Program Verification Using Cyclic Proof

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Computer Laboratory Programming Research Group Seminar
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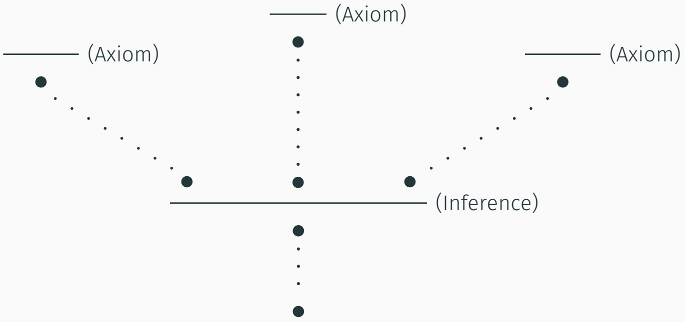
Prologue: It's Proof, But Not As We Know It



- We are all familiar with proofs as finite trees

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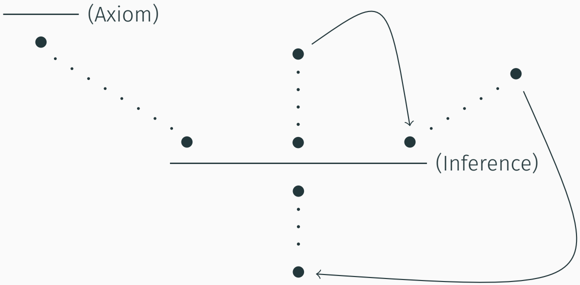

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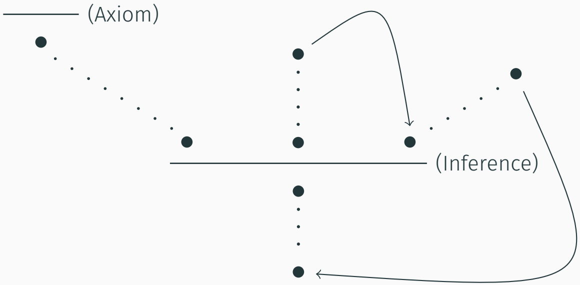
- But what if we allow proofs to be cyclic graphs instead?



- We are all familiar with proofs as finite trees
- But what if we allow proofs to be cyclic graphs instead?
- Cyclic proofs must satisfy a global soundness property

Introduction

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- Why cyclic proof?
  - It subsumes standard induction
  - It can help discover inductive hypotheses
  - Termination arguments can often be extracted from cyclic proofs

James Brotherston

Alex Simpson

Richard Bornat

Cristiano Calcagno

Dino Distefano

Nikos Gorogiannis

Example: First Order Logic

- Assume signature with zero, successor, and equality
- Allow inductive predicate definitions, e.g.

$$\frac{}{N\,0} \quad \frac{N\,x}{N\,sx} \qquad \frac{}{E\,0} \quad \frac{E\,x}{O\,sx} \quad \frac{O\,x}{E\,sx}$$

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- Allow inductive predicate definitions, e.g.

$$\frac{}{N\,0} \quad \frac{N\,x}{N\,sx} \qquad \frac{}{E\,0} \quad \frac{E\,x}{O\,sx} \quad \frac{O\,x}{E\,sx}$$

- These induce unfolding rules for the sequent calculus, e.g.

$$\frac{\Gamma \vdash \Delta, N\,t}{\Gamma \vdash \Delta, N\,st}\ (NR_2) \qquad \frac{\Gamma, t = 0 \vdash \Delta \quad \Gamma, t = sx, N\,x \vdash \Delta}{\Gamma, N\,t \vdash \Delta}\ (\text{Case } N)$$

where x is fresh

A Cyclic Proof of $N\ x \vdash E\ x, O\ x$

$$N\ x \vdash E\ x, O\ x$$

A Cyclic Proof of $N\,x \vdash E\,x, O\,x$

$$\dfrac{x = 0 \vdash E\,x, O\,x \qquad\qquad x = sy, N\,y \vdash E\,x, O\,x}{N\,x \vdash E\,x, O\,x}\ (\text{Case } N)$$

A Cyclic Proof of $N\,x \vdash E\,x, O\,x$

$$\dfrac{\dfrac{\vdash E\,0, O\,0}{x = 0 \vdash E\,x, O\,x}\,(=\text{L}) \qquad x = sy, N\,y \vdash E\,x, O\,x}{N\,x \vdash E\,x, O\,x}\,(\text{Case } N)$$

A Cyclic Proof of $N\,x \vdash E\,x, O\,x$

$$\dfrac{\dfrac{\dfrac{}{\vdash E\,0, O\,0}\,(E\mathrm{R}_1)}{x = 0 \vdash E\,x, O\,x}\,(=\mathrm{L}) \qquad x = sy, N\,y \vdash E\,x, O\,x}{N\,x \vdash E\,x, O\,x}\,(\text{Case } N)$$

A Cyclic Proof of $N\,x \vdash E\,x, O\,x$

$$
\dfrac{\dfrac{\dfrac{}{\vdash E\,0, O\,0}\,(E\mathrm{R}_1)}{x = 0 \vdash E\,x, O\,x}\,(=\mathrm{L}) \qquad \dfrac{N\,y \vdash E\,sy, O\,sy}{x = sy, N\,y \vdash E\,x, O\,x}\,(=\mathrm{L})}{N\,x \vdash E\,x, O\,x}\,(\text{Case } N)
$$

A Cyclic Proof of $N\ x \vdash E\ x, O\ x$

$$
\dfrac{\dfrac{\dfrac{}{\vdash E\ 0, O\ 0}\ (E\mathrm{R}_1)}{x = 0 \vdash E\ x, O\ x}\ (=\mathrm{L}) \qquad \dfrac{\dfrac{N\ y \vdash E\ y, O\ sy}{N\ y \vdash E\ sy, O\ sy}\ (E\mathrm{R}_2)}{x = sy, N\ y \vdash E\ x, O\ x}\ (=\mathrm{L})}{N\ x \vdash E\ x, O\ x}\ (\mathrm{Case}\ N)
$$

A Cyclic Proof of $N\,x \vdash E\,x, O\,x$

$$
\dfrac{
  \dfrac{
    \dfrac{}{\vdash E\,0, O\,0}\,(ER_1)
  }{x = 0 \vdash E\,x, O\,x}\,(=\!L)
  \qquad
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{N\,y \vdash E\,y, O\,y}{N\,y \vdash E\,y, O\,sy}\,(OR_1)
      }{N\,y \vdash E\,sy, O\,sy}\,(ER_2)
    }{x = sy, N\,y \vdash E\,x, O\,x}\,(=\!L)
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}{N\,y \vdash E\,y, O\,sy}\,(O\mathrm{R}_1)
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- Suppose $N\ x \vdash E\ x, O\ x$ is not valid:

  $[\![X]\!]_{m_1}$

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$$
\dfrac{
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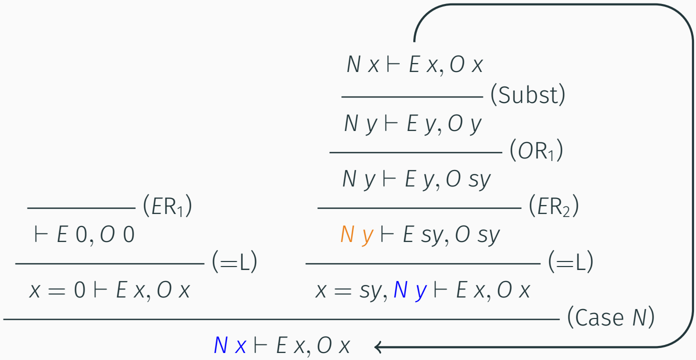
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{N\,x \vdash E\,x, O\,x}{N\,y \vdash E\,y, O\,y}\,(\mathrm{Subst})}{N\,y \vdash E\,y, O\,sy}\,(OR_1)}{N\,y \vdash E\,sy, O\,sy}\,(ER_2)}{x = sy, {\color{orange}N\,y} \vdash E\,x, O\,x}\,(=\mathrm{L})
}{
{\color{blue}N\,x} \vdash E\,x, O\,x
}\,(\mathrm{Case}\ N)
$$

- Suppose $N\,x \vdash E\,x, O\,x$ is not valid:

  $[\![x]\!]_{m_1} > {\color{orange}[\![y]\!]_{m_2}}$



- Suppose $N\ x \vdash E\ x, O\ x$ is not valid:

$$[\![x]\!]_{m_1} > [\![y]\!]_{m_2} = [\![y]\!]_{m_3}$$

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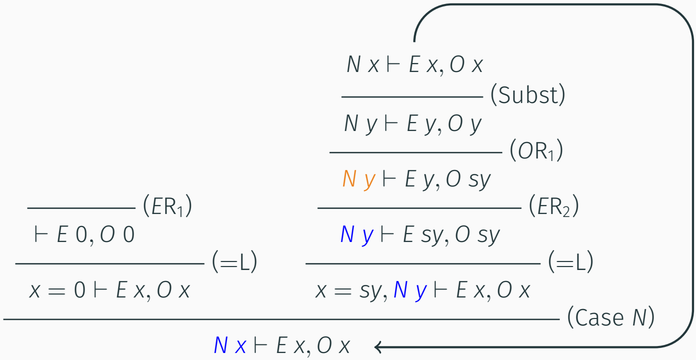

- Suppose $N\,x \vdash E\,x, O\,x$ is not valid:

$$[\![x]\!]_{m_1} > [\![y]\!]_{m_2} = [\![y]\!]_{m_3} = [\![y]\!]_{m_4}$$

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$$\cfrac{\cfrac{\cfrac{}{\vdash E\,0, O\,0}\,(ER_1)}{x = 0 \vdash E\,x, O\,x}\,(=\mathrm{L}) \qquad \cfrac{\cfrac{\cfrac{\cfrac{N\,x \vdash E\,x, O\,x}{N\,y \vdash E\,y, O\,y}\,(\text{Subst})}{N\,y \vdash E\,y, O\,sy}\,(OR_1)}{N\,y \vdash E\,sy, O\,sy}\,(ER_2)}{x = sy, N\,y \vdash E\,x, O\,x}\,(=\mathrm{L})}{N\,x \vdash E\,x, O\,x}\,(\text{Case } N)$$

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A Cyclic Proof of $N\,x \vdash E\,x, O\,x$

$$\dfrac{\dfrac{\dfrac{}{\vdash E\,0, O\,0}\,(ER_1)}{x = 0 \vdash E\,x, O\,x}\,(=L) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{N\,x \vdash E\,x, O\,x}{N\,y \vdash E\,y, O\,y}\,(\text{Subst})}{N\,y \vdash E\,y, O\,sy}\,(OR_1)}{N\,y \vdash E\,sy, O\,sy}\,(ER_2)}{x = sy, N\,y \vdash E\,x, O\,x}\,(=L)}{N\,x \vdash E\,x, O\,x}\,(\text{Case } N)$$

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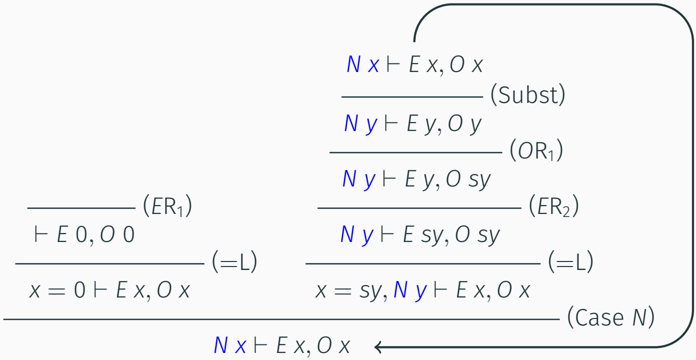
$$[\![x]\!]_{m_1} > [\![y]\!]_{m_2} = [\![y]\!]_{m_3} = [\![y]\!]_{m_4} = [\![y]\!]_{m_5} = [\![x]\!]_{m_6} > [\![y]\!]_{m_7} \dots$$



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$$n_1 > n_2 > n_3 > \ldots$$

Example: Separation Logic

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    - $F * G$ denotes a heap h that can be split into disjoint sub-heaps h_1 and h_2 which model F and G respectively
- Inductive predicates now represent data-structures, e.g. linked-list segments:

$$\frac{x = y \wedge \mathbf{emp}}{\mathsf{ls}(x, y)} \qquad \frac{x \mapsto z * \mathsf{ls}(z, y)}{\mathsf{ls}(x, y)}$$

A Cyclic Proof of List Segment Concatenation

$$\mathsf{ls}(x, y) * \mathsf{ls}(y, z) \vdash \mathsf{ls}(x, z)$$

A Cyclic Proof of List Segment Concatenation

$$\dfrac{\begin{array}{c}(x = y \land \mathsf{emp}) * \mathsf{ls}(y,z) \vdash \mathsf{ls}(x,z) \\ \vdots \end{array} \qquad x \mapsto v * \mathsf{ls}(v,y) * \mathsf{ls}(y,z) \vdash \mathsf{ls}(x,z)}{\mathsf{ls}(x,y) * \mathsf{ls}(y,z) \vdash \mathsf{ls}(x,z)}\ (\text{Case } \mathsf{ls})$$

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$$\dfrac{\dfrac{\mathsf{emp} * \mathsf{ls}(x, z) \vdash \mathsf{ls}(x, z)}{(x = y \wedge \mathsf{emp}) * \mathsf{ls}(y, z) \vdash \mathsf{ls}(x, z)}\ (=\mathrm{L}) \qquad \vdots \qquad x \mapsto v * \mathsf{ls}(v, y) * \mathsf{ls}(y, z) \vdash \mathsf{ls}(x, z)}{\mathsf{ls}(x, y) * \mathsf{ls}(y, z) \vdash \mathsf{ls}(x, z)}\ (\text{Case } \mathsf{ls})$$

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$$\dfrac{\dfrac{\dfrac{\mathsf{ls}(x,z) \vdash \mathsf{ls}(x,z)}{\mathsf{emp} * \mathsf{ls}(x,z) \vdash \mathsf{ls}(x,z)}\ (\equiv)}{(x = y \wedge \mathsf{emp}) * \mathsf{ls}(y,z) \vdash \mathsf{ls}(x,z)}\ (=\mathrm{L}) \qquad x \mapsto v * \mathsf{ls}(v,y) * \mathsf{ls}(y,z) \vdash \mathsf{ls}(x,z)}{\mathsf{ls}(x,y) * \mathsf{ls}(y,z) \vdash \mathsf{ls}(x,z)}\ (\text{Case } \mathsf{ls})$$

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$$\dfrac{\dfrac{\dfrac{}{\mathsf{ls}(x,z) \vdash \mathsf{ls}(x,z)}\;(\mathrm{Id})}{\mathsf{emp} * \mathsf{ls}(x,z) \vdash \mathsf{ls}(x,z)}\;(\equiv)}{(x = y \wedge \mathsf{emp}) * \mathsf{ls}(y,z) \vdash \mathsf{ls}(x,z)}\;(=\mathrm{L})$$

$$\dfrac{\vdots \qquad x \mapsto v * \mathsf{ls}(v,y) * \mathsf{ls}(y,z) \vdash \mathsf{ls}(x,z)}{\mathsf{ls}(x,y) * \mathsf{ls}(y,z) \vdash \mathsf{ls}(x,z)}\;(\text{Case } \mathsf{ls})$$

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$$\vdots$$

$$\dfrac{\vdots \qquad \dfrac{x \mapsto v * \mathsf{ls}(v,y) * \mathsf{ls}(y,z) \vdash x \mapsto v * \mathsf{ls}(v,z)}{x \mapsto v * \mathsf{ls}(v,y) * \mathsf{ls}(y,z) \vdash \mathsf{ls}(x,z)}\,(\mathsf{ls}\mathrm{R}_2)}{\mathsf{ls}(x,y) * \mathsf{ls}(y,z) \vdash \mathsf{ls}(x,z)}\,(\mathrm{Case}\ \mathsf{ls})$$

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$$
\dfrac{
\begin{array}{c}
\dfrac{\dfrac{\dfrac{}{\mathsf{ls}(x,z) \vdash \mathsf{ls}(x,z)}\ (\mathsf{Id})}{\mathbf{emp} * \mathsf{ls}(x,z) \vdash \mathsf{ls}(x,z)}\ (\equiv)}{(x = y \wedge \mathbf{emp}) * \mathsf{ls}(y,z) \vdash \mathsf{ls}(x,z)}\ (=\mathsf{L}) \\ \vdots
\end{array}
\qquad
\dfrac{\dfrac{x \mapsto v \vdash x \mapsto v \qquad \mathsf{ls}(v,y) * \mathsf{ls}(y,z) \vdash \mathsf{ls}(v,z)}{x \mapsto v * \mathsf{ls}(v,y) * \mathsf{ls}(y,z) \vdash x \mapsto v * \mathsf{ls}(v,z)}\ (*)}{x \mapsto v * \mathsf{ls}(v,y) * \mathsf{ls}(y,z) \vdash \mathsf{ls}(x,z)}\ (\mathsf{lsR}_2)
}{\mathsf{ls}(x,y) * \mathsf{ls}(y,z) \vdash \mathsf{ls}(x,z)}\ (\text{Case } \mathsf{ls})
$$

$$
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\dfrac{}{\mathsf{ls}(x,z) \vdash \mathsf{ls}(x,z)}\,(\mathrm{Id})
}{\mathsf{emp} * \mathsf{ls}(x,z) \vdash \mathsf{ls}(x,z)}\,(\equiv)
}{(x = y \wedge \mathsf{emp}) * \mathsf{ls}(y,z) \vdash \mathsf{ls}(x,z)}\,(=\mathrm{L})
\quad
\dfrac{
\dfrac{
\dfrac{}{x \mapsto v \vdash x \mapsto v}\,(\mathrm{Id})
\qquad
\mathsf{ls}(v,y) * \mathsf{ls}(y,z) \vdash \mathsf{ls}(v,z)
}{x \mapsto v * \mathsf{ls}(v,y) * \mathsf{ls}(y,z) \vdash x \mapsto v * \mathsf{ls}(v,z)}\,(*)
}{x \mapsto v * \mathsf{ls}(v,y) * \mathsf{ls}(y,z) \vdash \mathsf{ls}(x,z)}\,(\mathsf{ls}\mathrm{R}_2)
}{\mathsf{ls}(x,y) * \mathsf{ls}(y,z) \vdash \mathsf{ls}(x,z)}\,(\mathrm{Case}\ \mathsf{ls})
$$

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$$\dfrac{\dfrac{\dfrac{\dfrac{}{\mathsf{ls}(x,z) \vdash \mathsf{ls}(x,z)}\,(\mathrm{Id})}{\mathsf{emp} * \mathsf{ls}(x,z) \vdash \mathsf{ls}(x,z)}\,(\equiv)}{(x = y \wedge \mathsf{emp}) * \mathsf{ls}(y,z) \vdash \mathsf{ls}(x,z)}\,(=\mathrm{L}) \qquad \dfrac{\dfrac{\dfrac{}{x \mapsto v \vdash x \mapsto v}\,(\mathrm{Id}) \quad \dfrac{\mathsf{ls}(x,y) * \mathsf{ls}(y,z) \vdash \mathsf{ls}(x,z)}{\mathsf{ls}(v,y) * \mathsf{ls}(y,z) \vdash \mathsf{ls}(v,z)}\,(\mathrm{Subst})}{x \mapsto v * \mathsf{ls}(v,y) * \mathsf{ls}(y,z) \vdash x \mapsto v * \mathsf{ls}(v,z)}\,(*)}{x \mapsto v * \mathsf{ls}(v,y) * \mathsf{ls}(y,z) \vdash \mathsf{ls}(x,z)}\,(\mathsf{lsR}_2)}{\mathsf{ls}(x,y) * \mathsf{ls}(y,z) \vdash \mathsf{ls}(x,z)}\,(\text{Case } \mathsf{ls})$$

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$$\frac{\mathsf{ls}(x,z) \vdash \mathsf{ls}(x,z)}{\mathsf{emp} * \mathsf{ls}(x,z) \vdash \mathsf{ls}(x,z)}\,(\equiv)$$

$$\frac{\mathsf{emp} * \mathsf{ls}(x,z) \vdash \mathsf{ls}(x,z)}{(x = y \wedge \mathsf{emp}) * \mathsf{ls}(y,z) \vdash \mathsf{ls}(x,z)}\,(=\mathrm{L})$$

$$\frac{}{x \mapsto v \vdash x \mapsto v}\,(\mathrm{Id})$$

$$\frac{\mathsf{ls}(x,y) * \mathsf{ls}(y,z) \vdash \mathsf{ls}(x,z)}{\mathsf{ls}(v,y) * \mathsf{ls}(y,z) \vdash \mathsf{ls}(v,z)}\,(\mathrm{Subst})$$

$$\frac{x \mapsto v \vdash x \mapsto v \qquad \mathsf{ls}(v,y) * \mathsf{ls}(y,z) \vdash \mathsf{ls}(v,z)}{x \mapsto v * \mathsf{ls}(v,y) * \mathsf{ls}(y,z) \vdash x \mapsto v * \mathsf{ls}(v,z)}\,(*)$$

$$\frac{x \mapsto v * \mathsf{ls}(v,y) * \mathsf{ls}(y,z) \vdash x \mapsto v * \mathsf{ls}(v,z)}{x \mapsto v * \mathsf{ls}(v,y) * \mathsf{ls}(y,z) \vdash \mathsf{ls}(x,z)}\,(\mathsf{ls}\mathrm{R}_2)$$

$$\frac{(x = y \wedge \mathsf{emp}) * \mathsf{ls}(y,z) \vdash \mathsf{ls}(x,z) \qquad x \mapsto v * \mathsf{ls}(v,y) * \mathsf{ls}(y,z) \vdash \mathsf{ls}(x,z)}{\mathsf{ls}(x,y) * \mathsf{ls}(y,z) \vdash \mathsf{ls}(x,z)}\,(\mathrm{Case}\ \mathsf{ls})$$

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$$\dfrac{\vdots \qquad \vdots}{\mathsf{ls}(x,y) * \mathsf{ls}(y,z) \vdash \mathsf{ls}(x,z)}\,(\mathrm{Case}\ \mathsf{ls})$$

Global Soundness for Cyclic Proof: Elements

- Fix some values that we can trace along paths in the proof
  - In our examples: inductive predicate instances

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  - Inductive definitions induce a monotone operator φ on sets of models
  - Interpret the inductive definitions using the lfp

$$\varphi(\bot) \sqsubseteq \varphi(\varphi(\bot)) \sqsubseteq \ldots \sqsubseteq \varphi^{\omega}(\bot) \sqsubseteq \ldots \sqsubseteq \mu X.\varphi(X)$$

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$$\varphi(\bot) \sqsubseteq \varphi(\varphi(\bot)) \sqsubseteq \ldots \sqsubseteq \varphi^{\omega}(\bot) \sqsubseteq \ldots \sqsubseteq \mu X.\varphi(X)$$

- Map $(m, P\,\vec{t})$ to the least approximation $\varphi^{\alpha}(\bot)$ of P in which m appears

Global Soundness for Cyclic Proof: Elements

- Fix some values that we can trace along paths in the proof
  - In our examples: inductive predicate instances
- Map (model, trace-value) pairs to elements of a w.f. set
  - Inductive definitions induce a monotone operator φ on sets of models
  - Interpret the inductive definitions using the lfp

$$\varphi(\bot) \sqsubseteq \varphi(\varphi(\bot)) \sqsubseteq \ldots \sqsubseteq \varphi^{\omega}(\bot) \sqsubseteq \ldots \sqsubseteq \mu X.\varphi(X)$$

  - Map $(m, P\,\vec{t})$ to the least approximation $\varphi^{\alpha}(\bot)$ of P in which m appears
- Identify the progression points of the proof system, e.g.

$$\frac{x = y \wedge \mathbf{emp} \vdash F \quad x \mapsto v * \mathsf{ls}(v, y) \vdash F}{\mathsf{ls}(x, y) \vdash F} \text{ (Case } \mathsf{ls}\text{)}$$

Global Soundness for Cyclic Proof: General Principle

- Impose global soundness condition on proof graphs:
  - Every infinite path must have an infinitely progressing trace
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  - Assume the conclusion of the proof is invalid
  - Local soundness implies an infinite sequence of (counter) models
  - Global soundness then implies an infinite descending chain in a well-founded set

Cyclic Proof vs Explicit Induction

- Explicit induction requires induction hypothesis F up-front

$$\frac{}{N\ 0} \qquad \frac{N\ x}{N\ sx} \qquad \frac{\Gamma \vdash F[0] \quad \Gamma, F[x] \vdash F[sx], \Delta \quad \Gamma, F[t] \vdash \Delta}{\Gamma, N\ t \vdash \Delta}\ (\text{Ind } N)$$

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- Cyclic proof enables '*discovery*' of induction hypotheses
- Complex induction schemes naturally represented by nested and overlapping cycles
- The explicit induction rules are derivable in the cyclic system (cf. Brotherston & Simpson)

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- Cyclic proofs are the (strict) regular subset of the set of non-well-founded proof trees
- Theorem (Brotherston & Simpson): the full infinite system is cut-free complete
- Cut is likely not eliminable in the cyclic sub-system

A Simple Imperative Language

$$
\begin{aligned}
&\text{(Terms)} & t &::= \texttt{nil} \mid x \\
&\text{(Boolean Expressions)} & B &::= t\texttt{=}t \mid t\texttt{!=}t \\
&\text{(Programs)} & C &::= \varepsilon && \text{(stop)} \\
& & &\mid\ x\texttt{:=}t\texttt{;}C && \text{(assignment)} \\
& & &\mid\ x\texttt{:=[}y\texttt{];}C \mid \texttt{[}x\texttt{]:=}y\texttt{;}C && \text{(load/store)} \\
& & &\mid\ \texttt{free(}x\texttt{);}C \mid x\texttt{:=new;}C && \text{(de/allocate)} \\
& & &\mid\ \texttt{if}\ B\ \texttt{then}\ C\texttt{;}C && \text{(conditional)} \\
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- The following program deallocates a linked list

```
while x!=nil do y:=[x];free(x);x=y;
```

Program Verification by Symbolic Execution

- We use Hoare logic for proving triples $\{P\}\, C\, \{Q\}$ using Separation Logic as an assertion language

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Handling Loops in Cyclic Proofs

- The standard Hoare rule for handling `while` loops:

$$\frac{\{B \wedge P\}\, C_1\, \{P\} \quad \{\neg B \wedge P\}\, C_2\, \{Q\}}{\{P\}\, \texttt{while}\, B\, \texttt{do}\, C_1\, \texttt{;}\, C_2\, \{Q\}}$$

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- With cyclic proof, it is enough just to unfold loops

$$\frac{\{B \wedge P\}\, C_1\mathtt{;}\mathtt{while}\, B\, \mathtt{do}\, C_1\mathtt{;}\, C_2\, \{Q\} \quad \{\neg B \wedge P\}\, C_2\, \{Q\}}{\{P\}\, \mathtt{while}\, B\, \mathtt{do}\, C_1\mathtt{;}\, C_2\, \{Q\}}$$

Example: Deallocating the Linked List

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while x!=nil do y:=[x];free(x);x=y;
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$$\dfrac{\left\{\begin{array}{c}\mathtt{x}\neq\mathtt{nil}\\ \wedge\ \mathsf{ls}(\mathtt{x},\mathtt{nil})\end{array}\right\}\ \mathtt{y:=[x];free(x);x=y;while\,x!=nil\,do\,y:=[x];free(x);x=y;}\ \{\mathsf{emp}\} \quad \vdots \quad \left\{\begin{array}{c}\mathtt{x}=\mathtt{nil}\\ \wedge\ \mathsf{ls}(\mathtt{x},\mathtt{nil})\end{array}\right\}\ \varepsilon\ \{\mathsf{emp}\}}{\{\mathsf{ls}(\mathtt{x},\mathtt{nil})\}\ \mathtt{while}\ \ldots\ \{\mathsf{emp}\}}\ (\mathsf{while})$$

Example: Deallocating the Linked List

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Example: Deallocating the Linked List

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\dfrac{\dfrac{\{\mathsf{ls}(\texttt{y},\texttt{nil})\}\ \texttt{x=y;}\ \ldots\ \{\mathsf{emp}\}}{\left\{\begin{array}{c}\color{orange}{\texttt{x} \mapsto \texttt{y}}\\ *\ \mathsf{ls}(\texttt{y},\texttt{nil})\end{array}\right\} \texttt{free(x);}\ \ldots\ \{\mathsf{emp}\}}\,(\color{orange}{\text{free}})}{\left\{\begin{array}{c}\texttt{x} \mapsto v\\ *\ \mathsf{ls}(v,\texttt{nil})\end{array}\right\} \texttt{y:=[x];}\ \ldots\ \{\mathsf{emp}\}}\,(\text{load})
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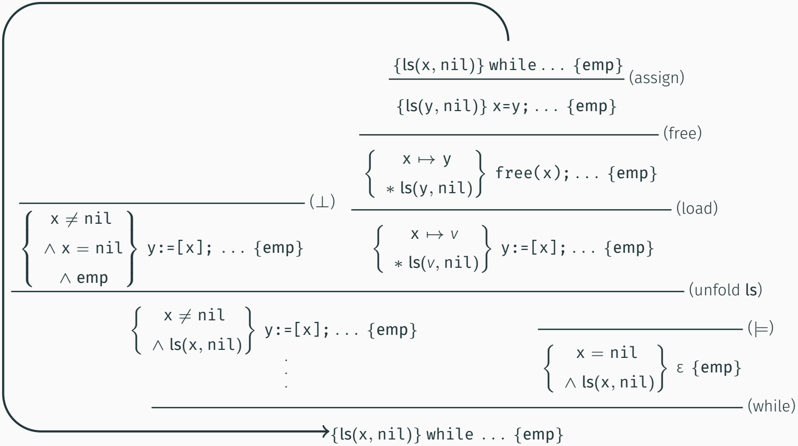
Example: Deallocating the Linked List

{ls(x, nil)} while x!=nil do y:=[x];free(x);x=y; {emp}



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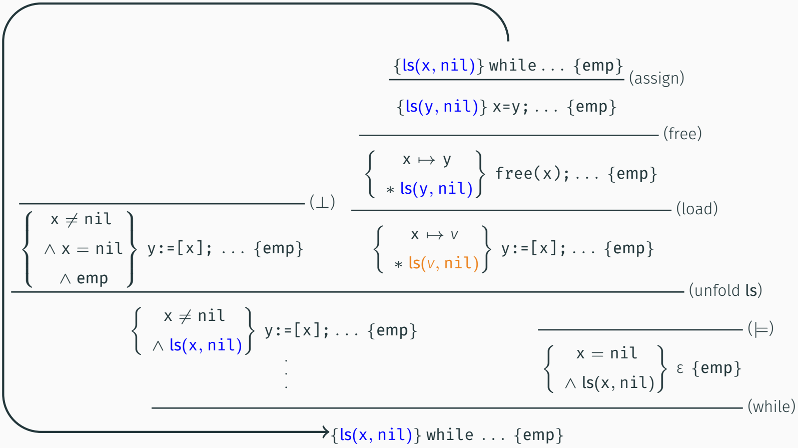

(Procedures)	$\texttt{proc}\, p(\vec{x})\, \{\, C \,\}$
(Programs)	$C ::= \ldots \mid p(\vec{t})\texttt{;}\, C$

Verifying Recursive Procedures

$$\begin{array}{rl} \text{(Procedures)} & \texttt{proc}\, p(\vec{x})\, \{\, C \,\} \\ \text{(Programs)} & C \ ::= \ \ldots \ \mid \ p(\vec{t})\,\texttt{;}\, C \end{array}$$

$$\text{(proc):} \quad \frac{\{P\}\, C\, \{Q\}}{\{P\}\, p(\vec{x})\, \{Q\}}\, (\text{body}(p) = C)$$

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$$\text{(param):} \quad \frac{\{P\}\, p(\vec{t})\, \{Q\}}{\{P[t/x]\}\, p(\vec{t})[t/x]\, \{Q[t/x]\}}$$

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- The following procedure recursively deallocates a linked list

```
proc dealloc(x) { if x!=nil then y:=[x]; free(x); dealloc(y); }
```

Example: Deallocating the Linked List (Recursively)

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proc dealloc(x) { if x!=nil then y:=[x]; free(x); dealloc(y); }
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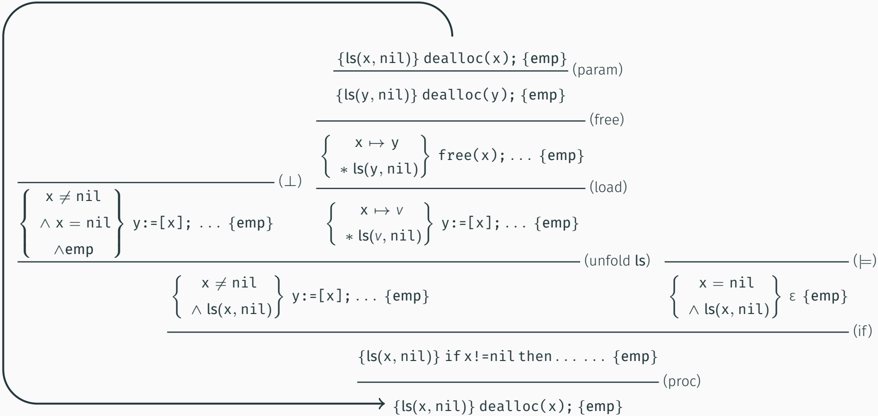

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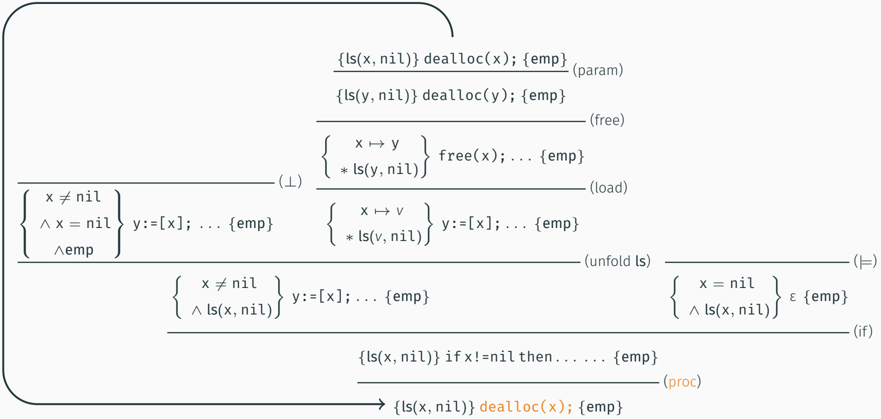

Example: Deallocating the Linked List (Recursively)

`proc dealloc(x) { if x!=nil then y:=[x]; free(x); dealloc(y); }`

{ls(x, nil)} dealloc(x); {emp} (param)

{ls(y, nil)} dealloc(y); {emp}

(free)

$\left\{ \begin{array}{c} \texttt{x} \mapsto \texttt{y} \\ * \ \mathsf{ls}(\texttt{y}, \texttt{nil}) \end{array} \right\}$ free(x); ... {emp}

(load)

$\left\{ \begin{array}{c} \texttt{x} \mapsto v \\ * \ \mathsf{ls}(v, \texttt{nil}) \end{array} \right\}$ y:=[x]; ... {emp}

(⊥)

$\left\{ \begin{array}{c} \texttt{x} \neq \texttt{nil} \\ \wedge \ \texttt{x} = \texttt{nil} \\ \wedge \mathsf{emp} \end{array} \right\}$ y:=[x]; ... {emp}

(unfold ls)

$\left\{ \begin{array}{c} \texttt{x} \neq \texttt{nil} \\ \wedge \ \mathsf{ls}(\texttt{x}, \texttt{nil}) \end{array} \right\}$ y:=[x]; ... {emp}

(⊨)

$\left\{ \begin{array}{c} \texttt{x} = \texttt{nil} \\ \wedge \ \mathsf{ls}(\texttt{x}, \texttt{nil}) \end{array} \right\}$ ε {emp}

(if)

{ls(x, nil)} if x!=nil then ... ... {emp}

(proc)

{ls(x, nil)} dealloc(x); {emp}

Example: Deallocating the Linked List (Recursively)

proc dealloc(x) { if x!=nil then y:=[x]; free(x); dealloc(y); }

{ls(x, nil)} dealloc(x); {emp} (param)

{ls(y, nil)} dealloc(y); {emp}

(free)

$\left\{\begin{array}{c} x \mapsto y \\ * \ \mathsf{ls}(y, \mathtt{nil}) \end{array}\right\}$ free(x); . . . {emp}

(⊥) (load)

$\left\{\begin{array}{c} x \neq \mathtt{nil} \\ \wedge\ x = \mathtt{nil} \\ \wedge \mathsf{emp} \end{array}\right\}$ y:=[x]; . . . {emp}

$\left\{\begin{array}{c} x \mapsto v \\ * \ \mathsf{ls}(v, \mathtt{nil}) \end{array}\right\}$ y:=[x]; . . . {emp}

(unfold ls) (⊨)

$\left\{\begin{array}{c} x \neq \mathtt{nil} \\ \wedge\ \mathsf{ls}(x, \mathtt{nil}) \end{array}\right\}$ y:=[x]; . . . {emp}

$\left\{\begin{array}{c} x = \mathtt{nil} \\ \wedge\ \mathsf{ls}(x, \mathtt{nil}) \end{array}\right\}$ ε {emp}

(if)

{ls(x, nil)} if x!=nil then . . . . . . {emp}

(proc)

{ls(x, nil)} dealloc(x); {emp}

A Subtlety with Procedures

```
proc shuffle(x){
  if x!=nil then
    y:=[x]; reverse(y); shuffle(y); [x]:=y;
}
```

A Subtlety with Procedures

```
proc shuffle(x){
  if x!=nil then
    y:=[x]; reverse(y); shuffle(y); [x]:=y;
}
```

$$\dfrac{\dfrac{\dfrac{\vdots}{\{\mathsf{ls}(y,\mathtt{nil})\}\ \mathtt{reverse(y);}\ \{\mathsf{ls}(y,\mathtt{nil})\}}}{\{x \mapsto y * \mathsf{ls}(y,\mathtt{nil})\}\ \mathtt{reverse(y);}\ \{x \mapsto y * \mathsf{ls}(y,\mathtt{nil})\}}\ (\text{frame}) \qquad \dfrac{\vdots}{\{x \mapsto y * \mathsf{ls}(y,\mathtt{nil})\}\ \mathtt{shuffle(y);}\ \ldots\ \{\mathsf{ls}(x,\mathtt{nil})\}}}{\dfrac{\{x \mapsto y * \mathsf{ls}(y,\mathtt{nil})\}\ \mathtt{reverse(y);shuffle(y);}\ \ldots\ \{\mathsf{ls}(x,\mathtt{nil})\}}{\vdots}}$$

A Subtlety with Procedures

```
proc shuffle(x) {
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$$\dfrac{\vdots}{\{x \mapsto y * \mathsf{ls}(y,\texttt{nil})\}\ \texttt{shuffle(y);}\ \ldots\ \{\mathsf{ls}(x,\texttt{nil})\}}$$

$$\dfrac{\{x \mapsto y * \mathsf{ls}(y,\texttt{nil})\}\ \texttt{reverse(y);}\ \{x \mapsto y * \mathsf{ls}(y,\texttt{nil})\} \qquad \{x \mapsto y * \mathsf{ls}(y,\texttt{nil})\}\ \texttt{shuffle(y);}\ \ldots\ \{\mathsf{ls}(x,\texttt{nil})\}}{\{x \mapsto y * \mathsf{ls}(y,\texttt{nil})\}\ \texttt{reverse(y); shuffle(y);}\ \ldots\ \{\mathsf{ls}(x,\texttt{nil})\}}$$

$$\vdots$$

A Subtlety with Procedures

```
proc shuffle(x){
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```

⋮

{ls(y, nil)} reverse(y); {ls(y, nil)}

——————————————————— (frame)

{x ↦ y ∗ ls(y, nil)} reverse(y); {x ↦ y ∗ ls(y, nil)}   ⋮ {x ↦ y ∗ ls(y, nil)} shuffle(y); ... {ls(x, nil)}

———————————————————————————————————

{x ↦ y ∗ ls(y, nil)} reverse(y); shuffle(y); ... {ls(x, nil)}

⋮

Solution: Explicit Approximation

- We explicitly label predicate instances, e.g. $\mathsf{ls}_{\alpha}(x, y)$
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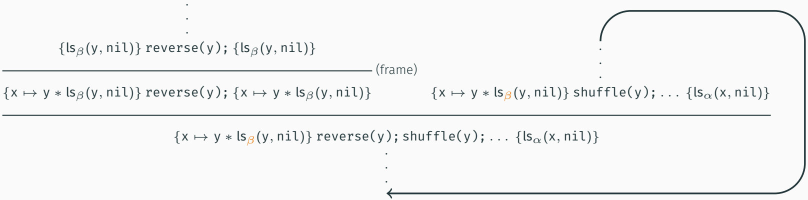

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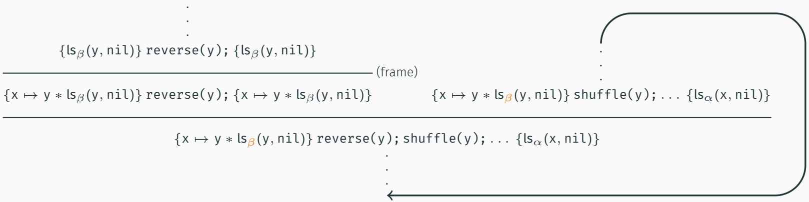

- We now need constraints on labels when unfolding, e.g.

$$\frac{\Gamma, \beta < \alpha, t = 0 \vdash \Delta \quad \Gamma, \beta < \alpha, t = sx, N_{\beta}\ x \vdash \Delta}{\Gamma, N_{\alpha}\ t \vdash \Delta}\ (\text{Case } N)$$

The Cyclist Verification Tool

- Our verification tool, Cyclist, is implemented in OCaml
- Generic cyclic proof-search procedure using iterated depth-first search
    - Cycles are formed eagerly and discarded if unsound
- The generic proof search is parametric
    - Different proof systems implemented as separate modules

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github.com/ngorogiannis/cyclist

Performance Results

Program	Time (ms)	LOC	Procs	Nodes	Back-links
list traverse	17	6	1	18	2
tree traverse	24	7	1	26	2
list deallocate	14	7	1	13	1
tree deallocate	21	8	1	24	2
tree reflect	20	9	1	22	2
list rev. deallocate	43	18	1	49	2
list append	28	21	1	34	1
list reverse	122	14	1	34	1
list reverse (tail rec.)	31	18	1	32	1
list reverse (with append)	47	28	2	56	2
list filter	27	16	1	29	1
list partition	31	25	1	40	1
list ackermann	126	17	1	50	3
queue	894	30	3	119	6
functional queue	254	28	3	62	1
shuffle	202	23	2	79	4

Results of Experimental Evaluation on 2.93GHz Intel Core i7-870, 8GB RAM

Concluding Remarks

- Ongoing work: inferring constraints on predicate labels automatically
- Some problems remain hard, of course
  - Generalisation of inductive hypotheses
  - Finding and applying lemmas
  - Synthesizing procedure summaries (see previous point!)

Thank You

Related Work

- Cyclic proofs for FOL with inductive predicates
  (Brotherston & Simpson, LICS 2007)
- Cyclic proofs for Separation Logic with inductive predicates
  (Brotherston, SAS 2007)
- Cyclic proofs verifying simple heap-manipulating **WHILE** language
  (Brotherston, Bornat & Calcagno, POPL 2008)
- Implementations in Isabelle/HOL, then OCaml
  (Brotherston, Distefano, Gorogiannis, CADE 2011/APLAS 2012)
- Abduction of inductive predicates using cyclic proof
  (Brotherston & Gorogiannis, SAS 2014)
- Current Work — cyclic proofs for verifying:
    - procedural heap-manipulating language (Rowe, Brotherston)
    - temporal properties (Tellez Espinosa, Brotherston)