Alternating cycles and paths in edge-coloured multigraphs: a survey

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Abstract

A path or cycle in an edge-coloured multigraph is called alternating if its successive edges differ in colour. We survey results of both theoretical and algorithmic character concerning alternating cycles and paths in edge-coloured multigraphs. We also show useful connections between the theory of paths and cycles in bipartite digraphs and the theory of alternating paths and cycles in edge-coloured graphs.

1 Introduction

We consider graphs (and sometimes multigraphs) so that each edge has a colour. A trail in such a graph is called *alternating* if its successive edges differ in colour. J. Petersen's famous paper [47] seems to be the first place where one can find applications of alternating trails (cf. [45]). Besides a number of applications in graph theory and algorithms (cf. [53], p. 58, [36]), the concept of alternating trails and the special cases, alternating paths and cycles, appears in various other fields: genetics (cf. [23, 24, 25]), social sciences (cf. [19]), etc.

The reader will see below bilateral connections between the theory of alternating cycles and paths in edge-coloured graphs and the theory of cycles

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and paths in directed and undirected graphs, matching theory, and other branches of graph theory (cf. [12, 29, 36, 37, 38, 43, 52]). These results have been scattered in the literature and seem not well known to a broader audience. In particular, in Section 4 we provide some examples of results on 2-edge-coloured complete bipartite graphs and bipartite tournaments which are either equivalent (with a very easy transformation from one to the other), or one can be derived from the other fairly easily. We describe some other examples when results on directed paths and cycles in bipartite and general digraphs imply results on alternating paths and cycles in edge-coloured graphs.

In applications to genetics (cf. [23, 24, 25]) researchers consider edge-coloured multigraphs which are unions of monochromatic graphs. From a theoretical point of view there is no good reason to restrict investigation to multigraphs without parallel edges when more general results may be available. Therefore, we shall sometimes deal with edge-coloured multigraphs rather than edge-coloured graphs.

Our paper is organized as follows. In the next section we provide the reader with necessary notation and terminology. Section 3 is devoted to problems on alternating cycles in general edge-coloured multigraphs. First, we consider the problem of characterizing c-edge-coloured graphs containing at least one alternating cycle. We point out that the well known theorem by Grossman and Häggkvist, solving the problem for $c = 2$, also holds for $c \geq 3$ (proved by Yeo [54]). We describe a number of applications of the result by Grossman and Häggkvist. In the second part of Section 3 we show how to find a maximum collection (with respect to the number of vertices covered) of disjoint alternating cycles in a c-edge-coloured multigraph in polynomial time. This construction, is a generalization of one used for the case $c = 2$ in [3]. In the end of Section 3, we consider results on colour-connectivity for edge-coloured multigraphs which is, in a sense, an analogue of strong connectivity for digraphs. The notion of colour-connectivity, first introduced by Saad in [48], has already proved its importance for the longest alternating cycle problem in certain 2-edge-coloured multigraphs [48].

In Section 4, we describe two known approaches which allow one to obtain results for bipartite 2-edge-coloured multigraphs using results on directed graphs. Applying the approaches we derive new results from known ones, as well as show equivalence, or similarity, of known results obtained separately for bipartite 2-edge-coloured graphs and bipartite digraphs.

In Section 5, we deal with cycles, trails and paths in 2-edge-coloured complete multigraphs. We first describe characterizations of 2-edge-coloured complete multigraphs containing a Hamiltonian alternating cycle as well as some of their extensions. Then we consider a generalization of problems treated by Saad [48], and Das and Rao [21]: determine the length of a longest closed alternating trail visiting each vertex v of a complete 2-edge-coloured multigraph at most $f(v) > 0$ times. In Section 5, we also consider pancyclicity problems (the main result on this topic, Theorem 5.10, is a slight generalization of a result in [20]) and some other problems on alternating cycles and paths in complete 2-edge-coloured multigraphs.

Section 6 is devoted to a number of interesting and difficult open problems on alternating cycles and paths in c-edge-coloured complete graphs for $c \geq 2$ as well as to some results on the topic. In Section 7, we describe results on alternating Eulerian trails in general edge-coloured graphs and Hamiltonian alternating cycles in a special family of 2-edge-coloured graphs which appear in an application to genetics.

2 Notation and terminology

The terminology is fairly standard, generally following [8] and [16]. All graphs and digraphs considered are finite and have no loops. A set with k elements is called a *k-set*.

In this paper we consider *edge-coloured multigraphs*, i.e. multigraphs so that each edge has a colour and no two parallel (i.e. joining the same pair of vertices) edges have the same colour. If the number of colours is restricted by an integer c, we speak about *c-edge-coloured multigraphs.* We usually use the integers $1, 2, ..., c$ to denote the colours in c-edge-coloured multigraphs. In case $c = 2$, we also use the names *red* and *blue* for colours 1 and 2, respectively. The *red subgraph* (*blue subgraph*, resp.) of a 2-edge-coloured multigraph G consists of the vertices of G and all red (blue, resp.) edges of G.

When multigraphs have no parallel edges, we call them *graphs*, as usual. A multigraph G is *complete k-partite*, if it is possible to partition $V(G)$ into k subsets (called *partite sets*) such that, for every pair $u \in V_i$ and $v \in V_j$, u is adjacent to v if and only if $i \neq j$. A multigraph G with n vertices is called *complete* if G is complete n-partite.

We consider only *simple* paths and cycles. Let G be a c-edge-coloured multigraph ($c \geq 2$). A cycle or path in G is called *alternating* if its successive edges differ in colour. Since, in edge-coloured multigraphs, we consider only alternating paths and cycles, we shall sometimes omit the adjective 'alternating' before the words 'path' and 'cycle'. An *m-path-cycle subgraph* F_m of G is a union of m (alternating) paths and a number of (alternating) cycles in G, all vertex disjoint. When $m = 0$, we shall call F_0 an *cycle subgraph*. An m-path-cycle subgraph F of G is *maximum* if F has maximum number of vertices among all m-path-cycle subgraphs of G. An alternating path or cycle is called *Hamiltonian* if it contains all the vertices of G. If G has a Hamiltonian alternating cycle, G is called *Hamiltonian*. An alternating path P is called an *(x, y)-path* if x and y are the end vertices of P.

The *j'th degree* of v, $d_{j,G}(v)$, is the number of vertices in G joined to v by an edge of colour j ($1 \leq j \leq c$). The *maximum monochromatic degree* of G (denoted by $\Delta(G)$) is $\max\{d_j(v) : \ v \in V(G), \ j = 1, 2, ..., c\}$. For a set $X \subset V(G)$, $d_j(X) = \sum_{x \in X} d_j(x)$.

The colour of an edge e in G will be denoted by $\chi_G(e)$. Let X and Y be two sets of the vertices of G. Then XY denotes the set of all edges having one end vertex in X and the other in Y. In case all the edges in XY have the same colour, say i, we write $\chi(XY) = i$.

An edge-coloured multigraph G is called *even-pancyclic* if G contains an even number of vertices and has an alternating cycle of length $2k$ for each $k = 2, 3, ..., m$, where $2m = |V(G)|$. G is *vertex even-pancyclic* if G has an even number of vertices and, for every $k = 2, ..., m$ and every $v \in V(G)$, G contains an alternating cycle through v of length $2k$, where $2m = |V(G)|$. Similarly one can define even-pancyclic and vertex even-pancyclic bipartite digraphs.

A digraph obtained by replacing each edge of a complete bipartite graph by an arc or a pair of mutually opposite arcs with the same end vertices is called a *semicomplete bipartite digraph*. A semicomplete bipartite digraph with no directed cycles of length two is called a *bipartite tournament*. A digraph D is called *strongly connected* (or *strong*, for short) if, for each pair x, y of distinct vertices of D, there is a path from x to y and a path from y to x.

3 General edge-coloured multigraphs

Problems on alternating cycles and paths in general 2-edge-coloured graphs are at least as difficult as the corresponding ones for directed cycles and paths in directed graphs. To see that, we consider the following simple transformation attributed to Häggkvist in [43]. Let D be a digraph. Replace each arc xy of D by two (unoriented) edges xz_{xy} and $z_{xy}y$ by adding a new vertex z_{xy} and then colour the edge xz_{xy} red and the edge $z_{xy}y$ blue. Let G be the 2-edge-coloured graph obtained in this way. It is easy to see that each alternating cycle in G corresponds to a directed cycle in D and vice versa. Hence, in particular, the following problems on paths and cycles in 2-edge-coloured graphs are NP-complete: the Hamiltonian alternating cycle problem and the problem to find an alternating cycle through a pair of vertices. Using the transformation above, we also see that the problem to check whether a c-edge-coloured graph has an alternating cycle is more general, even for $c = 2$, than the simple problem to verify whether a digraph contains a directed cycle. So, we start with the following:

Problem 3.1 *Given a c-edge-coloured graph G, check whether G contains an alternating cycle.*

Grossman and Häggkvist [29] were the first to study this problem. They proved Theorem 3.2 below in the case $c = 2$. The case $c \geq 3$ was proved recently by Yeo [54]. Let v be a cut vertex in a 2-edge-coloured graph G. We say that v *separates colours* if no component of $G - v$ is joined to v by at least two edges of different colours.

Theorem 3.2 *[29, 54] Let G be a c-edge-coloured graph, $c \geq 2$, such that every vertex of G is incident with at least two edges of different colours. Then either G has a cut vertex separating colours, or G has an alternating cycle.*

We shall see that Theorem 3.2 actually solves Problem 3.1. Indeed, neither a vertex incident with only edges of the same colour, nor a vertex that separates colours can be on any alternating cycle. Thus all such vertices may be deleted without destroying any alternating cycle.

The following theorem by Whitehead [52] shows that some stronger extension of Theorem 3.2 for $c = 2$ to general c does not hold.

Theorem 3.3 *Let G be a graph with minimum degree at least 2. Then we can colour the edges of G with three colours so that each of the following conditions hold:*

1. *Every vertex is incident with edges in exactly two colour classes.*

2. *There is no alternating cycle with edges of just two colours.*

We give three interesting corollaries of Theorem 3.2 (for the case $c = 2$), all proved in [29].

Corollary 3.4 *Let M be a perfect matching in a graph G. If no edge of M is a cut edge of G, then G has a cycle whose edges are taken alternately from M and $G - M$.*

Corollary 3.5 *Let G be a 2-edge-coloured Eulerian graph so that all monochromatic degrees are odd. Then G has an alternating cycle.*

Corollary 3.6 *Let G be a 2-edge-coloured graph so that both red and blue subgraphs of G are regular and non-trivial. Then G has an alternating cycle.*

Theorem 3.2 also implies:

Corollary 3.7 *[11, 40, 42] There does not exist a bridge-less graph that contains a unique perfect matching.*

The last result was generalized in [39].

It is easy to see that Theorem 3.2 implies that the problem of checking whether a c-edge-coloured graph has an alternating cycle is polynomially solvable. Another possibility to solve Problem 3.1 is to use the following well-known construction. Here, we can actually find a cycle subgraph with maximum number of vertices of a c-edge-colored multigraph in polynomial time. This result seems very useful, as a starting point, for a number of problems on alternating cycles and paths.

Let G be an arbitrary c-edge-colored multigraph (with colours 1,2,...,c). For each vertex v of G we form the following graph H_v: $V(H_v) = \{v_1, ..., v_{2c-2}\}$ and v_i is adjacent to v_j $(i < j)$ iff either both $i, j \in \{1, ..., c\}$ or $i \in$

$\{1, ..., c\}, j \in \{c+1, ..., 2c-2\}$. Construct a new graph R from the disjoint union of the graphs H_v $(v \in V(G))$ as follows. An edge $v_i u_j$ is in R iff $i = j = \chi_G(vu)$. Let the edges of R of the form $v_i v_j$ where both $i, j \in \{1, ..., c\}$ have the weight 0 and all other edges have the weight 1. Then, a maximum weight matching in R corresponds to a cycle subgraph F of G with maximum number of vertices. To see this, it suffices to note that for any perfect matching of R and any H_v (corresponding to one vertex v of G), all but two of the vertices $v_1, v_2, \ldots, v_c$ will be matched to vertices within H_v and with index at least $c+1$. Hence if the edge between the two remaining vertices in H_v is not in the matching, then in G this corresponds to v being on an alternating cycle and vice versa. This construction implies the existence of a polynomial algorithm for finding F since a maximum weight perfect matching in a weighted graph on n vertices can be found in time $O(n^3)$ (cf. [46], Ch. 11).

Sometimes, one needs to find a maximum 1-path-cycle subgraph of a c-edge-coloured multigraph G. We can easily transform the last problem to the maximum cycle subgraph problem as follows. Add an extra-vertex x to G and join x to every vertex of G by two edges of colour $c+1$ and $c+2$ respectively (new colours). Clearly, a maximum cycle subgraph of the new multigraph corresponds to a maximum 1-path-cycle subgraph of G. We formulate the obtained results as a theorem:

Theorem 3.8 *One can construct a maximum cycle subgraph and a maximum 1-path-cycle subgraph, respectively in a c-edge-coloured multigraph G on n vertices in time $O(n^3)$.*

In the rest of this section we consider the notion of colour-connectivity. This is in some sense an analogue of strong connectivity for (bipartite) digraphs. The notion of colour-connectivity was introduced by Saad [48] in the case when $c = 2$ (he used another name, but the definition was the same).

Definition 3.9 *A pair of vertices x, y of a c-edge-coloured multigraph G is called* colour-connected *if there exist alternating (x, y)-paths $P = xx' \ldots y'y$ and $P' = xu' \ldots v'y$ such that $\chi(xx') \neq \chi(xu')$ and $\chi(y'y) \neq \chi(v'y)$. We define a vertex x to be colour-connected to itself. We say that G is* colour-connected *if every pair of vertices of G are colour-connected.*

Clearly, every Hamiltonian edge-coloured multigraph is colour connected. This simple remark shows that colour-connectivity may be an important condition for alternating cycle problems. In Section 5, we shall see that this is indeed so at least when $c = 2$.

The following lemma from [3] shows that we can efficiently check whether two given vertices in a 2-edge-coloured multigraph are colour-connected.

Lemma 3.10 *[3] Let $G = (V, E)$ be a connected 2-edge-coloured multigraph and let x and y be distinct vertices of G. For each choice of $i, j \in \{1, 2\}$ we can find an alternating path $P = x_1x_2 \dots x_k$ with $x_1 = x$, $x_k = y$, $\chi(x_1x_2) = i$ and $\chi(x_{k-1}x_k) = j$ in time $O(|E|)$ (if one exists).*

We can use colour-connectivity more effectively if we know that this is an equivalence relation on the vertices of a given 2-edge-coloured multigraph. This leads us to the following:

Definition 3.11 *A 2-edge-coloured multigraph G is* convenient *if colour-connectivity is an equivalence relation on the vertices of G. If G is convenient, an equivalence class of colour-connectivity is called a* colour-connected component *of G.*

Unfortunately, there are non-convenient multigraphs. Consider the graph G on 5 vertices, $1, 2, 3, 4, 5$, and 6 edges, $13, 23, 45$ of colour 1 and $12, 34, 35$ of colour 2. It is easy to check that the vertices 1 and 2 are colour-connected to 4, but 1 and 2 are not colour-connected in G. On the other hand, all 2-edge-coloured bipartite multigraphs are convenient (see the next section). Thus Lemma 3.10 implies

Corollary 3.12 *[3] Let G be a convenient 2-edge-coloured multigraph $G = (V, E)$. Then we can check whether G is colour-connected in time $O(|V||E|)$ and find the colour-connected components of G in time $O(|V|^2|E|)$.*

4 2-edge-coloured bipartite multigraphs versus digraphs

The aim of this section is to describe some applications of two approaches which allow one to obtain results for bipartite 2-edge-coloured multigraphs using results on directed graphs.

Let D be a bipartite digraph with partite sets V_1, V_2. Define a 2-edge-coloured bipartite multigraph $M = M(D)$ in the following way: M has the same partite sets as D and $x_i x_{3-i}$ is an edge of colour i in M iff (x_i, x_{3-i}) is an arc in D and $x_1 \in V_1, x_2 \in V_2$. Moreover, $M^{-1}(G) = H$ if $M(H) = G$. This trivial correspondence which we shall call the *BB-correspondence* leads us to a number of easy and some more complex results which are described in this and the next sections. One example is the fact that the Hamiltonian alternating cycle problem for bipartite 2-edge-coloured graphs is NP-complete. In many of our results on cycles we shall exploit the following simple observation:

Proposition 4.1 *A bipartite digraph G is strongly connected iff $M(G)$ is colour-connected.*

The following correspondence which we shall call the *BD-correspondence* is less universal but may allow one to exploit the wider area of results on general digraphs (cf. [14]). The idea of the BD-correspondence can be traced back to Häggkvist [36]. Let G be a 2-edge-coloured bipartite multigraph with partite sets V_1 and V_2 so that $|V_1| = |V_2| = m$ and let G' be the red subgraph of G. Suppose that G' has a perfect matching $v_{11}v_{21}, v_{12}v_{22}, ..., v_{1m}v_{2m}$, where $v_{ij} \in V_i$ ($i = 1, 2$ and $1 \leq j \leq m$). Construct a digraph $D = D(G)$ as follows: $V(D) = \{1, 2, ..., m\}$ and, for $1 \leq i \neq j \leq m$, (i, j) is an arc of D iff $v_{1i}v_{2j} \in E(G) - E(G')$. It is easy to see that if D has a Hamiltonian cycle, then G has a Hamiltonian alternating cycle including all the edges of the perfect matching. Using BD-correspondence and Ghouila-Houri's theorem on Hamiltonian cycles in digraphs [28], Hilton [38] proved the following:

Theorem 4.2 *Let G be a 2-edge-coloured regular bipartite graph such that each of the partite sets of G has m vertices and let G'' be the blue subgraph of G. If the degree $d(G)$ of G is at least $\frac{m}{2} + 1$ and G'' is regular of degree at least $\frac{m}{2}$ and at most $d(G) - 1$, then G has a Hamiltonian alternating cycle.*

Although the last theorem is best possible (consider two disjoint copies of $K_{m/2,m/2}$ with perfect matchings in both copies in red and all other edges in blue), Hilton [38] guesses that the bound on $d(G)$ could be lowered considerably if we assume that G is connected. He also claims that the regularity condition could be weakened a great deal.

It was noticed in [18] that Theorem 4.2 follows easily from the following result by Häggkvist [36].

Theorem 4.3 *Let G be any bipartite graph so that each of the partite sets contains m vertices. If $d(v)+d(w) \geq m+1$, for every pair v, w of vertices from different partite sets, then every perfect matching of G lies in a Hamiltonian cycle.*

The following theorem is a straightforward application of the BB-correspondence to Theorem 3 in [2]. In this theorem, we do not indeed require any regularity but pay for that by stronger demands on minimum monochromatic degrees. To state Theorem 4.4, we need to define the following 2-edge-coloured bipartite multigraphs. Let $n_2 \geq 3$ be an integer and let $K_{r,p}$, $r \leq p \leq n_2 - r$ (K_{r-1,n_2-p}, respectively) be a complete bipartite graph with bipartition (X_1, Y_1) ((X_2, Y_2), respectively). Then each of the monochromatic subgraphs of $G_1(n_1, n_2)$ ($n_1 = 2r-1$) consists of the disjoint union of $K_{r,p}$ and K_{r-1,n_2-p}, together with all the edges between exactly one vertex of X_1 and the vertices of Y_2.

$G_2 = G_2(3, n_2)$ has partite sets $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, ..., y_{n_2}\}$ ($n_2 \geq 3$). The red (blue, resp.) subgraph of G_2 has edge set $\{x_i y_i : \ i = 1, 2\} \cup R$ ($\{x_i y_{3-i} : \ i = 1, 2\} \cup R$, resp.), where $R = \{x_3 y_i : \ i = 1, 2\} \cup \{x_j y_i : i = 3, 4, ..., n_2; j = 1, 2\}$.

Theorem 4.4 *Let G be a 2-edge-coloured bipartite multigraph with partite sets V_1 and V_2 so that $n_1 = |V_1| \leq |V_2| = n_2$. If the minimum monochromatic degree r of G is at least $(n_1+1)/2$, then G has an alternating cycle of length $2n_1$ unless:*

1. *$n_2 > n_1 = 2r-1$ and G is isomorphic to $G_1(n_1, n_2)$ or*
2. *$r = 2$, $n_1 = 3$, and G is isomorphic to $G_2(3, n_2)$.*

The BB-correspondence is very useful when we consider 2-edge-coloured complete bipartite multigraphs. In this case we can use the rich theory of semicomplete bipartite digraphs (cf. [9, 33]). In [30, 37, 44], the following characterization of Hamiltonian semicomplete bipartite digraphs was obtained.

Theorem 4.5 *A semicomplete bipartite digraph is Hamiltonian if and only if it is strong and has a spanning cycle subgraph. There is an algorithm for constructing a Hamiltonian cycle in a strong semicomplete bipartite digraph on n vertices in time $O(n^{2.5})$ (if one exists).*

Through the BB-correspondence and Proposition 4.1, this theorem implies

Theorem 4.6 *A 2-edge-coloured complete bipartite multigraph is Hamiltonian iff it is colour-connected and has a spanning cycle subgraph. There is an algorithm for constructing a Hamiltonian alternating cycle in a colour-connected 2-edge-coloured complete bipartite multigraph on n vertices in time $O(n^{2.5})$ (if one exists).*

Another condition for a 2-edge-coloured complete multigraph to contain a Hamiltonian alternating cycle was obtained by Chetwynd and Hilton [18]:

Theorem 4.7 *A 2-edge-coloured complete bipartite graph B with partite sets U and W ($|U| = |W| = n$) is Hamiltonian iff B has a spanning cycle subgraph and, for every $k = 2, ..., n-1$ and every pair of k-sets X and Y such that $X \subset U$, $Y \subset W$, $d_1(X) + d_2(Y) > k^2$ and $d_2(X) + d_1(Y) > k^2$.*

We point out that the proof of Theorem 4.7 is quite similar to that of Theorem 4.5 in [30] (see also [34]).

Using the corresponding result on longest cycles in semicomplete bipartite digraphs proved in [34], one obtains the following:

Theorem 4.8 *The length of the longest alternating cycle in a colour-connected 2-edge-coloured complete bipartite multigraph G is equal to the number of vertices in maximum cycle subgraph of G. There is an algorithm for finding a longest alternating cycle in a colour-connected 2-edge-coloured complete bipartite multigraph on n vertices in time $O(n^3)$.*

Therefore, the longest alternating cycle (longest directed cycle, resp.) problem for 2-edge-coloured complete bipartite multigraphs (semicomplete bipartite digraphs, resp.) is polynomially solvable. This provides a positive answer to a question by Saad [48].

The following result is a characterization of vertex even-pancyclic 2-edge-coloured complete bipartite multigraphs obtained from the corresponding characterization for semicomplete bipartite digraphs [55] (actually, Zhang [55] proved this characterization only for bipartite tournaments but it is valid for semicomplete bipartite digraphs as well, cf. [35].) Let r be a positive integer. The 2-edge-coloured complete bipartite graph $B(r)$ has partite sets $\{x_1, ..., x_{2r}\}$ and $\{y_1, ..., y_{2r}\}$. The set of red edges of $B(r)$ is $\{x_iy_j : \ 1 \leq i \leq r, \ 1 \leq j \leq r\} \cup \{x_iy_j : \ r+1 \leq i \leq 2r, \ r+1 \leq j \leq 2r\}$.

Theorem 4.9 *A 2-edge-coloured complete bipartite multigraph is vertex even-pancyclic iff it is Hamiltonian and not isomorphic to one of the graphs* $B(r)$ $(r = 2, 3, \ldots)$.

Corollary 4.10 *A 2-edge-coloured complete bipartite multigraph is even-pancyclic if and only if it is Hamiltonian and not isomorphic to one of the graphs* $B(r)$ $(r = 2, 3, \ldots)$.

The last result was obtained by Das [20]. The equivalent (via the BB-correspondence) claim was proved by Beineke and Little [10] for bipartite tournaments. (Both results were published in the same year!)

To save the space we shall not give any other 'BB-translations' of results obtained for cycles and paths in semicomplete bipartite digraphs (cf. [33]) into the alternating cycles and paths language. A less trivial application of the BB-correspondence is provided in the proof of Theorem 5.10.

5 2-edge-coloured complete multigraphs

In 1968, solving a problem by Erdős, M. Bankfalvi and Z. Bankfalvi [6] gave the following characterization of Hamiltonian 2-edge-coloured complete graphs.

Theorem 5.1 *A 2-edge-coloured complete graph G of order $2n$ is Hamiltonian iff it has a spanning cycle subgraph and, for every $k = 2, ..., n-1$ and every pair of disjoint k-sets X and Y, $d_1(X) + d_2(Y) > k^2$.*

Our formulation is somewhat different from the one in [6], but it is obviously equivalent and simpler to state. Saad [48] proved the following more general result, using the notion of colour-connectivity rather than a degree condition.

Theorem 5.2 *The length of a longest alternating cycle in a colour-connected 2-edge-coloured complete graph G is equal to the number of vertices in a maximum cycle subgraph of G.*

Corollary 5.3 *A 2-edge-coloured complete multigraph G is Hamiltonian if and only if G is colour-connected and contains a spanning cycle subgraph.*

Manoussakis (cf. [48]) posed the following problem: is there a polynomial algorithm for finding a longest alternating cycle in a 2-edge-coloured complete graph. Using Theorem 5.2, Saad [48] proved the existence of a randomized polynomial algorithm for Manoussakis' problem. An affirmative solution of Manoussakis' problem was recently obtained in [3]. Below, in this section, we provide a short schema of the solution of a more general problem treated in [3].

Das [20] and later Häggkvist and Manoussakis [37] observed that the Hamiltonian cycle problem for 2-edge-coloured complete bipartite multigraphs (or semicomplete bipartite digraphs) can be reduced to that for 2-edge-coloured complete multigraphs using the following simple construction. Consider a 2-edge-coloured complete bipartite multigraph L with bipartition (X, Y). Add to L the edges $\{x'x'', y'y'' : \ x', x'' \in X, \ y', y'' \in Y\}$ and set $\chi(XX) = 1, \ \chi(YY) = 2$. Let K be the 2-edge-coloured complete multigraph obtained in this way. It is not difficult to verify that K has no alternating cycle containing any of the edges from $XX \cup YY$. Hence, K is Hamiltonian iff L is Hamiltonian. Moreover, it is easy to check that K is colour-connected iff L is colour-connected. In the sequel, we shall call the construction above the *DHM-construction*.

The DHM-construction shows that (the non-algorithmic part of) Theorem 4.6 for graphs follows immediately from Corollary 5.3. On the other hand, Corollary 5.3 can be derived from a key lemma to prove Theorem 4.6 (Lemma 4.1 in [3]) using mostly the same ideas as in the proof of Theorem 4.6 in [30, 34]. This means that Theorem 4.6 and Corollary 5.3 are in some sense equivalent. The same can be said about the non-algorithmic part of Theorem 4.8 and Theorem 5.2.

As another indication of the usefulness of the DHM-construction, consider the following simple derivation of Theorem 4.7 from Theorem 5.1: Let $B = (U, V, E)$ be a 2-edge-coloured complete bipartite graph with partite sets U, V, each of size n. Suppose B has a spanning cycle subgraph. Because the number of edges between an arbitrary k-set $X \subset U$ and an arbitrary k-set $Y \subset V$ is equal to k^2, it follows that the condition $d_1(X) + d_2(Y) > k^2$ and $d_2(X)+d_1(Y) > k^2$ is necessary for the existence of a Hamiltonian alternating cycle in B.

To see the sufficiency we proceed as follows. Suppose

(*) for every k-set $X' \subset U$ and every k-set $Y' \subset V$ we have $d_{1,B}(X') + d_{2,B}(Y') > k^2$ and $d_{2,B}(X') + d_{1,B}(Y') > k^2$.

Construct a 2-edge-coloured complete graph G on $2n$ vertices from B by adding all edges inside U and colouring them i and adding all edges inside V and colouring them $3-i$, where $i=1$, or $i=2$. The actual choice of i is not important, but the proof is simpler if we do not specify the value of i. If G satisfies that for every $k \in \{2,3,\dots,n-1\}$ and every choice of disjoint k-sets X' and Y', $d_{1,G}(X') + d_{2,G}(Y') > k^2$, then, by Theorem 5.1 and the DHM-construction, B is Hamiltonian. Hence we may assume that there exists a $k \in \{2,3,\dots,n-1\}$ and disjoint k-sets X and Y, such that $d_{1,G}(X) + d_{2,G}(Y) = k^2$. Note that this means that there are no edges of colour 1 from X to $(U \cup V) - Y$ and no edges of colour 2 from Y to $(U \cup V) - X$. From this and the fact that all edges inside U and V respectively are monochromatic, it follows that if each of the sets $X \cap U$ $(X \cap V)$and $Y \cap U$ $(Y \cap V)$ are non-empty, then $U \subset X \cup Y$ $(V \subset X \cup Y)$. Thus, because each of X and Y have size at most $n-1$, we may assume without loss of generality that $X \subset U$ (here we used that i is not specified).

Suppose first that $Y \cap U \neq \emptyset$. Then, by the remark above, $U \subset X \cup Y$. It is easy to see that $|V - Y| > |Y \cap U|$ and since in B the set $V - Y$ only has edges of colour 1 to vertices in $Y \cap U$ (all edges of colour 1 from X go to Y), it follows from Hall's matching theorem that B has no spanning cycle subgraph, contradicting our assumption above. Thus we may assume that $Y \subset V$, but then the choice of k-sets $X' = X$ and $Y' = Y$ contradicts (*). $\square$.

Let f be a mapping from the vertex set of a 2-edge-coloured complete multigraph G into the set of all *positive* integers. A subgraph H of G is called an *$f_=$-subgraph (an $f_\geq$-subgraph*, resp.) of G if $d_{1,H}(x) = d_{2,H}(x) = f(x)$ ($d_{1,H}(x) = d_{2,H}(x) \leq f(x)$, resp.) for every vertex x in G. A connected $f_\geq$-subgraph H of G is called *maximum* if H has the maximum number of edges among all connected $f_\geq$-subgraphs of G.

Note that if $f(x) = 1$ for every $x \in V(G)$, then the problems of finding a connected $f_=$-subgraph and a maximum connected $f_\geq$-subgraph of G coincide with the Hamiltonian alternating cycle problem and longest alternating cycle problem, respectively. In general, every connected $f_\geq$-subgraph (or $f_=$-subgraph) of G can be viewed as a closed alternating trail in G (cf. Theorem 7.1).

Das and Rao [21] gave the following characterization of 2-edge-coloured complete graphs which contain connected $f_=$-subgraphs.

Theorem 5.4 *Let G be a 2-edge-coloured complete graph of order $2n$. G contains a connected $f_=$-subgraph iff G has a $f_=$-subgraph and, for each pair k_1, k_2 of positive integers with $k_1+k_2 \leq n-4$ and every disjoint pair consisting of a k_1-set X and a k_2-set Y, $d_1(X) + d_2(Y) > k_1k_2$.*

Consider the more general problem:

Problem 5.5 *Find a maximum connected $f_{\geq}$-subgraph in a 2-edge-coloured complete multigraph G on n vertices.*

Because $d_{1,H}(x) = d_{2,H}(x)$ in any $f_{\geq}$-subgraph H of G, we may assume that each $f(x) \leq |E(G)|/2$, since every value of $f(x) > |E(G)|/2$ can be replaced by $|E(G)|/2$ without changing the solution of the problem.

To describe a scheme (from [3]) of solving this problem in polynomial time, we need some more definitions.

Let H be a 2-edge-coloured multigraph with vertices $v_1, ..., v_k$ ($k \geq 2$). A multigraph L is called an *extension* of H if the vertex set of L can be partitioned into non-trivial subsets $V_1, ..., V_k$ (called *partite sets*) so that, for every pair i, j ($1 \leq i < j \leq k$) and every pair $x \in V_i$, $y \in V_j$, the number of edges between x and y coincides with the number of edges between v_i and v_j (in H), and if H has only one edge between v_i and v_j, then $\chi_L(V_iV_j) = \chi_H(v_iv_j)$. (If H has two edges between v_i and v_j (of different colours, of course; see the definition of edge-coloured multigraphs), then there are two edges between every $x \in V_i$ and $y \in V_j$.) An *extended 2-edge-coloured complete multigraph* is an extension of a 2-edge-coloured complete multigraph. We denote the set of all extended 2-edge-coloured complete multigraphs by $\mathcal{ECM}$.

It is easy to check that Problem 5.5 is equivalent to the problem of finding a longest alternating cycle in the extension of G with n partite sets $\{V_x : x \in V(G)\}$ of sizes $|V_x| = f(x)$. Therefore, a solution of Problem 5.5 can be obtained from a solution of the following problem:

Problem 5.6 *Find a longest alternating cycle in an extended 2-edge-coloured complete multigraph.*

Since each $f(x) \leq |E(G)|/2$, a polynomial algorithm for solving Problem 5.6 can be converted into a polynomial algorithm for Problem 5.5. It is easier for us to deal with Problem 5.6 than Problem 5.5. Hence, in the sequel, we shall consider only Problem 5.6.

Obviously, each alternating cycle of a convenient 2-edge-coloured multigraph G is contained in a colour-connected component of G (recall Definition 3.11). Hence, we may restrict our attention only to colour-connected multigraphs $G \in \mathcal{ECM}$ because of Theorem 3.12 and the following:

Theorem 5.7 *[3] Every multigraph $G \in \mathcal{ECM}$ is convenient.*

The main result of [3] is the next theorem.

Theorem 5.8 *The length of a longest alternating cycle in a colour-connected extended 2-edge-coloured complete multigraph G is equal to the number of vertices in a maximum cycle subgraph of G. Given a maximum cycle subgraph of a colour-connected $G \in \mathcal{ECM}$, a longest alternating cycle in G can be constructed in in time $O(n^3)$, where n is the number of vertices in G.*

By Theorem 3.8, Theorem 5.8 implies the next result which actually gives a solution to Problem 5.6.

Theorem 5.9 *A longest alternating cycle in a colour-connected multigraph $G \in \mathcal{ECM}$ with n vertices can be constructed in time $O(n^3)$.*

Our next topic is even-pancyclicity. Consider the following Hamiltonian 2-edge-coloured complete graphs which are not even-pancyclic (see the proof of this fact below). Let $r \geq 2$ be an integer. Each of the graphs $H(r),\ H'(r),\ H''(r)$ has a vertex set $A \cup B \cup C \cup D$ so that the sets A, B, C, D are mutually disjoint and each of these sets contains r vertices. Moreover, the edge set of the red subgraph of $H(r)$ consists of $AA \cup CC \cup AC \cup AD \cup CB$. The edge set of the red (blue, resp.) subgraph of $H'(r)$ ($H''(r)$, resp.) consists of $AC \cup CB \cup BD \cup DA$. Note that each of $H(r),\ H'(r),\ H''(r)$ contains $B(r)$ (see the definition just before Theorem 4.9).

By the DHM-construction, the following result is a generalization of Theorem 4.9.

Theorem 5.10 *A 2-edge-coloured complete multigraph G is vertex even-pancyclic if and only if G is Hamiltonian and not isomorphic to the graphs $H(r),\ H'(r),\ H''(r)$ for $r = 2, 3, ...$*

Since the graphs $H(r),\ H'(r),\ H''(r)$ are not even-pancyclic for $r = 2, 3, ...$, we obtain the following characterization proved in [20].

Corollary 5.11 *A 2-edge-coloured complete multigraph G is even-pancyclic if and only if G is Hamiltonian and not isomorphic to the graphs $H(r),\ H'(r),\ H''(r)$ for $r = 2, 3, ...$*

We prove Theorem 5.10 below. This proof is completely different from the proof of Corollary 5.11 in [20]. Moreover, our proof is shorter and somewhat simpler.

Proof of Theorem 5.10: We first show that none of $H(r), H'(r), H''(r)$ ($r = 2, 3, ...$) are even-pancyclic. By the DHM-construction, $H(r)$ is not even-pancyclic since $B(r)$ is not either (see Theorem 4.10). We prove that each of $H'(r), H''(r)$ ($r = 2, 3, ...$) does not contain a cycle subgraph with $4r-2$ vertices. Clearly, we only need to prove this fact for $H'(r)$ ($r = 2, 3, ...$). Assume that some $H'(r)$ ($r \geq 2$) has a cycle subgraph F with $4r-2$ vertices. Let L be the subgraph of $H'(r)$ induced by $V(F)$. Since the red subgraph of L is complete bipartite, it contains a perfect matching iff the partite sets of L have the same size. In such a case, the blue subgraph of L has no perfect matching since L consists of a disjoint union of two complete graphs such that each of them has odd number of vertices; a contradiction.

The rest of the proof is based on Theorem 4.9. Let G be a Hamiltonian 2-edge-coloured complete multigraph that is not isomorphic to the graphs $H(r), H'(r), H''(r)$ ($r = 2, 3, ...$). We show that G is vertex even-pancyclic. Let $W = x_1x_2...x_{4r-2j}x_1$ (where $j = 0$ or 1) be a Hamiltonian alternating cycle of G. Then, W is a Hamiltonian alternating cycle in the spanning complete bipartite subgraph Q of G with partite sets $A \cup C$ and $B \cup D$, where A (B, C, D, resp.) consists of all vertices of W having subscripts which are congruent to 1 (2,3,4, resp.) modulo 4. If Q is not isomorphic to $B(r)$, then we are done by Theorem 4.9. Therefore, we may assume that $|A| = |B| = |C| = |D| = r$ and $\chi(AB) = \chi(CD) = 2,\ \chi(AD) = \chi(BC) = 1$. Let a, b, c, d be arbitrary vertices from A, B, C and D, respectively. Observe that $B(r)$ contains alternating cycles of lengths $4, 8, 12, ..., 4r$ through any vertex. Therefore, it is sufficient to prove that, for every $k = 2, 3, ..., r$, G contains an alternating cycle Z_{4k-2} of length $4k-2$ through $\{a, b, c, d\}$.

Suppose that BD has edges of both colours such that b and d are joined by an edge e of colour i. Since $|B| = |D| = r \geq 2$, this implies that there exists an edge $e' = b'd'$ $(b' \in B,\ d' \in D)$ of colour $3-i$ so that either all the vertices of e and e' are distinct or $b \neq b'$, $d = d'$ or $b = b'$, $d \neq d'$ or $b = b'$ and $d = d'$. Because of symmetry between B and D, it is sufficient to consider the first two cases and, moreover, we may assume that $i = 1$.

In the first case (e and e' are disjoint), $Z_{4k-2} = bdcb'd'P$, where P is an alternating path of length $4k-7$ starting in a, finishing in b and visiting the sets A, B, C, D in the order $A, B, C, D, A, ...$ In the second case, $Z_{4k-2} = bdb'cd''P$, where $d'' \in D - d$ and P is an alternating path of length $4k-7$ starting in a, finishing in b and visiting the sets A, B, C, D in the order $A, B, C, D, A, ...$

We conclude that if at least one the sets BD and AC has edges of both colours, then G is vertex even-pancyclic. Therefore, we may assume that the sets BD and AC are monochromatic, and either $\chi(BD) = \chi(AC)$ or $\chi(BD) \neq \chi(AC)$.

Case 1: $\chi(BD) = \chi(AC) = i$. We may suppose, w.l.o.g, that $i = 1$. Since G is not isomorphic to $H'(r)$, at least one of the sets AA, BB, CC, DD has an edge of colour 1. Let, w.l.o.g, AA contains an edge of colour 1. We consider the following two subcases: 1) there is an edge aa', $a' \in A$, of colour 1, 2) there is an alternating path aa_1a_2 in AA so that $\chi(aa_1) = 2$. In the first subcase, $Z_{4k-2} = aa'bcdP$, where P is an alternating path of length $4k-7$ starting in a vertex of $B - b$, finishing in a and visiting the sets A, B, C, D in the order $B, A, C, D, B, ...$ The second subcase can be treated analogously to the first one, but now we start Z_{4k-2} with a_1a_2 instead of aa' and pick up a later. This approach is not valid only in the case $k = 2$. In this case, aa_1a_2bcda is the desired alternating cycle.

Case 2: $\chi(BD) \neq \chi(AC)$. We may suppose, w.l.o.g, that $\chi(BD) = 2, \chi(AC) = 1$. Since G is not isomorphic to $H(r)$, the following is not valid: $\chi(AA) = \chi(CC) = 1,\ \chi(BB) = \chi(DD) = 2$. We may assume, w.l.o.g, that AA has an edge of colour 2. Now we have two subcases analogous to those in Case 1. These two subcases can be treated similarly to the subcases of Case 1. $\square$.

The longest path problem for 2-edge-coloured complete multigraphs is much simpler than the longest cycle problem.

Theorem 5.12 *Let G be a 2-edge-coloured complete multigraph with n vertices. Then*

1. *for any 1-path-cycle subgraph F of G there is an alternating path P of G satisfying $V(P) = V(F)$ (if F is a maximum 1-path-cycle subgraph of G, then P is a longest alternating path in G);*

2. *there exists an $O(n^3)$ algorithm for finding a longest alternating path in G.*

Proof: Obviously, F is contained in a complete bipartite subgraph B of G. F corresponds to a collection of a directed path and directed cycles, all vertex disjoint, F' of $M^{-1}(B)$ (see the definition of the BB-correspondence in the beginning of Section 4) so that $V(F) = V(F')$. Therefore, by the main theorem of [32], there is a path P' in $M^{-1}(B)$ such that $V(P') = V(F')$. This path corresponds to an alternating path P of B so that $V(P') = V(P)$. Clearly, P is an alternating path in G and, moreover, $V(P) = V(F)$.

The complexity result easily follows from the construction above, the proof of the main theorem in [32] and Theorem 3.8. $\square$.

Corollary 5.13 *A 2-edge-coloured complete multigraph has a Hamiltonian alternating path if and only if it contains a spanning 1-path-cycle subgraph.*

Note that the last characterization is simpler and more intuitive than the corresponding one in [12]. Recently, Yeo (private communication) constructed an infinite family of examples showing that Corollary 5.13 cannot be extended to all 2-edge-coloured complete multipartite graphs.

We conclude this section with the following interesting problem from [12].

Problem 5.14 *Let x and y be two specified vertices in a 2-edge-coloured complete graph. Is there a polynomial algorithm for finding a Hamiltonian alternating (x, y)-path (if one exists)?*

An affirmative answer to the same question for 2-edge-coloured complete bipartite graphs follows from the corresponding result by Bang-Jensen and Manoussakis [5] for bipartite tournaments. In [5] a complete characterization is given of those bipartite tournaments that do not have a Hamiltonian path with endvertices x and y. Since this characterization is rather complicated, we shall not state the translation of it (via the BB-correspondence) to 2-edge-coloured complete bipartite graphs.

6 c-edge-coloured ($c \geq 3$) complete and complete bipartite graphs

Let K_n^c denote a c-edge-coloured complete graph with n vertices.

The Hamiltonian cycle problem for c-edge-coloured complete graphs seems to be much more difficult in the case when $c \geq 3$, than in the case $c = 2$ treated above.

Problem 6.1 *[12] Determine the complexity of the Hamiltonian alternating cycle problem for c-edge-coloured complete graphs when $c \geq 3$.*

We raise the following analogous problem for Hamiltonian alternating paths.

Problem 6.2 *Determine the complexity of the Hamiltonian alternating path problem for c-edge-coloured complete graphs when $c \geq 3$.*

There is a polynomial time algorithm for the Hamiltonian alternating path problem above if the following generalization of Corollary 5.13 is true.

Conjecture 6.3 *A K_n^c ($c \geq 2$) has a Hamiltonian alternating path if and only if K_n^c contains a spanning 1-path-cycle subgraph.*

We state a weaker result proved in [4].

Theorem 6.4 *If a K_n^c ($c \geq 2$) contains a spanning cycle subgraph, then it has a Hamiltonian alternating path.*

We conclude our discussion of Hamiltonian alternating paths by the following simple but perhaps surprising result of Barr [7].

Theorem 6.5 *Every K_n^c without monochromatic triangles has a Hamiltonian alternating path.*

Another interesting problem is to find a non-trivial characterization of Hamiltonian c-edge-coloured ($c \geq 3$) complete graphs. In the rest of this section, we consider results from [12] related to Problem 6.1, we give an example showing that the obvious analogue of Corollary 5.3 is not valid for $c \geq 3$. Later we present some conditions which guarantee the existence of a Hamiltonian alternating cycle in a c-edge-coloured complete graph. We also describe results on disjoint alternating paths with fixed end vertices in c-edge-coloured complete graphs as well as some related problems.

A *strictly alternating cycle* in K_n^c is a cycle of length pc (p is an integer) so that the sequence of colours $(12...c)$ is repeated p times.

Theorem 6.6 *[12] Let $c \geq 3$. The problem of determining the existence of a Hamiltonian strictly alternating cycle in K_n^c is NP-complete.*

The following result shows that if we relax the strict places of colours, but maintain the number of their appearances in a Hamiltonian cycle, then we still have an NP-complete problem.

Theorem 6.7 *[12] Given positive integers p and $c \geq 3$, the problem of determining the existence of a Hamiltonian alternating cycle C of K_{cp}^c so that each colour appears p times in C is NP-complete.*

The following example shows that the obvious analog of Corollary 5.3 is not valid for $c \geq 3$. G_6 is 3-edge-coloured complete graph on vertices 1,2,3,4,5,6. All the edges of G_6 has colour 1 except of the following: the triangles 2342 and 2562 have colours 2 and 3, resp., $\chi(36) = \chi(45) = 2$, $\chi(12) = 3$. It is easy to check that G_6 is colour-connected and has the spanning cycle subgraph $1231 \cup 4564$, but G_6 is not Hamiltonian. Note that the paths that show that G_6 is colour-connected may be chosen so that for each choice of vertices x and y the two paths P and P' described in Definition 3.9 are internally disjoint. Hence it will not enough to change Definition 3.9 to require that P and P' are disjoint, a condition which is obviously necessary for the existence of a Hamiltonian alternating cycle. Using the definition of G_6, given an even number n, one can easily construct a 3-edge-coloured complete graph on $n \geq 6$ vertices which is colour-connected and has a spanning cycle subgraph, but has no Hamiltonian alternating cycle.

An edge-coloured graph G on n vertices is *pancyclic* if G has an alternating cycle of length i, for every $i = 3, 4, ..., n$.

In [22], Daykin posed the following interesting problem: Find a positive constant d such that every K_n^c with $\Delta(K_n^c) \leq dn$ is Hamiltonian.

This problem was independently solved by Bollobás and Erdös [15], and Chen and Daykin [17]. In [15] (in [17]), it was proved that if $69\Delta(K_n^c) < n$ ($17\Delta(K_n^c) \leq n$, resp.), then K_n^c is pancyclic. Shearer improved the last result showing that if $7\Delta(K_n^c) < n$, then K_n^c is pancyclic. So far, the best asymptotic estimate was obtained by Alon and the second author [1].

Theorem 6.8 *[1] For every $\epsilon > 0$ there exists an $n_0 = n_0(\epsilon)$ so that for every $n > n_0$, every K_n^c satisfying*

$$\Delta(K_n^c) \leq (1 - \frac{1}{\sqrt{2}} - \epsilon)n \quad (\quad = (0.2928... - \epsilon)n \quad)$$

is pancyclic.

However, Theorem 6.8 seems to be far from the best possible, at least, if the following is true.

Conjecture 6.9 *[15] Every K_n^c with $\Delta(K_n^c) \leq \lfloor n/2 \rfloor - 1$ is Hamiltonian.*

Clearly, for every $n = 4k+1$, there exists a K_n^2 so that both its monochromatic graphs is regular of degree $2k$. Since such a K_n^2 is not Hamiltonian (n is odd), Conjecture 6.9 is sharp. The results above indicate that the graphs satisfying the conditions of Conjecture 6.9 may be pancyclic.

Chen and Daykin [17] proved that if $K_{m,m}^c$ is an edge-coloured complete bipartite graph with partite sets of cardinality m each and $\Delta(K_{m,m}^c) \leq m/25$ then $K_{m,m}^c$ contains a Hamiltonian alternating cycle. This result was improved in [1]:

Theorem 6.10 *For every $\epsilon > 0$ there exists an $m_0 = m_0(\epsilon)$ so that for every $m > m_0$, every $K_{m,m}^c$ satisfying*

$$\Delta(K_{m,m}^c) \leq (1 - \frac{1}{\sqrt{2}} - \epsilon)m \quad (\quad = (0.2928... - \epsilon)m \quad)$$

contains an alternating cycle of every even length between 4 and $2m$.

The estimate of the last theorem again seems to be far from best possible. Therefore, we pose the following:

Problem 6.11 *Determine the maximum constants t_1 and t_2 such that $K^c_{m,m}$ satisfying $\Delta(K^c_{m,m}) \leq t_1$ ($\Delta(K^c_{m,m}) \leq t_2$, resp.) is Hamiltonian (even-pancyclic, resp.)*

The following problem was considered in [13]:

Problem 6.12 *Is there a polynomial algorithm for finding an alternating cycle through p fixed vertices $x_1, ..., x_p$ in a c-edge-coloured complete graph?*

The same problem for general graphs is proved to be NP-complete [13] (note that this is a trivial consequence of the BB-correspondence, as described in Section 4). The authors of [13] gave the positive answer to Problem 6.12 for two cases: $c \geq 2,\ p = 1$ and $c = 2,\ p = 2$. In the first case, the solution follows from the next non-difficult result:

Theorem 6.13 *Let x be a vertex in a c-edge-coloured complete graph G. There exists an alternating cycle through x in G if and only if there exists an alternating cycle of length three or four containing x.*

For the case $c = 2$, $p = 2$, Benkouar, Manoussakis and Saad [13] proved the following:

Theorem 6.14 *There is an $O(n^2)$-algorithm for finding an alternating cycle (if one exists) through two given vertices in a 2-edge-coloured complete graph with n vertices.*

By the DHM-construction, we obtain:

Corollary 6.15 *[13] There is an $O(n^2)$-algorithm for finding a cycle (if one exists) through two given vertices in a bipartite tournament with n vertices.*

The last corollary improves the $O(n^3)$-algorithm of Manoussakis and Tuza [44] for finding a cycle through two given vertices in a bipartite tournament.

The authors of [19] investigated the complexity of the following problem related to Problem 6.12:

Problem 6.16 *[19] Given a pair of distinct vertices x and y in a c-edge-coloured complete graph G, find an alternating (x, y)-path through p fixed vertices $x_1, ..., x_p$ in G (if one exists).*

In [19], it was shown that Problem 6.16 is polynomially solvable for $p = 1$ and $c = 2$. The complexity of Problem 6.16 for $p = 1$ and $c \geq 3$ is still open.

Manoussakis [43] proved the existence of a polynomial algorithm for finding the maximum number of internally pairwise vertex-disjoint alternating (x, y)-paths in a c-edge-coloured complete graph K_n^c with distinct vertices x and y. He also posed:

Problem 6.17 *Is there a polynomial algorithm for finding the maximum number of internally pairwise edge-disjoint alternating (x, y)-paths in K_n^c?*

Manoussakis [43] also considered the so-called linkage problems: for a given positive integer p and given $2p$ distinct vertices $x_1, y_1, x_2, y_2, ..., x_p, y_p$ in K_n^c, construct p pairwise vertex-disjoint (edge-disjoint, resp.) alternating (x_i, y_i)-paths, $i = 1, 2, ..., p$, in K_n^c. He proved that there exists an $O(n^p)$-algorithm for the "vertex-disjoint" version and an $O((n^7/\log n)^{1/2})$-algorithm for the "edge-disjoint" version of this problem.

7 Miscellaneous results

In this section, we describe some results on the Eulerian trail problem for edge-coloured graphs as well as results on a special family of 2-edge-coloured multigraphs that appear in genetics [24].

In [41], Kotzig proved the following characterization of edge-coloured graphs which contain alternating closed Eulerian trails (see also [27], Theorem VI-1).

Theorem 7.1 *An edge-coloured graph G has a closed alternating Eulerian trail if and only if G is connected, and for each vertex x and each colour i, $d(x)$ is even and $d_i(x) \leq \sum_{j \neq i} d_j(x)$.*

Obviously, the conditions above are necessary. To see their sufficiency, consider an outline of the proof of Theorem 7.1 that is due to the authors of [12]:

Suppose G satisfies the conditions of Theorem 7.1. We shall show that, for every vertex x, the edges of G incident to x can be partitioned into disjoint pairs of distinct edges so that the colours of the edges in each pair are different. This guarantees that each time we visit x through an edge e we can leave it through the edge f forming one of the above pairs with e. In order to determine this partition, for each vertex x we define a new graph G_x so that the vertices of G_x are the edges incident to x. Two vertices are connected in G_x if their corresponding edges in G have different colours. It is easy to see that the above partition exists iff each G_x has a perfect matching. Since each G_x is a complete multipartite graph, it is not difficult to prove, using Tutte's 1-factor theorem, that each G_x indeed has a perfect matching. Such a matching can be found in time $O(n \log n)$ [12]. The ideas above lead to an $O(n^2 \log n)$-algorithm for finding an alternating Eulerian trail in an edge-coloured graph G on n vertices that satisfies the conditions of Theorem 7.1.

In [23, 24], Dorninger considers the following family of 2-edge-coloured multigraphs that appear in genetics. Let π be a permutation of the set $N = \{1, 2, ..., 2n\}$ and k be a positive integer. The 2-edge-coloured multigraph $G(\pi, k)$ has $V(G(\pi, k)) = N$, the red edges $\{ij : \ |i - j| \leq k\}$ and the blue edges $\{ij : \ |\pi(i) - \pi(j)| \leq k\}$. Dorninger [24] proved the following:

Theorem 7.2 *For $k = 3$ and every permutation π of N, the multigraph $G(\pi, k)$ is Hamiltonian.*

It was shown [24] that the analogous result does not hold for $k = 2$ and an algorithm for checking Hamiltonicity in this case was provided.

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