Transitive Closure Logic

Infinitary and Cyclic Proof Systems

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Transitive Closure (**TC**) Logic extends FOL with formulas:

- $(RTC_{x,y}\,\varphi)(s,t)$
  - φ is a formula
  - x and y are distinct variables (which become bound in φ)
  - s and t are terms

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whose intended meaning is an infinite disjunction

$$\begin{aligned} & s = t \lor \varphi[s/x, t/y] \\ & \qquad \lor (\exists w_1 \,.\, \varphi[s/x, w_1/y] \land \varphi[w_1/x, t/y]) \\ & \qquad \lor (\exists w_1, w_2 \,.\, \varphi[s/x, w_1/y] \land \varphi[w_1/x, w_2/y] \land \varphi[w_2/x, t/y]) \\ & \qquad \lor \ \ldots \end{aligned}$$

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- M is a (standard) first-order model with domain D
- v is a valuation of terms in M:

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&\quad \wedge M, v[x := a_i, y := a_{i+1}] \models \varphi \quad \text{for all } i < n
\end{aligned}
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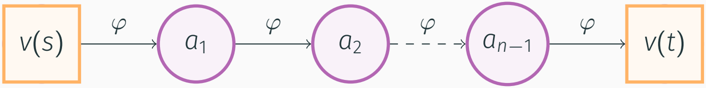

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- $(RTC_{x,y}\, \varphi)$ 'denotes' the reflexive, transitive closure of φ:

$$M, v \models (RTC_{x,y}\, \varphi)(s, t) \Leftrightarrow (v(s), v(t)) \in ([\![\varphi(x, y)]\!]_{M,v})^*$$

Why Transitive Closure logic?

- It is a minimal extension of FOL
- It has an intuitive, easy-to-understand semantics
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Theorem (Avron '03)

All finitely inductively defined relations are definable in* TC.†

A. Avron, *Transitive Closure and the Mechanization of Mathematics*, 2003.

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*as defined in: S. Feferman, *Finitary Inductively Presented Logics*, 1989

†with signatures containing a pairing function

Example: Arithmetic

- Take a signature $\Sigma = \{0, \mathsf{s}\}$ + equality and pairing

$$\mathsf{Nat}(x) \equiv (RTC_{v,w}\, \mathsf{s}\, v = w)(0, x)$$

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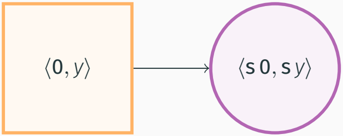

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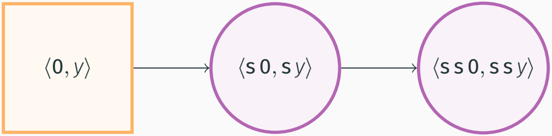

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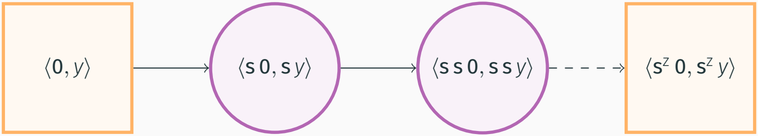

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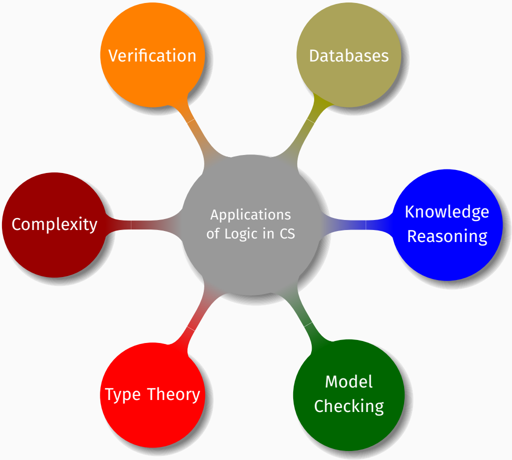
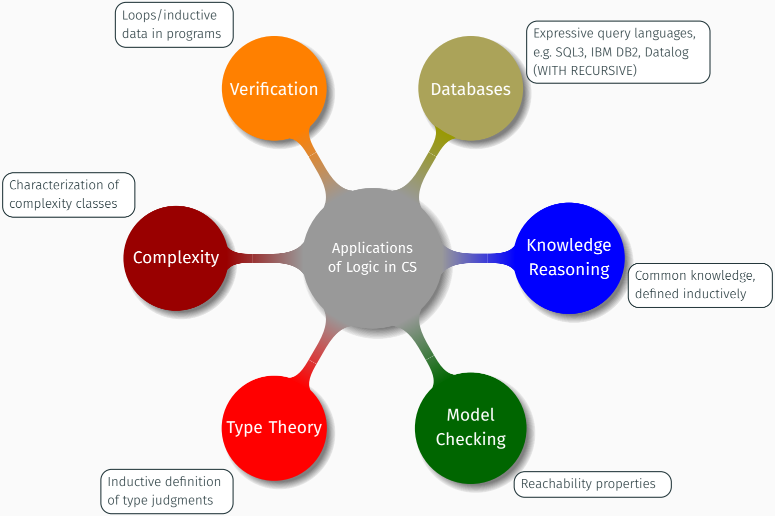
- The following axioms categorically characterise the natural numbers in TC:

$$\forall x \,.\, \mathsf{s}\, x \neq \mathsf{0}$$
$$\forall x, y \,.\, \mathsf{s}\,(x) = \mathsf{s}\,(y) \rightarrow x = y$$
$$\forall x \,.\, \mathsf{Nat}(x)$$



J. Halpern Et Al, *On the Unusual Effectiveness of Logic in Computer Science*, 2001



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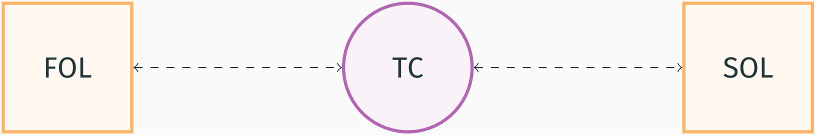
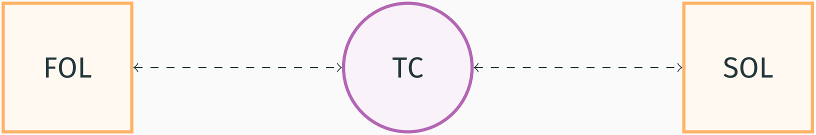
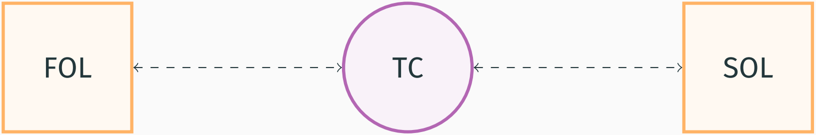

FOL
TC
SOL



"Everything should be made as simple as possible but not simpler"

—Albert Einsten

Weak **SOL**
ω-logic
Cardinality logic
FOL + Henkin Quantifiers



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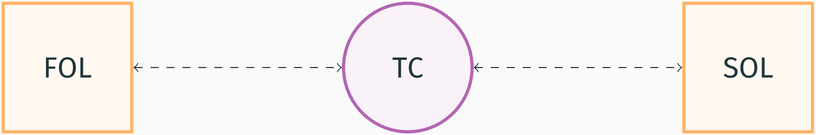

FOL
+ ML Ind.
Defs

FOMμ

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The transitive closure

$$R^+ = \bigcup_{i \geq 0} R_i, \quad \text{where } R_0 = R$$
$$R_{i+1} = R_i \circ R \quad (i \geq 0)$$

is a particular kind of fixed point:

$$R^+ = \mu X.\Psi_R(X)$$

where, for binary relations R and S, we define

$$\Psi_R(S) = R \cup (R \circ S)$$

FOL + Martin-Löf inductive definitions:

- For each predicate symbol $P_1, \ldots, P_n$, we give a set of productions of the form:

$$\frac{Q_1(\vec{s_1}) \quad \ldots \quad Q_n(\vec{s_n})}{P_i(\vec{t})}$$

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TC has all possible inductive definitions 'available' using only a finite signature

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FOL$_{\text{ID}}$ productions only allow for Horn clauses

- The semantics of the logic uses the least fixed point

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EFFECTIVE

COMPLETE

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Effective	Complete

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Effective	Complete
Finitary RTC_G	

What about the proof theory?

Effective

Finitary
RTC_G

Complete

Infinitary
RTC_G^ω

What about the proof theory?

Effective

Finitary
RTC_G

Cyclic
CRTC^{ω}_G

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↓

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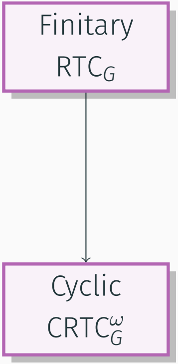
Effective Henkin-Complete



Complete

Infinitary

RTC^{ω}_G

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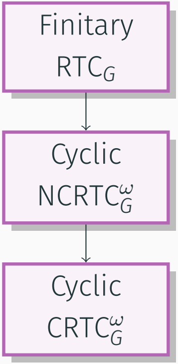
EFFECTIVE HENKIN-COMPLETE



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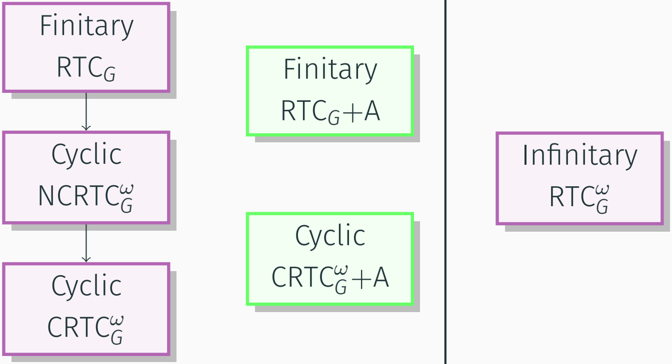

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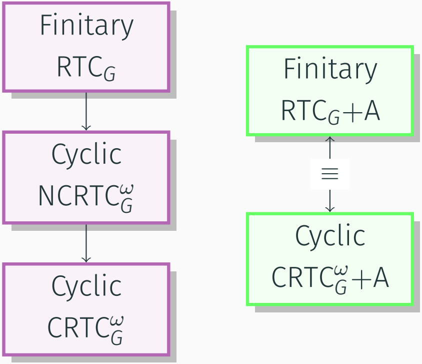

Complete

RTC_G: A Finitary Proof System with ‘Explicit’ Induction

We add the following rules to Gentzen’s sequent calculus for CL with substitution and equality:

reflexivity
$$\frac{}{\vdash (RTC_{x,y}\,\varphi)(t,t)}$$

step
$$\frac{\Gamma \vdash \Delta, (RTC_{x,y}\,\varphi)(s,r) \quad \Gamma \vdash \Delta, \varphi[r/x, t/y]}{\Gamma \vdash \Delta, (RTC_{x,y}\,\varphi)(s,t)}$$

induction
$$\frac{\Gamma, \psi(x), \varphi(x,y) \vdash \Delta, \psi[y/x]}{\Gamma, \psi[s/x], (RTC_{x,y}\,\varphi)(s,t) \vdash \Delta, \psi[t/x]}$$

$x \notin \mathbf{fv}(\Gamma, \Delta)$ and $y \notin \mathbf{fv}(\Gamma, \Delta, \psi)$

RTC$_G$ ‘captures’ TC:

$$\frac{\Gamma \vdash \Delta, (RTC_{x,y}\,\varphi)(s,t)}{\Gamma \vdash \Delta, (RTC_{x,y}\,\varphi)(t,s)} \qquad \frac{\Gamma \vdash \Delta, \varphi[s/x, r/y] \quad \Gamma \vdash \Delta, (RTC_{x,y}\,\varphi)(r,t)}{\Gamma \vdash \Delta, (RTC_{x,y}\,\varphi)(s,t)}$$

$$\frac{\Gamma \vdash \Delta, (RTC_{x,y}\,\varphi)(s,t)}{\Gamma \vdash \Delta, (RTC_{v,w}\,\varphi[v/x, w/y])(s,t)} \qquad \frac{\Gamma, \varphi \vdash \Delta, \psi}{\Gamma, (RTC_{x,y}\,\varphi)(s,t) \vdash \Delta, (RTC_{x,y}\,\psi)(s,t)}$$

$$\frac{\Gamma, \varphi[s/x] \vdash \Delta}{\Gamma, (RTC_{x,y}\,\varphi)(s,t) \vdash \Delta, s = t} \qquad \frac{\Gamma, (RTC_{x,y}\,\varphi)(s,t) \vdash \Delta}{\Gamma, (RTC_{v,w}\,(RTC_{x,y}\,\varphi)(v,w))(s,t) \vdash \Delta}$$

$$\frac{\Gamma \vdash \Delta, (RTC_{x,y}\,\varphi)(s,t)}{\Gamma \vdash \Delta, s = t, \exists z\,.\,(RTC_{x,y}\,\varphi)(s,z) \wedge \varphi[z/x, t/y]}$$

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- A **TC** Henkin-frame H is a triple $\langle D, I, \mathcal{D} \rangle$
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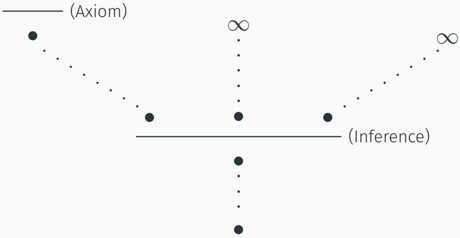
- A **TC** Henkin structure is a **TC** Henkin-frame closed under parametric definability, i.e.

$$\{a \in D \mid H, v[x := a] \models \varphi\} \in \mathcal{D} \text{ for all } \varphi, v, \text{ and } H$$

In non-well-founded proof theory we allow infinite height derivations:



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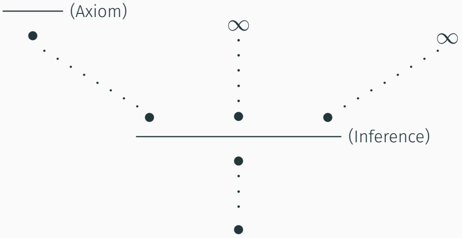

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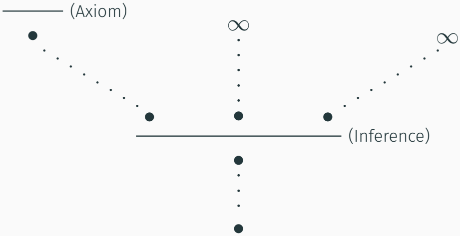

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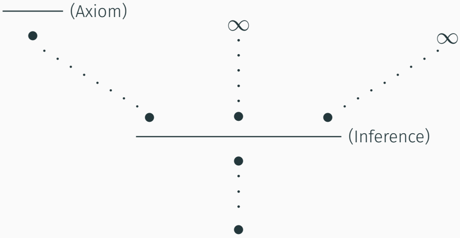

- We only accept proofs for which every path admits some infinite descent
- This is witnessed by tracing terms/formulas corresponding to elements of a well-founded set
- This global trace condition is an ω-regular property (i.e. decidable using Büchi automata)

RTC_G^{ω}: An Infinitary Proof System with 'Implicit' Induction

We simply replace the explicit induction rule of RTC_G with:

$$\text{case-split}\quad \frac{\Gamma, s = t \vdash \Delta \qquad \Gamma, (RTC_{x,y}\,\varphi)(s,z), \varphi[z/x, t/y] \vdash \Delta}{\Gamma, (RTC_{x,y}\,\varphi)(s,t) \vdash \Delta}\ (z \text{ fresh})$$

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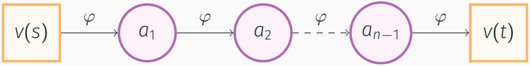
The trace progresses when it traverses the principal formula of a case-split rule.

Soundness of RTC_G^ω

- Define a measure function for *RTC*-formulas:

$$\delta_{(RTC_{x,y}\,\varphi)(s,t)}(M, v) = \begin{cases} \text{minimal no. of } \varphi\text{-steps} \\ \text{from } v(s) \text{ to } v(t) \text{ in } M \end{cases}$$



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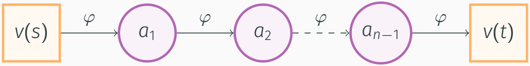

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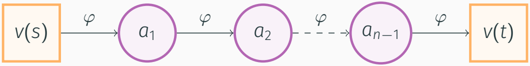

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$$\frac{\Gamma_1 \vdash \Delta_1 \quad \dots \quad (M',v') \not\models \Gamma_i \vdash \Delta_i \quad \dots \quad \Gamma_n \vdash \Delta_n}{(M,v) \not\models \Gamma \vdash \Delta}$$

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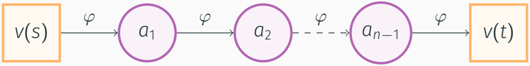

- The proof rules have the following property:

$$\frac{\ldots \quad (M',v') \not\models \Gamma_i, (RTC_{v,w}\,\varphi')(r,u) \vdash \Delta_i \quad \ldots}{(M,v) \not\models \Gamma, (RTC_{x,y}\,\varphi)(s,t) \vdash \Delta}$$

$$\delta_{(RTC_{v,w}\,\varphi')(r,u)}(M',v') \leq \delta_{(RTC_{x,y}\,\varphi)(s,t)}(M,v)$$

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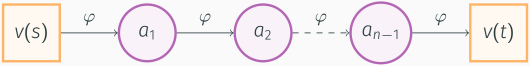

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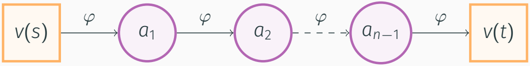

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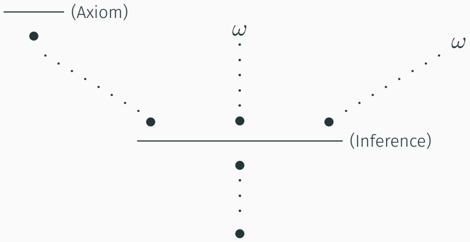
- Global trace condition $\Rightarrow n_1 > n_2 > n_3 > \ldots$

Cut-free Completeness of RTC_G^ω

Obtained using a variation of the standard technique:

1. Construct an infinite (cut-free) pre-proof via an exhaustive search tree
2. If not a valid proof, then it is possible to construct a counter-model
3. Thus search tree gives a valid proof for every valid sequent

CRTC_G^ω: A Cyclic Subsystem



- Restricting to all and only regular infinite pre-proofs gives an effective system

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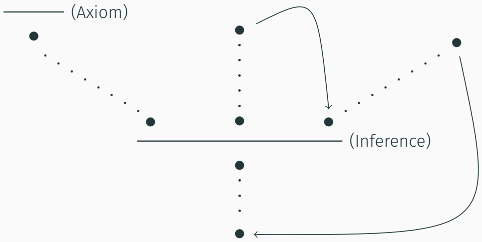

- Restricting to all and only regular infinite pre-proofs gives an effective system
- Regular pre-proofs can be represented as finite, possibly cyclic graphs

Implicit induction subsumes explicit induction

$$
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\dfrac{}{\Gamma, \psi[v/x] \vdash \Delta, \psi[v/x]}\,(\mathsf{Ax})
}{
\Gamma, \psi[v/x], v = w \vdash \Delta, \psi[w/x]
}\,({=}\mathsf{L})
\qquad
\begin{array}{c}
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\quad
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\vdots
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}{
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$$

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$NCRTC^{\omega}_{G}$, the subsystem of non-overlapping cyclic proofs, is a Henkin-complete

Equivalence Under Arithmetic

Obtain RTC_G+A and CRTC_G^ω+A by adding the following schemas:

1. $\mathsf{s}\,\mathbf{0} \vdash$
2. $\mathsf{s}\,x = \mathsf{s}\,y \vdash x = y$
3. $\vdash x + \mathbf{0} = x$
4. $\vdash x + \mathsf{s}\,y = \mathsf{s}\,(x + y)$
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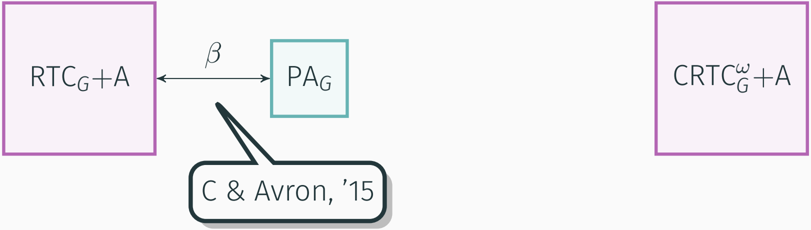

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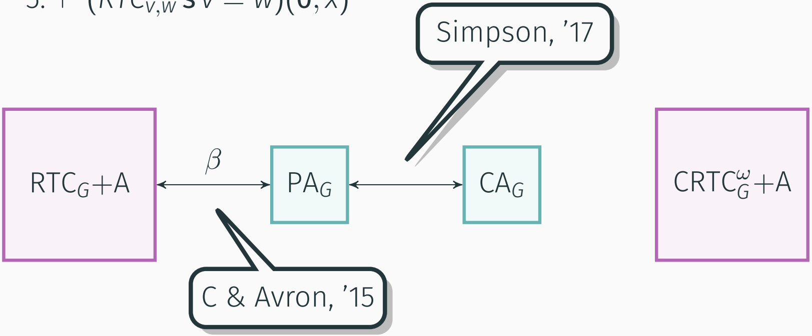

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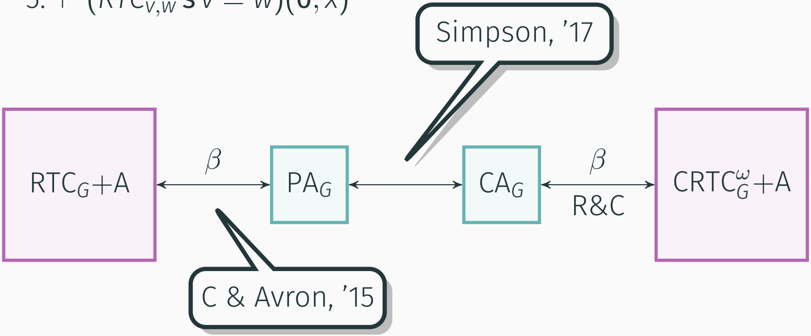

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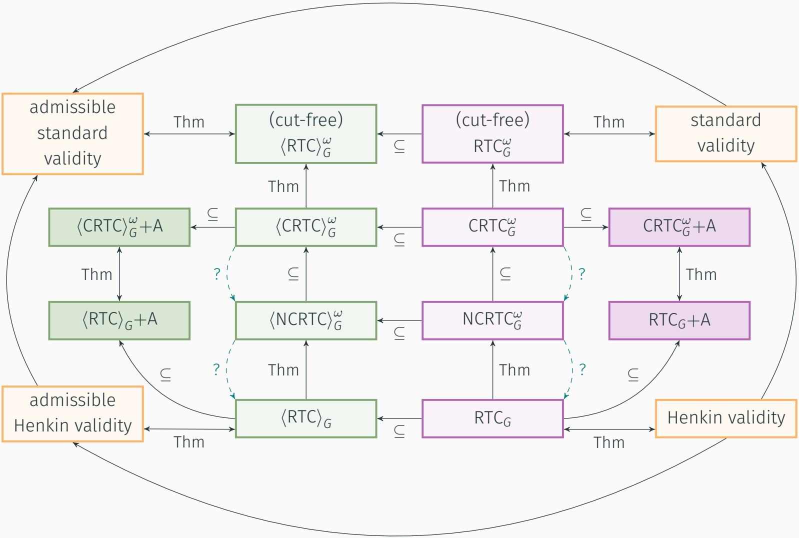
So this does not serve to show RTC_G and CRTC^ω_G inequivalent

- **TC** has all inductive definitions available

Summary of Results



Future Work

- Resolving the open question of the (in)equivalence of RTC_G, NCRTC_G^ω and CRTC_G^ω.
- Implementing CRTC_G^ω and investigating the practicalities of **TC**-logic to support automated inductive reasoning.
- Using the uniformity of **TC**-logic to better study the relationship between implicit and explicit induction.
  - Cuts required in each system
  - Relative complexity of proofs
- A uniform framework for coinductive reasoning?

Recall transitive closure as a fixed point:

$$R^{+} = \mu X.\Psi_R(X) \qquad\qquad \Psi_R(S) = R \cup (R \circ S)$$

The greatest fixed point gives the transitive co-closure

- Pairs (s, t) in $\nu X.\Psi_R(X)$ are those connected by a possibly infinite number of R-steps
- We can write $(RTC^{\mathrm{op}}_{x,y}\,\varphi)(s, t)$ to denote that (s, t) is in the reflexive, transitive co-closure of φ
- E.g. The following formula defines possibly infinite lists

$$(RTC^{\mathrm{op}}_{x,y}\,\exists z\,.\,x = \mathsf{cons}(z, y))(v, [\,])$$

We have the following standard semantics

$$M, v \models (RTC^{\mathrm{op}}_{x,y}\,\varphi)(s, t) \Leftrightarrow$$
$$\exists(\vec{a}_i)_{i\geq 0}\,.\,\forall i \geq 0\,.\,a_i = v(t) \vee M, v[x := a_i, y := a_{i+1}] \models \varphi$$

We have the following Henkin-semantics

$$H, v \models_{\mathcal{H}} (RTC^{\mathrm{op}}_{x,y}\,\varphi)(s, t) \Leftrightarrow$$
$$\text{there exists } A \in \mathcal{D} \text{ such that } v(s) \in A \text{ and}$$
$$\forall a \in A\,.\,\text{either } a = v(t) \text{ or } \exists b \in A\,.\,H, v[x := a, y := b] \models_{\mathcal{H}} \varphi$$