Uniform Inductive Reasoning in Transitive Closure Logic via Infinite Descent

Liron Cohen [1] Reuben N. S. Rowe [2]

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Transitive Closure (**TC**) Logic extends FOL with formulas:

- $(RTC_{x,y}\,\varphi)(s,t)$
  - φ is a formula
  - x and y are distinct variables (which become bound in φ)
  - s and t are terms

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  - φ is a formula
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  - s and t are terms

whose intended meaning is an infinite disjunction

$$
\begin{aligned}
s = t \lor \varphi[s/x, t/y] \\
&\lor (\exists w_1 \,.\, \varphi[s/x, w_1/y] \land \varphi[w_1/x, t/y]) \\
&\lor (\exists w_1, w_2 \,.\, \varphi[s/x, w_1/y] \land \varphi[w_1/x, w_2/y] \land \varphi[w_2/x, t/y]) \\
&\lor \;\ldots
\end{aligned}
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The formal semantics:

- M is a (standard) first-order model with domain D
- v is a valuation of terms in M:

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&\exists a_0, \ldots, a_n \in D \,.\, v(s) = a_0 \wedge v(t) = a_n \\
&\qquad \wedge M, v[x := a_i, y := a_{i+1}] \models \varphi \quad \text{for all } i < n
\end{aligned}
$$

$$
\begin{tikzcd}
\boxed{v(s)} \arrow[r, "\varphi"] & a_1 \arrow[r, "\varphi"] & a_2 \arrow[r, dashed, "\varphi"] & a_{n-1} \arrow[r, "\varphi"] & \boxed{v(t)}
\end{tikzcd}
$$

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- Consider the binary relation induced by φ (wrt. x and y):

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- $(RTC_{x,y}\,\varphi)$ ‘denotes’ the reflexive, transitive closure of φ:

$$M, v \models (RTC_{x,y}\,\varphi)(s,t) \Leftrightarrow (v(s), v(t)) \in ([\![\varphi(x,y)]\!]_{M,v})^*$$

Why Transitive Closure logic?

- It is a minimal extension of FOL
- It has an intuitive, easy-to-understand semantics
  - e.g. $\mathsf{Ancestor}(x, y) \equiv (RTC_{v,w}\, \mathsf{Parent}(v, w))(x, y)$ ... and $x \neq y$!
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Theorem (Avron '03)

All finitely inductively defined relations are definable in* TC.†

A. Avron, *Transitive Closure and the Mechanization of Mathematics*, 2003.

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*as defined in: S. Feferman, *Finitary Inductively Presented Logics*, 1989
†with signatures containing pairing/all $2n$-ary closure operators

Example: Arithmetic

- Take a signature $\Sigma = \{\mathsf{0}, \mathsf{s}\} + \text{equality}$

$$\mathsf{Nat}(x) \equiv (RTC_{v,w}\, \mathsf{s}\, v = w)(\mathsf{0}, x)$$

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$$\text{“}x = y + z\text{”} \equiv (RTC^2_{v_1,v_2,v_3,v_4}\, v_3 = \mathsf{s}\, v_1 \wedge v_4 = \mathsf{s}\, v_2)(\mathsf{0}, y, z, x)$$

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$$\boxed{(\mathbf{0}, y)}$$

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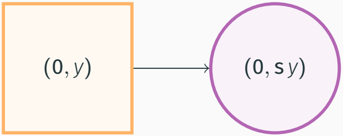

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$$\boxed{(\mathbf{0}, y)} \longrightarrow (\mathbf{0}, \mathsf{s}\, y) \longrightarrow (\mathsf{s}\,\mathsf{s}\, 0, \mathsf{s}\,\mathsf{s}\, y)$$

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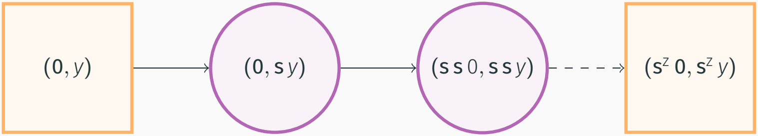

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- Take a signature $\Sigma = \{\mathbf{0}, \mathbf{s}\}$ + equality and pairing

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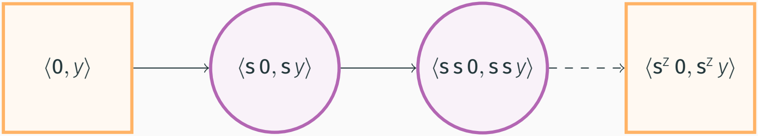

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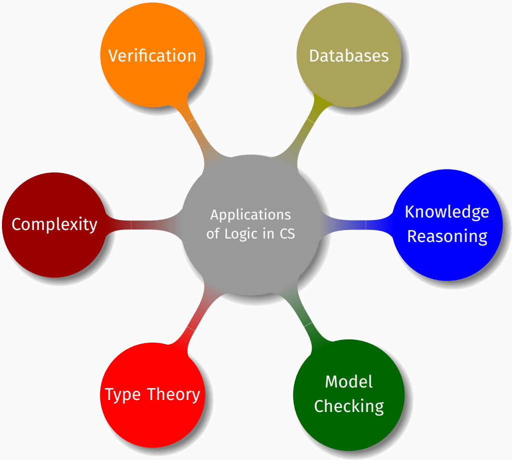
- The following axioms categorically characterise the natural numbers in TC:

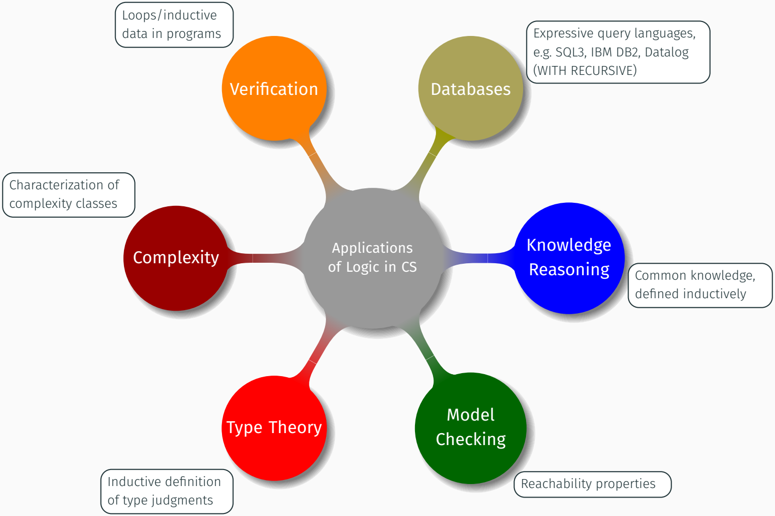
$$\forall x \,.\, \mathsf{s}\, x \neq \mathsf{0}$$

$$\forall x, y \,.\, \mathsf{s}\,(x) = \mathsf{s}\,(y) \rightarrow x = y$$

$$\forall x \,.\, \mathsf{Nat}(x)$$



J. Halpern Et Al, *On the Unusual Effectiveness of Logic in Computer Science*, 2001



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"Everything should be made as simple as possible but not simpler"

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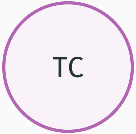

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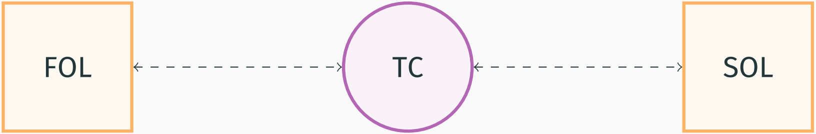

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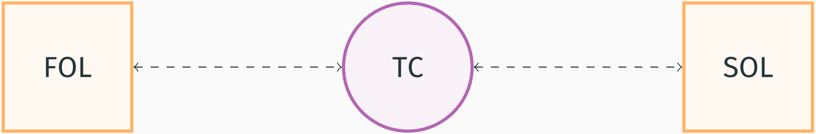
Weak **SOL**
ω-logic
Cardinality logic
FOL + Henkin Quantifiers



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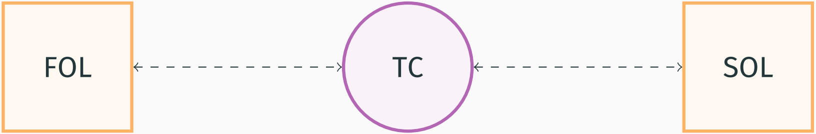

FOL +
Martin-Löf
Ind. Defs

First-order
Modal
μ-calculus

Modal μ-calculus has general fixed points, but (non-reflexive) transitive closure

$$R^+ = \bigcup_{i \geq 0} R_i, \quad \text{where} \quad \begin{aligned} R_0 &= R \\ R_{i+1} &= R_i \circ R \quad (i \geq 0) \end{aligned}$$

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is a particular kind of fixed point:

$$R^+ = \mu X . \Psi_R(X)$$

where, for binary relations R and S, we define

$$\Psi_R(S) = R \cup (R \circ S)$$

In **FOL** + Martin-Löf inductive definitions:

- We pick a (finite) set of 'inductive' predicates $P_1, \ldots, P_n$
- For each predicate we give a set of productions:

$$\frac{Q_1(\vec{s_1}) \quad \ldots \quad Q_n(\vec{s_n})}{P_i(\vec{t})}$$

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- We take the smallest interpretation closed under the rules
  - That is, a least fixed point

$$\frac{}{\mathsf{N}\,0} \qquad \frac{\mathsf{N}\,x}{\mathsf{N}\ \mathsf{s}\,(x)} \qquad [\![\mathsf{N}]\!] = \{0, \mathsf{s}\,0, \mathsf{s}\,\mathsf{s}\,0, \ldots, \mathsf{s}^n\,0, \ldots\}$$

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TC has all possible inductive definitions 'available' using only a finite signature

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$$[\![\mathsf{N}]\!] = \{0, \mathsf{s}\,0, \mathsf{s}\,\mathsf{s}\,0, \ldots, \mathsf{s}^n\,0, \ldots\}$$

$\mathbf{FOL}_{\mathrm{ID}}$ productions only allow for Horn clauses

So we have a logic, but we need a proof theory …

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Effective

Complete

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Effective	Complete
Finitary RTC_G	

So we have a logic, but we need a proof theory ...

Effective	Complete
Finitary RTC_G	Infinitary RTC_G^{ω}

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Effective	Complete
Finitary RTC_G	
	Infinitary RTC_G^{ω}
Cyclic CRTC_G^{ω}	

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Finitary
RTC_G

↓

Cyclic
CRTC^{ω}_G

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Infinitary
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Effective Henkin-Complete	Complete
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Henkin Sound

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Finitary RTC_G ↓	Finitary RTC_G+A	
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$\downarrow$

Cyclic $\mathrm{NCRTC}^{\omega}_G$

$\downarrow$

Cyclic CRTC^{ω}_G

Finitary RTC_G+A

$\equiv$

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So we have a logic, but we need a proof theory …

	Effective Henkin-Complete		Complete
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Henkin Sound?	Cyclic NCRTC_G^ω	$\equiv$	Infinitary RTC_G^ω
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RTC_G: A Finitary Proof System with ‘Explicit’ Induction

We add the following rules to Gentzen’s sequent calculus for CL with substitution and equality:

$$\text{reflexivity}\quad \frac{}{\vdash (RTC_{x,y}\,\varphi)(t,t)}$$

$$\text{step}\quad \frac{\Gamma \vdash \Delta, (RTC_{x,y}\,\varphi)(s,r) \qquad \Gamma \vdash \Delta, \varphi[r/x, t/y]}{\Gamma \vdash \Delta, (RTC_{x,y}\,\varphi)(s,t)}$$

$$\text{induction}\quad \frac{\Gamma, \psi(x), \varphi(x,y) \vdash \Delta, \psi[y/x]}{\Gamma, \psi[s/x], (RTC_{x,y}\,\varphi)(s,t) \vdash \Delta, \psi[t/x]}$$

$x \notin \mathsf{fv}(\Gamma, \Delta)$ and $y \notin \mathsf{fv}(\Gamma, \Delta, \psi)$

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RTC_G 'captures' TC:

$$\frac{\Gamma \vdash \Delta, (RTC_{x,y}\,\varphi)(s,t)}{\Gamma \vdash \Delta, (RTC_{x,y}\,\varphi)(t,s)} \qquad \frac{\Gamma \vdash \Delta, \varphi[s/x, r/y] \quad \Gamma \vdash \Delta, (RTC_{x,y}\,\varphi)(r,t)}{\Gamma \vdash \Delta, (RTC_{x,y}\,\varphi)(s,t)}$$

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$$\frac{\Gamma, \varphi[s/x] \vdash \Delta}{\Gamma, (RTC_{x,y}\,\varphi)(s,t) \vdash \Delta, s = t} \qquad \frac{\Gamma, (RTC_{x,y}\,\varphi)(s,t) \vdash \Delta}{\Gamma, (RTC_{v,w}\,(RTC_{x,y}\,\varphi)(v,w))(s,t) \vdash \Delta}$$

$$\frac{\Gamma \vdash \Delta, (RTC_{x,y}\,\varphi)(s,t)}{\Gamma \vdash \Delta, s = t, \exists z\,.\,(RTC_{x,y}\,\varphi)(s,z) \wedge \varphi[z/x, t/y]}$$

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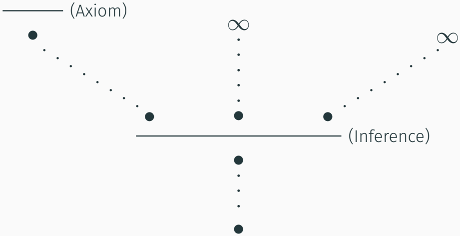
- A **TC** Henkin structure is a **TC** Henkin-frame closed under parametric definability, i.e.

$$\{a \in D \mid H, v[x := a] \models \varphi\} \in \mathcal{D} \text{ for all } \varphi, v, \text{ and } H$$

Non-well-founded Proof Theory



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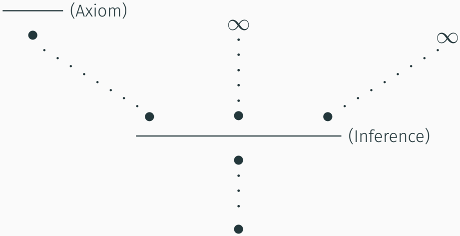

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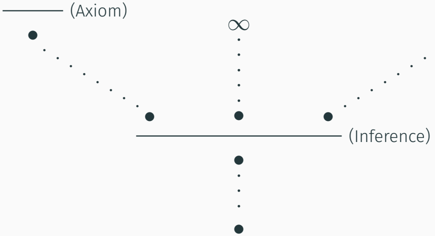

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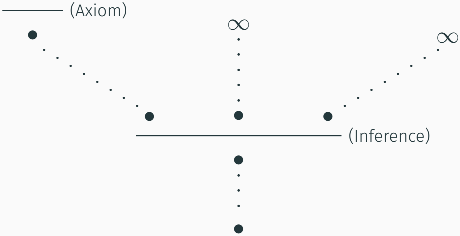

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- This is witnessed by tracing terms/formulas corresponding to elements of a well-founded set
- This global trace condition is an ω-regular property (i.e. decidable using Büchi automata)

We simply replace the explicit induction rule of RTC_G with:

$$\text{case-split}\quad \frac{\Gamma, s = t \vdash \Delta \quad \Gamma, (RTC_{x,y}\,\varphi)(s,z), \varphi[z/x, t/y] \vdash \Delta}{\Gamma, (RTC_{x,y}\,\varphi)(s,t) \vdash \Delta}\ (z \text{ fresh})$$

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The trace progresses when it traverses the principal formula of a case-split rule.

Soundness of RTC_G^ω

- Define a measure function for *RTC*-formulas:

$$\delta_{(RTC_{x,y}\,\varphi)(s,t)}(M, v) = \begin{cases} \text{minimal no. of } \varphi\text{-steps} \\ \text{from } v(s) \text{ to } v(t) \text{ in } M \end{cases}$$

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$$\frac{\Gamma_1 \vdash \Delta_1 \quad \ldots \quad (M', v') \not\models \Gamma_i \vdash \Delta_i \quad \ldots \quad \Gamma_n \vdash \Delta_n}{(M, v) \not\models \Gamma \vdash \Delta}$$

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$$\delta_{(RTC_{v,w}\,\varphi')(r,u)}(M', v') \leq \delta_{(RTC_{x,y}\,\varphi)(s,t)}(M, v)$$

Soundness of RTC_G^ω

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$$\delta_{(RTC_{x,y}\,\varphi)(s,t)}(M,v) = \begin{cases}\text{minimal no. of } \varphi\text{-steps} \\ \text{from } v(s) \text{ to } v(t) \text{ in } M\end{cases}$$

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- The proof rules have the following property:

$$\frac{\Gamma, s = t \vdash \Delta \quad (M', v') \not\models \Gamma, (RTC_{x,y}\,\varphi)(s,z), \varphi[z/x, t/y] \vdash \Delta}{(M,v) \not\models \Gamma, (RTC_{x,y}\,\varphi)(s,t) \vdash \Delta}$$

$$\delta_{(RTC_{v,w}\,\varphi')(r,u)}(M',v') < \delta_{(RTC_{x,y}\,\varphi)(s,t)}(M,v)$$

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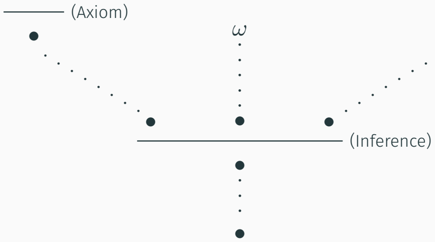
- Global trace condition $\Rightarrow n_1 > n_2 > n_3 > \ldots$

Cut-free Completeness of RTC_G^ω

Obtained using a variation of the standard technique:

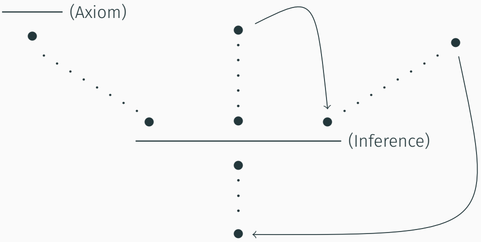
1. Construct an infinite (cut-free) pre-proof via an exhaustive search tree
2. If not a valid proof, then it is possible to construct a counter-model
3. Thus search tree gives a valid proof for every valid sequent

CRTC_G^ω: A Cyclic Subsystem



- Restricting to all and only regular infinite pre-proofs gives an effective system

CRTC$^{\omega}_{G}$: A Cyclic Subsystem



- Restricting to all and only regular infinite pre-proofs gives an effective system
- Regular pre-proofs can be represented as finite, possibly cyclic graphs

Implicit induction subsumes explicit induction

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\dfrac{\Gamma, \psi[v/x], (RTC_{x,y}\,\varphi)(v,w) \vdash \Delta, \psi[w/x]}{\Gamma, \psi[v/x], (RTC_{x,y}\,\varphi)(v,z) \vdash \Delta, \psi[z/x]}\;(\text{Subst})
\qquad
\dfrac{\begin{array}{c}\vdots \\ \Gamma, \psi, \varphi \vdash \Delta, \psi[y/x]\end{array}}{\Gamma, \psi[z/x], \varphi[z/x, w/y] \vdash \Delta, \psi[w/x]}\;(\text{Subst})
}{
\Gamma, \psi[v/x], (RTC_{x,y}\,\varphi)(v,z), \varphi[z/x, w/y] \vdash \Delta, \psi[w/x]
}\;(\text{Cut})
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}\;(\text{case-split})
}{
\begin{array}{c}\Gamma, \psi[s/x], (RTC_{x,y}\,\varphi)(s,t) \vdash \Delta, \psi[t/x] \\ \vdots\end{array}
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  \vdots
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}\,(\mathrm{Subst})
}\,(\text{case-split})
$$

The axiom $\Gamma, \psi[v/x] \vdash \Delta, \psi[v/x]$ is closed by (Ax).

$\mathrm{NCRTC}^{\omega}_{G}$, the subsystem of non-overlapping cyclic proofs, is Henkin-complete

Obtain RTC_G+A and CRTC^{ω}_G+A by adding the following schemas:

1. $\mathsf{s}\,0 \vdash$
2. $\mathsf{s}\,x = \mathsf{s}\,y \vdash x = y$
3. $\vdash x + \mathbf{0} = x$
4. $\vdash x + \mathsf{s}\,y = \mathsf{s}\,(x + y)$
5. $\vdash (RTC_{v,w}\, \mathsf{s}\, v = w)(\mathbf{0}, x)$

$$\boxed{\mathrm{RTC}_G\text{+A}} \longleftrightarrow \boxed{\mathrm{CRTC}^{\omega}_G\text{+A}}$$

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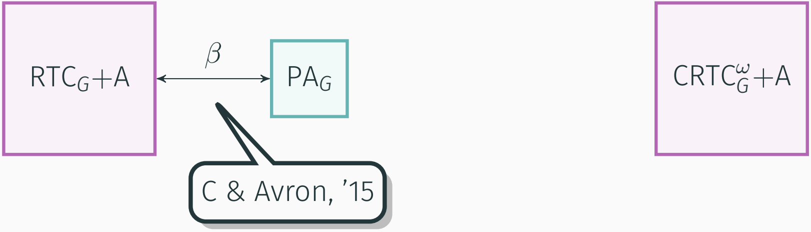

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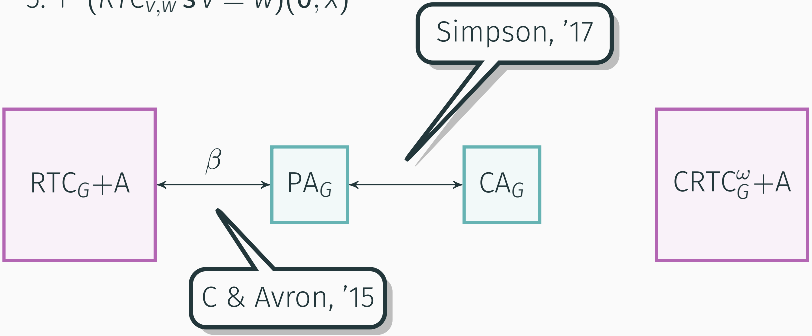

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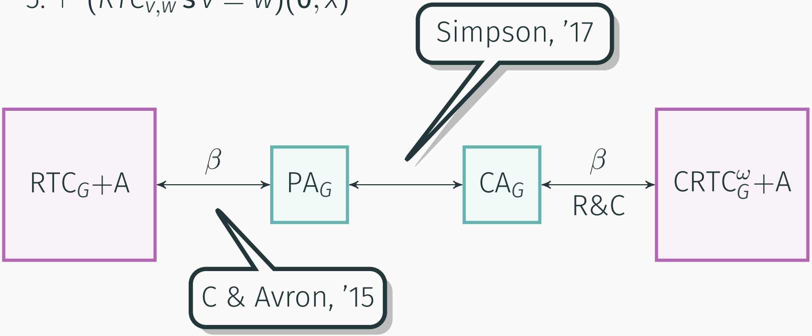

Equivalence: The General Case

For $\mathsf{FOL}_{\mathsf{ID}}$, implicit (cyclic) induction generally stronger than explicit induction [Berardi & Tatsuta, '17]

Equivalence: The General Case

For $\mathsf{FOL_{ID}}$, implicit (cyclic) induction generally stronger than explicit induction [Berardi & Tatsuta, '17]

- For signature $\{0, s\} + \{N\}$:
  - **0**,**s**-axioms $\vdash_{\mathsf{CLKID}^\omega}$ "2-hydra"

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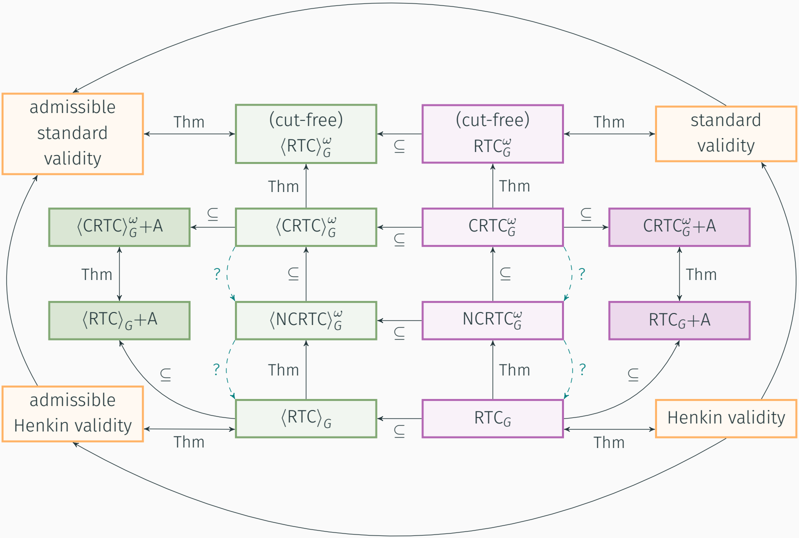
So this does not serve to show RTC_G and CRTC^ω_G inequivalent

- **TC** has all inductive definitions available

Summary of Results



Future Work

- Resolving the open question of the (in)equivalence of RTC_G, NCRTC_G^ω and CRTC_G^ω (with/out pairs).
- Implementing CRTC_G^ω and investigating the practicalities of **TC**-logic to support automated inductive reasoning.
- Using the uniformity of **TC**-logic to better study the relationship between implicit and explicit induction.
  - Cuts required in each system
  - Relative complexity of proofs
    - Das, A. (2018). On The Logical Complexity of Cyclic Arithmetic (*Submitted*)
- A uniform framework for coinductive reasoning?

Recall transitive closure as a fixed point:

$$R^{+} = \mu X.\Psi_R(X) \qquad\qquad \Psi_R(S) = R \cup (R \circ S)$$

The greatest fixed point gives the transitive co-closure

- Pairs (s, t) in $\nu X.\Psi_R(X)$ are those connected by a possibly infinite number of R-steps
- We can write $(RTC^{\mathrm{op}}_{x,y}\,\varphi)(s, t)$ to denote that (s, t) is in the reflexive, transitive co-closure of φ
- E.g. The following formula defines possibly infinite lists

$$(RTC^{\mathrm{op}}_{x,y}\,\exists z\,.\,x = \mathbf{cons}(z, y))(v, [\,])$$

We have the following standard semantics

$$M, v \models (RTC^{\mathrm{op}}_{x,y}\,\varphi)(s,t) \Leftrightarrow$$
$$\exists(\vec{a}_i)_{i\geq 0}\,.\,\forall i \geq 0\,.\,a_i = v(t) \vee M, v[x := a_i, y := a_{i+1}] \models \varphi$$

We have the following Henkin-semantics

$$H, v \models_{\mathcal{H}} (RTC^{\mathrm{op}}_{x,y}\,\varphi)(s,t) \Leftrightarrow$$
$$\text{there exists } A \in \mathcal{D} \text{ such that } v(s) \in A \text{ and}$$
$$\forall a \in A\,.\,\text{either } a = v(t) \text{ or } \exists b \in A\,.\,H, v[x := a, y := b] \models_{\mathcal{H}} \varphi$$