MINIMIZING AND MAXIMIZING THE DIAMETER IN ORIENTATIONS OF GRAPHS

G. Gutin

Raymond and Beverly Sackler
Faculty of Exact Sciences
School of Mathematical Sciences
Tel Aviv University
Ramat-Aviv 69978, Israel

Abstract.
For a graph G, let $G'(G'')$ denote an orientation of G having maximum (minimum respectively) finite diameter. We show that the length of the longest path in any 2-edge connected (undirected) graph G is precisely $\mathrm{diam}(G')$. Let $K(m_1, m_2, \ldots, m_n)$ be the complete n-partite graph with parts of cardinalities $m_1, m_2, \ldots, m_n$. We prove that if $m_1 = m_2 = \cdots = m_n = m$, $n \geq 3$, then $\mathrm{diam}(K''(m_1, m_2, \ldots, m_n)) = 2$, unless $m = 1$ and $n = 4$.

1. Introduction

The following is a well known Theorem of Robbings [1]; a connected graph G has a strongly connected orientation if and only if G has no bridge.

Therefore, we consider here only (connected) graphs without bridges (an edge e of a (connected) graph G is called a bridge if $G - e$ is not connected). For a graph G, let $G'(G'')$ denote an orientation of G having maximum (minimum, respectively) finite diameter.

In this work we prove that for any graph G $\mathrm{diam}(G')$ is equal to the length of the longest path of G (denoting here by $\ell p(G)$). This implies the inequality of Ghouila-Houri (cf. [2], page 72) for oriented graphs.

Define $f(m_1, m_2, \ldots, m_n) = \mathrm{diam}(K''(m_1, m_2, \ldots, m_n))$. Boesh and Tindell [3] proved that $f(m, m) = 3$ for $m \geq 2$. Plesnik (cf. [4]) showed that if $m_1, m_2 \geq 2$, then $f(m_1, m_2) \leq 4$. Finally, Soltes [4] determined the exact value of $f(m_1, m_2)$ for all m_1, m_2. If $m_1 \geq m_2 \geq 2$, then $f(m_1, m_2) = 3$ for $m_1 \leq \binom{m_2}{\lfloor m_2/2 \rfloor}$, and otherwise $f(m_1, m_2) = 4$. A short proof of this result, using the well known theorem of Sperner is given in [5].

In the present paper we prove that if $n \geq 3$, then $f(m_1, m_2, \ldots, m_n) \leq 3$ for all $m_i (i = 1, 2, \ldots, n)$ and determine $f(m_1, \ldots, m_n)$ precisely for all $m_1 = m_2 = \cdots = m_n = m$; if $n \geq 3$ then $f(m_1, m_2, \ldots, m_n) = 2$ unless $n = 4$ and $m = 1$.

2. Maximum Diameter

Let G be a graph or a diagraph. Then the symbol $V(G)(E(G), A(G))$ denotes the set of all vertices (edges, arcs, respectively) of G. For any $X, Y \subseteq V(G)$ a path $y_1 y_2 \ldots y_p$ is called an (X, Y)-path if $y_1 \in X$, $y_p \in Y$, and $y_2, y_3, \ldots, y_{p-1} \notin X \cup Y$.

Theorem 1. *Let G be a 2-edge-connected graph. Then* $\mathrm{diam}(G') = \ell p(G)$.

Proof: For any strongly connected orientation G_0 of G we obviously have $\mathrm{diam}(G_0) \leq \ell p(G)$. Hence we must construct only some orientation G_1 of G with the property $\mathrm{diam}(G_1) = \ell p(G)$. This is done by a process similar to the one known as ear-decomposition of a graph [6].

Let $P = x_1 x_2 \ldots x_n$ be the longest path of G, and associate each vertex x_i with a mark $m(x_i) = i$. Since G has no bridges the edge $x_{n-1} x_n$ is not a bridge. Consequently, there exists an $(\{x_1, x_2, \ldots, x_{n-1}\},\ \{x_n\})$-path P_1 different from the path $x_{n-1} x_n$. Let x_i be the first vertex of P_1. Define $m(v) = i$ for all vertices $v \in V(P_1) \backslash \{x_n\}$. Since $x_{i-1} x_i$ is not a bridge there exists an $(\{x_1, x_2, \ldots, x_{i-1}\}, \{x_i, x_{i+1}, \ldots, x_n\} \cup V(P_1))$-path P_2 different from the path $x_{i-1} x_i$. Similarly if x_j is the first vertex of P_2 (note that $j < i$), then define $m(v) = j$ for all vertices in P_2 besides the last one. Analogously, we can build paths $P_3, P_4, \ldots$, and define mark m of the vertices of $P_3, P_4, \ldots$ until we obtain a path P_s with the first vertex x_1.

Now, we orientate path P from x_1 to x_n (we obtain the dipath Q), and each path $P_i (i = 1, 2, \ldots, s)$ from its endvertex having a bigger mark to its other end vertex (with the smaller mark), we derive the dipath Q_i. It is easy to check that the oriented graph induced by the arcs of the paths $\bigcup_{i=1}^{s} Q_i \cup Q$ is a strongly connected digraph. Define

$$X = V(G) \backslash (V(P) \cup \bigcup_{i=1}^{s} V(P_i))$$

and suppose $X \neq \emptyset$ (the case $X = \emptyset$ is easier). Since G has no bridges there exists some vertex $v \in X$ and a pair of $(\{v\},\ V(G)\backslash X)$-paths with no common vertices (besides v). We unite these two paths to one (path S_1). Now orientate the last path from its end vertex having the bigger mark to the one having the smaller mark. If the marks of the two end vertices coincide then the orientation is arbitrary.

If $X \backslash V(S_1) \neq \emptyset$ we shall continue the construction of paths $S_2, S_3, \ldots$ passing over the rest of the vertices of X until $\bigcup_{i=1}^{t} V(S_i) = X$, where the orientation is chosen in the same manner. Finally orient each unoriented edge uv from u to v if $m(n) \geq m(v)$ and from v to u otherwise.

Let D denote the obtained oriented graph. D contains a strongly connected spanning subgraph. Therefore, D is strongly connected. Since all the arcs (u, w) of D, besides those in P, are oriented such that $m(v) \geq m(w)$, there is no path from x_1 to x_n having length less than $n-1$. Hence, $\mathrm{diam}(D) = n - 1$. □

Corollary 1. *If $m_1 \geq m_i (i = 2, \ldots, n)$, and $M = \sum_{i=2}^{n} m_i$, $p = m_1 + M$ then $\mathrm{diam}(K'(m_1, m_2, \ldots, m_n)) = p - 1 - \max\{m_1 - 1 - M, 0\}$.*

Proof: If $m_1 > M$, then it is easy to see that

$$\ell p(K(m_1, \ldots, m_n)) = 2M = p - 1 - (m_1 - 1 - M)\ .$$

Otherwise, $K(m_1, \ldots, m_n)$ is Hamiltonian (by Dirac's theorem, or by exhibiting an explicit Hamilton cycle) and $\ell p(K(m_1, \ldots, m_n)) = p - 1$. □

3. Minimum Diameter

Let $V_1, V_2, \ldots, V_n$ be the parts of $K(m_1, m_2, \ldots, m_n)$, where $V_i = \{v_j^{(i)} : j = 1, 2, \ldots, m_i\}$; we use the following notation for a digraph D and $X, Y \subseteq V(D)$ $X \times Y = \{(x, y) : x \in X,\ y \in Y\}$, $A_D(X, Y) = A(D) \cap (X \times Y \cup Y \times X)$; we write down

$X \to Y$ iff $A_D(X,Y) = X \times Y$, $X \mapsto Y$ iff for every $y \in Y$ there exists $x \in X$ such that $(x,y) \in A(D)$; and define the distance

$$d(X,Y) = \max_{x \in X} \max_{y \in Y} d(x,y) \ ;$$

if $m_1 = m_2 = \cdots m_n = m$, then $f(m^{(n)}) = f(m_1, m_2, \ldots, m_n)$, $K(m^{(n)}) = K(m_1, m_2, \ldots, m_n)$,

$R(m^{(n)}) = R(m_1, \ldots, m_n)$, where R is defined below.

Theorem 2. *If $n \geq 3$, then $f(m_1, m_2, \ldots, m_n) \leq 3$ for all positive integers $m_1, m_2, \ldots, m_n$.*

Proof: Let for any odd n $R(m_1, m_2, \ldots, m_n)$ means an orientation of $K(m_1, m_2, \ldots, m_n)$ such that $V_i \to V_j$ if and only if

$$j - i \equiv 1, 2, \ldots, \lfloor n/2 \rfloor (\operatorname{mod} n) \ .$$

If n is even, then $R(m_1, m_2, \ldots, m_n)$ is determined by the following

$$R(m_1, m_2, \ldots, m_n) - V_n \cong R(m_1, m_2, \ldots, m_{n-1}) \ ,$$

$$V_n \to V_i (i = 1, 3, 5, \ldots, n-1), \ V_j \to V_n \quad (j = 2, 4, 6, \ldots, n-2) \ .$$

We prove that $\operatorname{diam} R(m_1, m_2, \ldots, m_n) \leq 3$.

Case 1. $n \equiv 1 (\operatorname{mod} 2)$, $n \geq 3$. It is sufficient to prove that $d(V_1, V_i) \leq 3$ for all $i = 1, 2, \ldots, n$. If $1 < j \leq \lfloor \frac{n}{2} \rfloor + 1$, then $V_1 \to V_j$ by the definition. If $\lfloor \frac{n}{2} \rfloor + 1 < j \leq n$, then $V_{\lfloor \frac{n}{2} \rfloor + 1} \to V_j$, hence $d(V_1, V_j) = 2$. Since $V_1 \to V_{\lfloor \frac{n}{2} \rfloor + 1} \to V_{\lfloor \frac{n}{2} \rfloor + 2} \to V_1 \quad d(V_1, V_1) \leq 3$.

Case 2. $n \equiv 0 (\operatorname{mod} 2)$, $n \geq 4$. Since $R(m_1, \ldots, m_n) - V_n \cong R(m_2, \ldots, m_{n-1})$ we have $d(V_i, V_j) \leq 3$ for all $1 \leq i, \ j \leq n-1$. Besides, $V_n \to V_i \to V_{i+1}$ for $i = 1, 3, 5, \ldots, n-3$; and $V_n \to V_{n-1}$, therefore $d(V_n, V_t) \leq 2$ for $t = 1, 2, \ldots, n-1$. Analogously, $V_i \to V_{i+1} \to V_n$ for $i = 1, 3, 5, \ldots, n-3$; $V_{n-1} \to V_1 \to V_2 \to V_n$, hence $d(V_t, V_n) \leq 3$ for $t = 1, 2, \ldots, n-1$. Finally, $V_n \to V_1 \to V_2 \to V_n$, therefore $d(V_n, V_n) \leq 3$. □

Lemma 1.

$$f(1^{(n)}) = \begin{cases} 2, & \textit{if} \quad n \geq 3, \ n \neq 4; \\ 3, & \textit{if} \quad n = 4 \ . \end{cases}$$

Proof: Clearly $f(1^{(n)}) > 1$ for all $n \geq 3$. We prove that

$$\operatorname{diam} R(1^{(n)}) = \begin{cases} 2, & \text{if} \quad n \geq 3,\ n \neq 4\ , \\ 3, & \text{if} \quad n = 4\ . \end{cases} \tag{1}$$

If the integer n is odd (1) follows from the proof of case 1 in Theorem 2 (if all $m_i = 1$ we do not need 3-cycles).

If $n \equiv 0 (\operatorname{mod} 2)$, $n \geq 6$, then we can use discussion in case 2 in the proof of Theorem 2, but we change $V_{n-1} \to V_1 \to V_2 \to V_n$ to $V_{n-1} \to V_2 \to V_n$. If $n = 4$, then $d(V_3, V_4) = 3$, hence $\operatorname{diam} R(1,1,1,1) = 3$. But $R(1,1,1,1)$ is the unique strongly connected tournament on 4 vertices (up to isomorphism), therefore $f(1,1,1,1) = 3$. □

Define $V_i' = V_i \backslash \{v_1^{(i)}\},\ i = 1, 2, \ldots, n$.

Lemma 2. *For* $m \geq 3,\ n \neq 4,\ n \geq 3\ f(m^{(n)}) = 2$.

Proof: We change the direction of all arcs of the form $\left(v_t^{(i)},\ v_t^{(j)}\right)$ $(t = 1, 2, \ldots, m;$ $1 \leq i \neq j \leq n)$ in $R(m^{(n)})$ (see the proof of Theorem 2) and obtain $R_1(m^{(n)})$. We next show that $\operatorname{diam} R_1(m^{(n)}) = 2$ for $n \geq 3,\ n \neq 4,\ m \geq 3$. Note that $X \to Y,\ Y \mapsto Z$ implies $d_D(X, Z) \leq 2 (X, Y, Z \subseteq V(D))$.

Case 1. $n \equiv 1 (\operatorname{mod} 2),\ n \geq 3$. Put $q = \lfloor \frac{n}{2} \rfloor$. It is easy to check that

$$\begin{aligned}
&v_1^{(1)} \to V_2' \cup V_3' \cup \cdots \cup V_{q+1}'\ , \\
&v_1^{(1)} \to \left\{v_2^{(q+1)}, v_3^{(q+1)}\right\} \mapsto V_{q+2} \cup V_{q+3} \cup \cdots \cup V_n \\
&v_1^{(1)} \to v_1^{(q+2)} \to v_1^{(s)}\ , \text{ where } \quad s = 2, 3, \ldots, q+1, \text{ and} \\
&v_1^{(1)} \to v_t^{(2)} \to v_t^{(1)}\ , \text{ where } \quad t = 2, 3, \ldots, m\ .
\end{aligned}$$

Hence, $d\left(v_1^{(1)},\ V(R_1(m^{(n)}))\right) = 2$. By the symmetry of $R_1(m^{(n)})$ the distance from any vertex of $R_1(m^{(n)})$ to any other vertex less or equal to 2. Therefore, $f(m^{(n)}) = 2$.

Case 2. $n \equiv 0 (\operatorname{mod} 2),\ n \geq 6$. Since $R_1(m^{(n)}) - V_n \cong R_1(m^{(n-1)})$ the distance $d(V_i, V_j) \leq 2$ for all $1 \leq i,\ j \leq n-1$. We note that

$$v_1^{(i)} \to \left\{v_2^{(i+1)},\ v_3^{(i+1)}\right\} \mapsto V_n \qquad (i = 1, 3, 5, \ldots, n-3)\ .$$

Consequently, $d\left(v_1^{(i)}, V_n\right) = 2 \quad (i = 1, 3, \ldots, n-3)$. For $j = 2, 4, \ldots, n-2$ $v_1^{(j)} \to V_n'$, $v_1^{(j)} \to v_1^{(j-1)} \to v_1^{(n)}$, thus,

$$d\left(v_1^{(j)}, V_n\right) = 2 \; .$$

Since $n \geq 6$, $v_1^{(n-1)} \to \{v_2^{(2)}, v_3^{(2)}\} \mapsto V_n$. Hence, $d\left(v_1^{(n-1)}, V_n\right) = 2$. Obviously, $d(V_n, V_n) = 2$. Thus, $d(V_1 \cup V_2 \cup \cdots \cup V_n, V_n) = 2$. It is easy to see that

$$v_1^{(n)} \to V_1' \cup V_3' \cup \cdots \cup V_{n-1}' \; ,$$

$$v_1^{(n)} \to v_1^{(i+1)} \to v_1^{(i)} (i = 1, 3, \ldots, n-3), \; v_1^{(n)} \to v_1^{(2)} \to v_1^{(n-1)} \; .$$

Therefore, $d\left(V_n, V_1 \cup V_3 \cup \cdots \cup V_{n-1}\right) = 2$.

$$v_1^{(n)} \to \left\{v_2^{(i)}, v_3^{(i)}\right\} \mapsto V_{i+1} (i = 1, 3, \ldots, n-3)\} \; .$$

Hence, $d\left(V_n, V_2 \cup V_4 \cup \cdots \cup V_{n-2}\right) = 2$. □

Lemma 3. *For $m \geq 2$ $f(m^{(4)}) = 2$.*

Proof: The orientation $Q = Q(m^{(4)})$ of $K(m^{(4)})$ is determined by the following

$$A(Q) = V_2 \times V_1 \cup V_1 \times V_3 \cup V_1 \times V_4 \cup V_3 \times V_2 \cup V_2 \times V_4 \cup V_3 \times V_4 \; .$$

We change the direction of all arcs of the form $\left(v_t^{(i)}, v_t^{(j)}\right)$ $(t = 1, 2, \ldots, m;\ 1 \leq i \neq j \leq n)$ in Q and obtain $Q_1\left(m^{(4)}\right)$. We next show that $\operatorname{diam} Q_1\left(m^{(4)}\right) = 2$ for $m \geq 3$. It is easy to check that

$$v_1^{(1)} \to V_3' \mapsto V_2 \cup V_4, \; v_1^{(1)} \to v_1^{(2)} \to v_1^{(3)} \; .$$

Hence, $d\left(V_1, V_2 \cup V_3 \cup V_4\right) = 2$. Analogously, $v_1^{(2)} \to V_1' \mapsto V_3 \cup V_4$, $v_1^{(2)} \to v_1^{(3)} \to v_1^{(1)}$,

$$v_1^{(3)} \to V_2' \mapsto V_1 \cup V_4, \; v_1^{(3)} \to v_1^{(1)} \to v_1^{(2)} \; .$$

Therefore, $d\left(V_i, V(Q_1(m^{(4)})) \setminus V_i\right) = 2$ for $i = 2, 3$. Obviously, $v_1^{(4)} \to v_1^{(1)} \to V_3'$, $v_1^{(4)} \to v_1^{(2)} \to V_1'$, $v_1^{(4)} \to v_1^{(3)} \to V_2'$. Hence, $d\left(V_4, V_1 \cup V_2 \cup V_3\right) = 2$. Finally, if

$$\left\{\left(v_t^{(i)}, v_t^{(j)}\right) : i = 1, 2, \ldots, m\right\} \subset A\left(Q_1(m^{(4)})\right) \; , \tag{2}$$

then $v_t^{(i)} \to v_t^{(j)} \to V_i \backslash \left\{v_t^{(i)}\right\}$, $t = 1, 2, \ldots, m$ (by the definition of $Q_1\left(m^{(4)}\right)$). Since for any $i = 1, 2, 3, 4$ there exists j such that (2) holds, we have

$$d\left(V_i, V_i\right) \leq 2 \quad \text{for} \quad i = 1, 2, 3, 4 \; .$$

Similarly one can consider the case $m = 2$. □

Lemma 4. *If*

$$2 \leq m_1 \leq m_2 \leq \binom{m_1}{\lfloor m_1/2 \rfloor}, \tag{3}$$

then $f(m_1, m_2, 2) = 2$.

Proof: By (3) and the result of Soltes [4] (see section 1) one can construct an orientation B of $K(m_1, m_2)$ with diameter three. Hence $d_B(V_1 \cup V_2,\ V_1 \cup V_2) = 3$. But $d_B(V_i, V_i) \equiv 0 (\mathrm{mod}\, 2)$ $(i = 1, 2)$, therefore $d_B(V_i, V_i) = 2$ $(i = 1, 2)$.

We add to B a new party V_3 $(|V_3| = 2)$ and the arcs

$$\left\{ \left(v_j^{(3)}, x^{(j)} \right) : x^{(j)} \in V_j, \quad j = 1, 2 \right\} \cup$$
$$\left\{ \left(x^{(j)}, v_{j+1}^{(3)} \right) : x^{(j)} \in V_j, \quad j = 1, 2;\ v_3^{(3)} = v_1^{(3)} \right\},$$

and obtain the oriented graph $D(m_1, m_2, 2)$ with diameter 2.

In fact, $V_1 \to v_2^{(3)} \to V_2,\ V_2 \to v_1^{(3)} \to V_1$, and since $d(V_1, V_1) = d(V_2, V_2) = 2$ we have

$$d(V_i, V_j) = 2 \qquad (i, j \in \{1, 2\}).$$

Since $v_1^{(3)} \to V_1 \mapsto V_2,\ v_2^{(3)} \to V_2 \mapsto V_1,\ V_1 \to v_2^{(3)}, V_2 \to v_1^{(3)}$, and for $i = 1, 2$ $d\left(V_i, v_i^{(3)}\right) = 2$ (the outdegree of any vertex in B is positive, hence for any $v_k^{(i)}$ $(k = 1, 2, \ldots, m)$ there exists a path $v_k^{(i)} v_j^{(i+1)} v_i^{(3)}$), we have

$$d(V_3, V_1 \cup V_2) = d(V_1 \cup V_2, V_3) = 2.$$

Besides, $v_1^{(3)} \to V_1 \to v_2^{(3)}, v_2^{(3)} \to V_2 \to v_1^{(3)}$, i.e. $d(V_3, V_3) = 2$. □

Lemma 5. *If* $n \geq 3,\ n \neq 4$, *then* $f\left(2^{(n)}\right) = 2$.

Proof: If $n = 3$, then by Lemma 4 $f\left(2^{(3)}\right) = 2$. If $n \geq 5$ and $|V_i| = 2 (i = 1, 2, \ldots, n)$ we can construct an oriented graph, isomorphic to $D(M_1, M_2, 2)$ (see the proof of Lemma 4) where

$$M_2 = m_1 + m_2 + \cdots + m_{\lfloor n/2 \rfloor}, \quad M_1 = m_{\lfloor n/2 \rfloor + 1} + m_{\lfloor n/2 \rfloor + 2} + \cdots + m_{n-1}.$$

It is easy to check that if $n \geq 5$, then

$$M_1 \leq M_2 \leq \binom{M_1}{\lfloor M_1/2 \rfloor}.$$

Hence, by virtue of Lemma 4, $\operatorname{diam} D(M_1, M_2, 2) = 2$, and therefore $f\left(2^{(n)}\right) = 2$ for $n \geq 5$. $\square$

Lemmas 1-3,5 imply immediately the next theorem.

Theorem 3. *$f\left(m^{(n)}\right) = 2$ for any integer $m \geq 1$, and any integer $n \geq 3$, except the pair $(m, n) = (1, 4)$, for which $f\left(1^{(4)}\right) = 3$.*

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