Anti-matroids

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Abstract

We introduce an anti-matroid as a family $\mathcal{F}$ of subsets of a ground set E for which there exists an assignment of weights to the elements of E such that the greedy algorithm to compute a maximal set (with respect to inclusion) in $\mathcal{F}$ of minimum weight finds, instead, the unique maximal set of maximum weight. We introduce a special class of anti-matroids, I-anti-matroids, and show that the Asymmetric and Symmetric TSP as well as the Assignment Problem are I-anti-matroids.

Keywords: Greedy algorithm, combinatorial optimization, TSP, Assignment Problem, matroids.

1 Introduction

Many combinatorial optimization problems can be formulated as follows. We are given a pair $(E, \mathcal{F})$, where E is a finite set and $\mathcal{F}$ is a family of subsets of E, and a weight function c that assigns a real weight $c(e)$ to every element of E. The weight $c(S)$ of $S \in \mathcal{F}$ is defined as the sum of the weights of the elements of S. It is required to find a maximal (with respect to inclusion) set $B \in \mathcal{F}$ of minimum weight. The *greedy algorithm* starts from the element of E of minimum weight that belongs to a set in $\mathcal{F}$. In

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every iteration the greedy algorithm adds a minimum weight unconsidered element e to the current set X provided $X \cup \{e\}$ is a subset of a set in $\mathcal{F}$.

It is well known that the greedy algorithm produces an optimal solution to the problem above when $(E, \mathcal{F})$ is a matroid. While the greedy algorithm does not necessarily find optima for non-matroidal pairs $(E, \mathcal{F})$ one might think that the greedy algorithm always produces a solution that is better than many others. It was shown by Gutin, Yeo and Zverovich [3] that for every $n \geq 2$ there is an instance of the Asymmetric TSP (ATSP) on n vertices for which the greedy algorithm finds the unique worst possible tour. The same result holds for the Symmetric TSP (STSP). These results contradict somewhat our intuition on the greedy algorithm.

In the present paper we introduce the notion of an anti-matroid. An *anti-matroid* is a pair $(E, \mathcal{F})$ such that there is an assignment of weights to the elements of E for which the greedy algorithm for finding a maximal set B in $\mathcal{F}$ of minimum weight constructs the unique maximal set of maximum weight. The above mentioned results on the TSP indicate that both STSP and ATSP are anti-matroids.

Similarly to matroids, we introduce I-anti-matroids and prove that every non-trivial I-anti-matroid is an anti-matroid. I-anti-matroids are of interest not only since they are somewhat close to matroids, but also because they include the STSP, ATSP and the Assignment Problem (AP). Thus, in particular, we obtain an easy and uniform proof that the above mentioned problems are anti-matroids. The fact that the AP is an I-anti-matroid is of particular interest since the AP is polynomial time solvable (unlike the ATSP and STSP provided P$\neq$NP).

2 I-Anti-matroids

An *I-independence family* is a pair consisting of a finite set E and a family $\mathcal{F}$ of subsets (called *independent sets*) of E such that (I1)-(I3) are satisfied.

(I1) the empty set is in $\mathcal{F}$;

(I2) If $X \in \mathcal{F}$ and Y is a subset of X, then $Y \in \mathcal{F}$;

(I3) All maximal sets of $\mathcal{F}$ (called *bases*) are of the same cardinality k.

If $S \in \mathcal{F}$, then let $I(S) = \{x : \ S \cup \{x\} \in \mathcal{F}\} - S$. This means that $I(S)$ contains all elements (different from S), which can be added to S, in order to have an independent set. An I-independence family $(E, \mathcal{F})$ is an I-*anti-matroid* if

(I4) There exists a base $B' \in \mathcal{F}$, $B' = \{x_1, x_2, \ldots, x_k\}$, such that the following holds for every base $B \in \mathcal{F}$, $B \neq B'$,

$$\sum_{j=0}^{k-1} |I(x_1, x_2, \ldots, x_j) \cap B| < k(k+1)/2.$$

An I-anti-matroid is *non-trivial* if $k \geq 2$.

Note that if we replace (I4) in (I1)-(I4) by the following condition, we obtain one of the definitions of a matroid [5]:

(I5) If U and V are in $\mathcal{F}$ and $|U| > |V|$, then there exists $x \in U - V$ such that $V \cup \{x\} \in \mathcal{F}$.

(In fact, (I1), (I2) and (I5) define a matroid, with (I3) being an implication of the three conditions.)

Theorem 2.1 *For every non-trivial I-anti-matroid $(E, \mathcal{F})$, there exists a weight function c from E to the set of positive integers such that the greedy algorithm finds the unique worst solution for the problem of finding a minimum weight base.*

Proof: Let $B' = \{x_1, \ldots, x_k\}$ be a base that satisfies (I4). Let $M > k$ and let $c(x_i) = iM$ and $c(x) = 1 + jM$ if $x \notin B'$, $x \in I(x_1, x_2, \ldots, x_{j-1})$ but $x \notin I(x_1, x_2, \ldots, x_j)$. Clearly, the greedy algorithm constructs B' and $c(B') = Mk(k+1)/2$.

Let $B = \{y_1, y_2, \ldots, y_k\}$. By the choice of c made above, we have that $c(y_i) \in \{aM, aM+1\}$ for some positive integer a. Then clearly

$$y_i \in I(x_1, x_2, \ldots, x_{a-1}),$$

but $y_i \notin I(x_1, x_2, \ldots, x_a)$, so y_i lies in $I(x_1, x_2, \ldots, x_j) \cap B$, provided $j \leq a - 1$. Thus, y_i is counted a times in the sum in (I4). Hence,

$$\begin{aligned} c(B) &= \sum_{i=1}^{k} c(y_i) \leq k + M \sum_{j=0}^{k-1} |I(x_1, x_2, \ldots, x_j) \cap B| \\ &\leq k + M(k(k+1)/2 - 1) = k - M + c(B'), \end{aligned}$$

which is less than the weight of B' as $M > k$. Since the greedy algorithm finds B', and B is arbitrary, we see that the greedy algorithm finds the unique heaviest base. $\square$

Theorem 2.1 implies that no non-trivial I-anti-matroid is a matroid. Let us consider one of the differences between matroids and I-anti-matroids. For a matroid $(E, \mathcal{F})$ and two distinct bases B and $B' = \{x_1, x_2, \ldots, x_k\}$, by (I5), we have that $|I(x_1, x_2, \ldots, x_j) \cap B| \geq k - j$ for $j = 0, 1, \ldots, k$. Thus,

$$\sum_{j=0}^{k-1} |I(x_1, x_2, \ldots, x_j) \cap B| \geq k(k+1)/2.$$

This inequality becomes equality for the matroid $(E, \mathcal{F})$ of the matrix whose columns are of I and $2I$, where I is the identity matrix (E consists of columns of $(I|2I)$ and $\mathcal{F}$ of sets of linearly independent columns). This shows that (I4) is sharp, in a sense, in the definition of I-anti-matroid.

Corollary 2.2 *Non-trivial ATSP, STSP and AP are all anti-matroids.*

Proof: By Theorem 2.1, it is enough to show that the three problems are non-trivial I-anti-matroids. For the non-trivial AP, E consists of edges of a complete bipartite graph G (with partite sets of cardinality $k \geq 2$) and $\mathcal{F}$ of matchings in G (including the empty one). Clearly, (I1)-(I3) hold. Let $B' = \{x_1z_1, \ldots, x_kz_k\}$, $B = \{u_1v_1, \ldots, u_kv_k\}$ be a pair of distinct perfect matchings in G. Observe that $I(x_1z_1, \ldots, x_jz_j) \cap B$ consists of edges of B belonging to the subgraph G' of G induced by the vertices $\{x_p, z_p : j+1 \leq p \leq k\}$. The cardinality of the perfect matching in G' is $k - j$ and, hence, $|I(x_1z_1, \ldots, x_jz_j) \cap B| \leq k - j$ for $j = 0, 1, \ldots, k-1$. Moreover, $|I(x_1z_1, \ldots, x_jz_j) \cap B| < k - j$ for some j since $B \neq B'$. This inequalities imply (I4) and, thus, the non-trivial AP is an I-anti-matroid.

For the STSP (ATSP), E consists of edges of the complete undirected (directed) graph K on $k \geq 2$ vertices $z_1, \ldots, z_k$ and every set in $\mathcal{F}$ is a subset of edges of a Hamilton cycle in K. Clearly, (I1)-(I3) hold for both ATSP and STSP.

To verify (I4) for the ATSP, transform K into a bipartite graph G with partite sets $z_1, \ldots, z_k$ and $z'_1, \ldots, z'_k$, and edges $\{z_pz'_q : 1 \leq p \neq q \leq k\}$. An arc z_pz_q of K corresponds to the edge $z_pz'_q$ in G. Observe that every Hamilton cycle in K corresponds to a perfect matching in G (but not vice versa, in general). It follows that (I4) is satisfied for the ATSP since (I4) holds for the non-trivial AP. Thus, the ATSP is an I-anti-matroid.

A similar proof can be provided for the STSP, but now every edge z_pz_q of K corresponds to the pair $z_pz'_q, z'_pz_q$ of edges in G. Hence, the STSP is an I-anti-matroid. □

It would be interesting to have general examples of anti-matroids that are not I-anti-matroids and are related to well-studied combinatorial optimization problems.

It is worth noting that the results of this paper can be placed within the topic of domination analysis of combinatorial optimization algorithms. For the sake of simplicity and clarity, we define the domination number only for a heuristic for the ATSP. The reader can easily extend this definition to other combinatorial optimization problems. The *domination number* of a heuristic $\mathcal{A}$ for the ATSP is the maximum integer $d(n)$ such that, for every instance $\mathcal{I}$ of the ATSP on n vertices, $\mathcal{A}$ produces a tour T which is not worse than at least $d(n)$ tours in $\mathcal{I}$ including T itself. Observe that an exact algorithm for the ATSP has domination number $(n-1)!$. For a survey on domination number of the ATSP and STSP, see [4]. The main result of our paper is that the domination number of the greedy algorithm for non-trivial I-anti-matroids is 1. It is worth noting that there are polynomial time heuristics for the ATSP and STSP of much larger domination number; for heuristics of domination number at least $(n-2)!/2$ see, e.g., [1, 2, 3, 4, 6, 7, 8, 9].

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