* no. 5 (1989) 101–106. (in Russian)

[4] G. Gutin, Cycles and paths in semicomplete multipartite digraphs, theorems and algorithms: a survey. *J. Graph Theory* **19** (1995) 481–505.

[5] K.M. Koh and B.P. Tan, Kings in multipartite tournaments. *Discrete Math.* **147** (1995) 171–183.

[6] K.M. Koh and B.P. Tan, Number of 4-kings in bipartite tournaments with no 3-kings, *Discrete Math.* **154** (1996) 281–287.

[7] K.M. Koh and B.P. Tan, The number of kings in a multipartite tournament. *Discrete Math.* **167/168** (1997) 411–418.

[8] L. Lovász, *Combinatorial Problems and Exercises.* North-Holland, Amsterdam, 1979.

[9] V. Petrovic, Kings in bipartite tournaments, *Discrete Math.* **173** (1997) 187–196.

[10] V. Petrovic and C. Thomassen, Kings in k-partite tournaments. *Discrete Math.* **98** (1991) 237–238.

[11] K.B. Reid, Tournaments: scores, kings, generalizations and special topics. *Congressus Nummerantium* **115** (1996) 171–211.