Assignment Problem based algorithms are impractical for the Generalized TSP

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Abstract

In the Generalized Traveling Salesman Problem (GTSP), given a weighted complete digraph D and a partition $V_1, \dots, V_k$ of the vertices of D, we are to find a minimum weight cycle containing exactly one (at least one) vertex from each set V_i, $i = 1, \dots, k$. Assignment Problem based approaches are extensively used for the Asymmetric TSP. To use analogs of these approaches for the GTSP, we need to find a minimum weight 1-regular subdigraph that contains exactly one (at least one) vertex from each V_i. We prove that, unfortunately, the corresponding problems are NP-hard. In fact, we show the following stronger result: Let $D = (V, A)$ be a digraph and let $V_1, V_2, \dots, V_k$ be a partition of V. The problem of checking whether D has 1-regular subdigraph containing exactly one vertex from each $V_1, V_2, \dots, V_k$ is NP-complete even if $|V_i| \leq 2$ for every $i = 1, 2, \dots, k$.

Keywords: Generalized TSP, Assignment Problem, combinatorial optimization, 1-regular subdigraphs

1 Introduction

A collection $X_1, X_2, \dots, X_k$ of subsets of a set X is called a partition of X if $\cup_{i=1}^{k} X_i = X$, $X_i \cap X_j = \emptyset$ and none of the sets X_i are empty. In the *Generalized Traveling Salesman Problem* (GTSP), given a weighted complete digraph D and a partition $V_1, \dots, V_k$ of the vertices of D, we are to find a minimum weight cycle containing exactly one (at least one) vertex from each set V_i, $i = 1, \dots, k$. We will use the abbreviation GTSP$_=$ (GTSP$_\geq$) for the "exactly one" ("at least one") variant and GTSP for both variants. Clearly, the (Asymmetric) TSP is simply the GTSP for $|V_i| = 1$, $i = 1, \dots, k$. In this paper we consider the GTSP with no restrictions imposed on the weights of the complete digraph, i.e., the asymmetric versions of the problem. We call the Asymmetric TSP simply the TSP.

The GTSP has applications in the design of ring networks, routing of welfare customers through governmental agencies, sequencing of computer files, flexible manufacturing scheduling, airport selection and routing for courier planes, and postal routing; see, e.g., Noon [13], Noon and Bean [14], and Laporte, Asef-Vaziri and Sriskandarajan [9].

Both types of the GTSP have been studied by Laporte, Mercure and Norbert [10], and Noon and Bean [14]; their symmetric weight versions were investigated, among others, by Fischetti, Salazar and Toth [4, 5], Laporte and Norbert [11], Salazar [15], and Sepehri [16]; an informative account on the symmetric GTSP is in [6]. Transformations from the $\mathrm{GTSP}_=$ to the TSP were provided by Noon and Bean [14], Lien, Ma and Wah [12] and Dimitrijevic and Saric [2]. Notice that while the transformations are of value for small size instances of the $\mathrm{GTSP}_=$, for larger ones they produce difficult TSP instances (containing large numbers). Thus, the transformations are not of use for larger instances of the $\mathrm{GTSP}_=$.

A digraph H is 1-regular if every vertex of H is tail (head) of exactly one arc of H, i.e., H is a collection of vertex-disjoint cycles. One of the most successful approaches to construct algorithms and (lower and upper) bounds for the TSP are the ones based on applications of the Assignment Problem (AP). The AP-based approaches start from computing a minimum weight spanning 1-regular subdigraph F in the given weighted complete digraph. Subdigraph F provides a relatively quickly calculated starting point for branch-and-bound type exact algorithms and various successful construction heuristics (see, e.g., Cirasella, Johnson, McGeoch and Zhang [1], Fischetti, Lodi and Toth [3], Glover, Gutin, Yeo and Zverovich [7], and Johnson et al. [8]). In [8], AP-based heuristics are considered as a special class of heuristics and the best among them, `Zhang`, `Patch` and `COP`, are shown to perform very well in computational experiments.

Naturally one asks whether the obvious analogs of the AP-based approaches can be used for the GTSP. For the $\mathrm{GTSP}_=$ ($\mathrm{GTSP}_\geq$) we need to find a minimum weight 1-regular subdigraph that contains exactly (at least) one vertex from each V_i. Unfortunately, the corresponding problems are actually NP-hard, see Corollary 2.2. Thus, the analogs of the AP-based approaches are impractical for the GTSP.

2 Impossibility of AP-based approaches

Theorem 2.1 *Let $D=(V,A)$ be a digraph and let $V_1,V_2,\dots,V_k$ be a partition of V. The problem of checking whether D has 1-regular subdigraph containing exactly one vertex from each $V_1,V_2,\dots,V_k$ is NP-complete even if $|V_i|\le 2$ for every $i=1,2,\dots,k$.*

Proof: We describe a polynomial time transformation from the well known 3-SAT problem. We may assume that in each instance of 3-SAT every variable is used in both positive and negative forms (otherwise, the instance can be reduced to the desired form). Suppose that an instance $(W,\mathcal{C})$ of this problem is given, where W is the set of variables and $\mathcal{C}$ is the set of three-variable clauses over W; $|\mathcal{C}|=k$. We denote by t_i^j the ith term in the jth clause.

Construct a digraph D_1 as follows. Let

$$V(D_1)=\{x_1,x_2,\dots,x_k\}\cup\{c_i^j : j=1,2,\dots,k,\ i=1,2,3\}$$

and $A(D_1)=\{x_jc_i^j : j=1,2,\dots,k,\ i=1,2,3\}\cup\{c_i^jx_{j+1} : j=1,2,\dots,k,\ i=1,2,3\}\cup\{c_i^jc_h^j : j=1,2,\dots,k,\ i=1,2,3,\ h=1,2,3,\ i\ne h\}$, where $x_{k+1}=x_1$ by definition.

Construct a digraph D_2 as follows. Let $|W|=r$ be the number of variables, and let n_i be the number of clauses containing the ith variable (not negated) and let $\bar{n_i}$ be the number of clauses containing the negated version of the ith variable. Note that $\sum_{i=1}^r n_i+\bar{n_i}=3k$. Let $V(D_2)=\{y_1,y_2,y_3,\dots,y_r\}\cup\{v_i^j : j=1,2,\dots,r,\ i=1,2,\dots,n_j\}\cup\{\bar{v}_i^j : j=1,2,\dots,r,\ i=1,2,\dots,\bar{n_j}\}$ and let the arc set of D_2 be defined as follows. Consider the cycle $C=y_1y_2\dots y_ry_1$. Now duplicate each arc in C and in one copy of the arc y_jy_{j+1} we insert the vertices $v_1^j,v_2^j,\dots,v_{n_j}^j$ (so that we get the path $y_jv_1^jv_2^j\dots v_{n_j}^jy_{j+1}$), and in the other copy of the arc y_jy_{j+1} we insert the vertices $\bar{v}_1^j,\bar{v}_2^j,\dots,\bar{v}_{\bar{n_j}}^j$.

Let a digraph D be the disjoint union of D_1 and D_2. Partition the vertex set of D into the following partite sets. The vertices $\{x_1,x_2,\dots,x_k\}$ and $\{y_1,y_2,\dots,y_r\}$ are placed into sets of size one. All other sets have size two, and each set contains a vertex of the form c_i^j and a vertex of the form v_a^b (or $\bar{v}_a^b$), such that t_i^j is the ath appearance of the bth variable (or its negation, if we use $\bar{v}_a^b$). By our construction we pair all c_i^j's with the v_a^b's and $\bar{v}_a^b$'s.

Thus, the number of classes in our partition is $k+r+3k$. We now claim that there is a 1-regular subdigraph containing exactly one vertex from each partite set if and only if our original instance of 3-SAT is satisfiable.

We assume that there is a 1-regular subdigraph F containing exactly one vertex from each partite set in D and prove now that $(W, \mathcal{C})$ is satisfiable. Observe that F must contain at least 2 cycles. One cycle, which we denote by C_1, contains all $\{x_1, x_2, \ldots, x_k\}$, and a number of c_i^j's and one cycle, which we denote by C_2, contains all $\{y_1, y_2, \ldots, y_r\}$ and a number of v_a^b's (and/or $\bar{v}_a^b$'s). All other cycles (if there are any) are 2-cycles and have the form $c_i^j c_h^j c_i^j$, where $j \in \{1, 2, \ldots, k\}$ and $i \neq h$.

Observe that C_1 uses at least one vertex in $\{c_i^j, c_2^j, c_3^j\}$ for every $j = 1, 2, \ldots, k$. If c_i^j lies on C_1, then we assign TRUE to t_i^j. If this is a valid (partial) assignment, we are done, and thus it remains to prove that no two terms t_i^j, t_p^q are assigned TRUE and $t_i^j = \bar{t}_p^q$. Assume that t_i^j, t_p^q are assigned TRUE and $t_i^j = \bar{t}_p^q$. This means that c_i^j and c_p^q belong to C_1 and, without loss of generality, the sth variable equals t_i^j and its negation equals t_p^q. However, by the definition of partite sets in D, C_2 can use neither v_1^s nor $\bar{v}_1^s$, which is impossible.

Now assume that $(W, \mathcal{C})$ is satisfiable and prove that D has a 1-regular subdigraph containing exactly one vertex from each partite set. Without loss of generality, we may assume that $t_1^j =$TRUE, $j = 1, 2, \ldots, k$, is a valid partial assignment. We can extend this partial assignment in such a way that every variable in W is assigned either TRUE or FALSE. Now we create a cycle C_2, containing all the vertices $\{y_1, y_2, \ldots, y_r\}$. We furthermore insert the path $v_1^j, v_2^j, \ldots, v_{n_j}^j$ between y_j to y_{j+1}, if the jth variable is FALSE. If the jth variable is TRUE, then we insert the path $\bar{v}_1^j, \bar{v}_2^j, \ldots, \bar{v}_{\bar{n}_j}^j$ between y_j to y_{j+1} instead. To construct C_1 we take the vertices

$$\{x_1, x_2, \ldots, x_k\} \cup \{c_1^j : j = 1, 2, \ldots, k\}$$

and add to them the vertices from $\{c_2^j, c_3^j : j = 1, 2, \ldots, k\}$, whose pairs from the partite sets do not belong to C_2. Clearly, $C_1 \cup C_2$ is the desired 1-regular subdigraph of D. □

Corollary 2.2 *Let $D = (V, A)$ be a weighted complete digraph and let $V_1, V_2, \ldots, V_k$ be a partition of V. The problem is NP-hard of finding a minimum weight 1-regular subdigraph F in D such that F contains exactly (at least) one vertex from each partite set V_i.*

Proof: Let $H = (V, E)$ be a digraph and let $V_1, V_2, \ldots, V_k$ be a partition of V. Let D be a weighted complete digraph obtained from H by assigning weight 1 to every arc of H and adding arcs of weight 2 to make D complete. Clearly, H has a 1-regular subdigraph containing exactly one vertex in each partite set if and only if D has a 1-regular subdigraph of weight k containing exactly (or, at least) one vertex in each partite set. It remains to apply Theorem 2.1. $\square$

Similarly to Theorem 2.1 one can prove the following:

Theorem 2.3 *Let $D = (V, A)$ be a digraph and let $V_1, V_2, \ldots, V_k$ be a partition of V. The problem of checking whether D has 1-regular subdigraph containing at least one vertex from each $V_1, V_2, \ldots, V_k$ is NP-complete even if $|V_i| \leq 2$ for every $i = 1, 2, \ldots, k$.*

References

[1] J. Cirasella, D.S. Johnson, L.A. McGeoch, and W. Zhang. The asymmetric traveling salesman problem: Algorithms, instance generators, and tests. In A.L. Buchsbaum and J. Snoeyink, editors, *Algorithm Engineering and Experimentation, Third International Wor kshop, ALENEX 2001*, Lect. Notes Comput. Sci., Vol. 2153, pages 32–59. Springer-Verlag, Berlin, 2001.

[2] V. Dimitrijević and Z. Sarić. An efficient transformation of the generalized traveling salesman problem into the traveling salesman problem on digraphs. *Information Sciences*, 102:105–110, 1997.

[3] M. Fischetti, A. Lodi, and P. Toth. Exact methods for the ATSP. In G. Gutin and A.P. Punnen, editors, *Traveling Salesman Problem and its Variations.* Kluwer, to appear in 2002.

[4] M. Fischetti, J.J. Salazar, and P. Toth. The symmetric generalized traveling salesman polytope. *Networks*, 26:113–123, 1995.

[5] M. Fischetti, J.J. Salazar, and P. Toth. A branch-and-cut algorithm for the symmetric generalized traveling salesman problem. *Operations Research*, 45:378–394, 1997.

[6] M. Fischetti, J.J. Salazar, and P. Toth. The generalized traveling salesman and orienteering problem. In G. Gutin and A.P. Punnen, editors, *Traveling Salesman Problem and its Variations.* Kluwer, to appear in 2002.

[7] F. Glover, G. Gutin, A. Yeo, and A. Zverovich. Construction heuristics and domination analysis for the asymmetric TSP. *European J. Oper. Res.*, 129:555–568, 2001.

[8] D.S. Johnson, G. Gutin, L.A. McGeoch, A. Yeo, W. Zhang, and A. Zverovich. Experimental analysis of heuristics for the ATSP. In G. Gutin and A.P. Punnen, editors, *Traveling Salesman Problem and its Variations.* Kluwer, to appear in 2002.

[9] G. Laporte, A. Asef-Vaziri, and C. Sriskandarajah. Some applications of the generalized traveling salesman problem. *J. Oper. Res. Soc.*, 47:1461–1467, 1996.

[10] G. Laporte, H. Mercure, and Y. Nobert. Generalized Travelling Salesman Problem through n sets of nodes: the asymmetrical cases. *Discrete Applied Mathematics*, 18:185–197, 1987.

[11] G. Laporte and Y. Nobert. Generalized Travelling Salesman through n sets of nodes: an integer programming approach. *INFOR*, 21:61–75, 1983.

[12] Y. Lien, E. Ma, and B.W.S. Wah. Transformation of the generalized traveling salesman problem into the standard traveling salesman problem. *Information Sciences*, 74:177–189, 1993.

[13] C.E. Noon. The Generalized Traveling Salesman Problem. Unpublished Dissertation, Department of Industrial and Operations Research, University of Tennessee, 1988.

[14] C.E. Noon and J.C. Bean. A Lagrangian based approach for the asymmetric generalized traveling salesman problem. *Operations Research*, 39:623–632, 1991.

[15] J.J. Salazar. *Algoritmi per il problema del Commesso Viaggiatore Generalizzato*. PhD thesis, Royal Spanish College in Bologna, 1992. (in Italian).

[16] M.M. Sepehri. *The symmetric generalized travelling salesman problem*. PhD thesis, University of Tennessee, Knoxville, 1991.