A note on the cardinality of certain classes of unlabeled multipartite tournaments

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Abstract

A multipartite tournament is an orientation of a complete multipartite graph. Simple derivations are obtained of the numbers of unlabeled acyclic and unicyclic multipartite tournaments, and unlabeled bipartite tournaments with exactly k cycles, which are pairwise vertex-disjoint.

Keywords: Multipartite tournaments; Bipartite tournaments; Enumeration

Bollobás, Frank and Karoński [1] enumerated labeled acyclic bipartite tournaments. Rousseau [4] obtained a short elementary proof of this result; the proof is based on certain bijections. Another proof is given by Moon [3] who also enumerated unlabeled acyclic bipartite tournaments.

In this note, we enumerate unlabeled acyclic and unicyclic multipartite tournaments. We partly generalize these results by counting unlabeled *strictly* k*-cyclic* bipartite tournaments, that is, bipartite tournaments with exactly k cycles, which are pairwise vertex-disjoint. Our proofs are short and simple and based on certain bijections from classes of multipartite tournaments into sets of integral sequences or other classes of multipartite tournaments; unlike the proofs in [3] for the number of unlabeled acyclic bipartite tournaments, no calculations are required in the proofs of our results.

A p*-partite (multipartite) tournament* [2] T is an orientation of a complete p-partite graph G. The *colour classes* of T are the colour classes of G, i.e., the maximal independent sets of vertices in G. An *unlabeled* p-partite tournament is an ordered $(p+1)$-tuple $(T, V_1, ..., V_p)$, where T is a p-partite tournament and $(V_1, ..., V_p)$ an ordered p-tuple of its colour classes. (When $(V_1, ..., V_p)$ can be determined from the context we shall write T rather than $(T, V_1, ..., V_p)$.) If the colour classes of T are of order $n_1, ..., n_p$ respectively ($n_i > 0,\ i = 1, ..., p$), then T is called an $(n_1, ..., n_p)$*-tournament.* We say that unlabeled $(n_1, ..., n_p)$-tournaments $(T, V_1, ..., V_p)$ and $(M, U_1, ..., U_p)$ are *equivalent* if there exists an isomorphism f from T to M such that $f(V_i) = U_i$ for every $i = 1, ..., p$. Intuitively, this means that vertices in the same colour class are interchangeable, but the colour classes themselves are not.

In what follows, $n = n_1 + ... + n_p$. Let $t_k(n_1, ..., n_p)$ denote the number of inequivalent unlabeled strictly k-cyclic $(n_1, ..., n_p)$-tournaments ($k \geq 0$). A sequence $s_1, s_2, ..., s_n$ is called an $(n_1, ..., n_p)$*-sequence* if it contains n_j elements equal to j, for every $j = 1, ..., p$, and no other elements. Clearly, the number of $(n_1, ..., n_p)$-sequences equals the multinomial coefficient $\binom{n}{n_1,...,n_p}$. The following result provides a graph-theoretical interpretation of multinomial coefficients.

Theorem 1. *The number $t_0(n_1, ..., n_p)$ of (inequivalent) unlabeled acyclic $(n_1, ..., n_p)$-tournaments equals the number of $(n_1, ..., n_p)$-sequences. Thus $t_0(n_1, ..., n_p) = \binom{n}{n_1,...,n_p}$.*

Proof: Let T be an acyclic $(n_1, ..., n_p)$-tournament with colour classes $V_1, ..., V_p$. We can assign to T an $(n_1, ..., n_p)$-sequence $s(T) = s_1, s_2, ..., s_n$ as follows. The vertices of zero in-degree in T are all in the same colour class: let them be $x_1, ..., x_{r_1}$, all in V_{j_1}, and set $s_1 = ... = s_{r_1} = j_1$. Let the vertices of zero in-degree in $T - \{x_1, ..., x_{r_1}\}$ be $x_{r_1+1}, ..., x_{r_2}$, all in V_{j_2}, and set $s_{r_1+1} = ... = s_{r_2} = j_2$. Continue in this way until all elements of $s(T) = s_1, ..., s_n$ are defined.

Conversely, given an $(n_1, ..., n_p)$-sequence $s = s_1, s_2, ..., s_n$, we construct an acyclic $(n_1, ..., n_p)$-tournament $T(s)$ as follows. For every $i = 1, 2, ..., n$, the ith vertex x_i of $T(s)$ belongs to V_{s_i}, and it dominates (is dominated by) all vertices x_k not in V_{s_i} such that $k > i$ ($i > k$).

It is easy to see that these two constructions are inverses of each other, that is, $T(s(T)) = T$ for each T and $s(T(s)) = s$ for each s. □

It is easy to see that the formula in Theorem 1 is also valid when some of the cardinalities n_i are zero. This remark will be used in applications of Theorem 1.

Let T be a strictly k-cyclic multipartite tournament and let $C_1, ..., C_k$ be its cycles. Contracting every cycle C_i into a single vertex w_i gives an acyclic digraph T'. Let $T^*(C_1, ..., C_k)$ denote the digraph obtained from T' by deleting all arcs between pairs of vertices in $\{w_1, ..., w_k\}$.

Now we obtain a simple formula for $t_k(n_1, n_2), k \geq 0$. The problem to obtain a compact formula for $t_k(n_1, ..., n_p)$ ($p \geq 3$) for every $k \geq 0$ seems to be much more difficult. We prove a relatively compact formula for $t_1(n_1, ..., n_p)$ in Theorem 3.

Theorem 2. *For every integer k such that $0 \leq k \leq \frac{1}{2}\min\{n_1, n_2\}$, $t_k(n_1, n_2) =$*

$\binom{n-3k}{n_1-2k, n_2-2k, k}$.

Proof: For $k=0$, the formula follows from Theorem 1. Thus we may assume that $k \geq 1$. Let T be a strictly k-cyclic (n_1, n_2)-tournament, and let $C_1, ..., C_k$ be the cycles of T. Every cycle C_i is of length four, since otherwise the chord joining two vertices distance 3 apart around C_i would complete another cycle. Thus, the cycles are 'interchangeable'. Therefore, $t_k(n_1, n_2)$ equals $t_0(n_1 - 2k, n_2 - 2k, k)$, the number of unlabeled acyclic $(n_1 - 2k, n_2 - 2k, k)$-tournaments of the form $T^*(C_1, ..., C_k)$. The result now follows by Theorem 1. □

Let $S(p,k)$ denote the set of all unordered k-subsets of $\{1, ..., p\}$. In what follows, we assume that $\binom{m}{m_1, ..., m_p} = 0$ if one of the integers m_i is negative. Note that

$$\binom{m}{m_1, ..., m_p, 1} = m\binom{m-1}{m_1, ..., m_p} \tag{1}$$

if $m_1 + ... + m_p = m - 1$.

Theorem 3. *The number of unlabeled unicyclic $(n_1, ..., n_p)$-tournaments ($p \geq 3$) is*

$$t_1(n_1, ..., n_p) = (n-3)\sum_{\pi \in S(p,2)} \binom{n-4}{n_1^2(\pi), ..., n_p^2(\pi)} + 2(n-2)\sum_{\pi \in S(p,3)} \binom{n-3}{n_1^1(\pi), ..., n_p^1(\pi)},$$

where $n_j^c(\pi) = n_j - c$ if $j \in \pi$, and $n_j^c(\pi) = n_j$ otherwise.

Proof: Let T be a unicyclic $(n_1, ..., n_p)$-tournament with colour classes $V_1, ..., V_p$ and let C be the unique cycle in T. Two vertices of C that are not consecutive in C must be in the same colour class, since otherwise the chord between them would complete another cycle. Thus C is of length three, or of length four with vertices from two alternating colour classes.

Let us first assume that C has four vertices from V_i and V_j, $i < j$, and $\pi = \{i, j\}$. Then the number of unlabeled unicyclic $(n_1, ..., n_p)$-tournaments containing C equals the number of unlabeled acyclic $(n_1, ..., n_{i-1}, n_i - 2, n_{i+1}, ..., n_{j-1}, n_j - 2, n_{j+1}, ..., n_p, 1)$-tournaments of the form $T^*(C)$, which is $t_0(n_1^2(\pi), \ldots, n_p^2(\pi), 1)$. By Theorem 1 and (1), this gives the first term in the formula for $t_1(n_1, ..., n_p)$.

Now let C be a cycle with three vertices from classes V_i, V_j and V_k, respectively, and in this order. Let also $\pi = \{i, j, k\}$. Then the number of unlabeled unicyclic $(n_1, ..., n_p)$-tournaments containing C equals $t_0(n_1^1(\pi), ..., n_p^1(\pi), 1)$. This fact and the possibility to have two unlabeled triangles C with vertices from classes V_i, V_j and V_k (in this order and in the opposite one) gives the second term in the formula for $t_1(n_1, ..., n_p)$. □

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