Uniform Inductive Reasoning in Transitive Closure Logic via Infinite Descent

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Computer Science Logic

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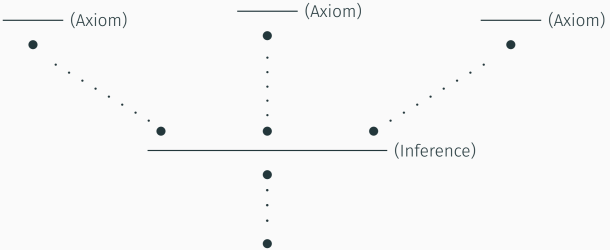
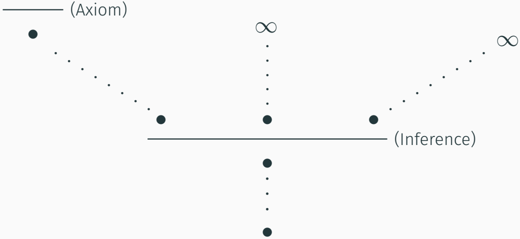
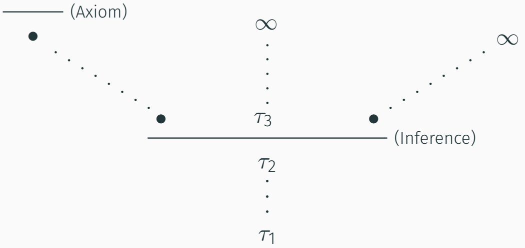
Non-well-founded Proofs: Syntactic Principles



Non-well-founded Proofs: Syntactic Principles

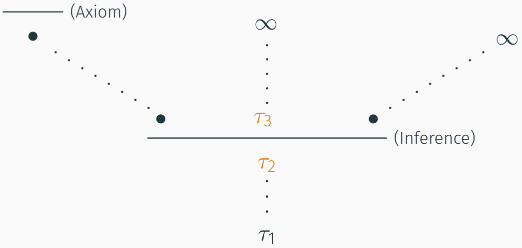


Non-well-founded Proofs: Syntactic Principles



- We trace syntactic elements τ through judgements

Non-well-founded Proofs: Syntactic Principles



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    - At certain points, there is a notion of ‘progression’

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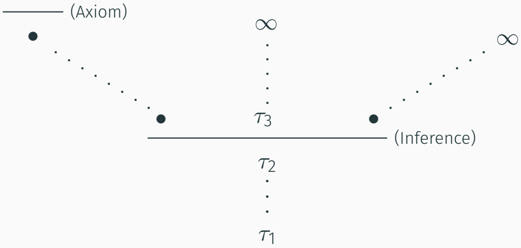

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- Each infinite path must admit some infinite descent

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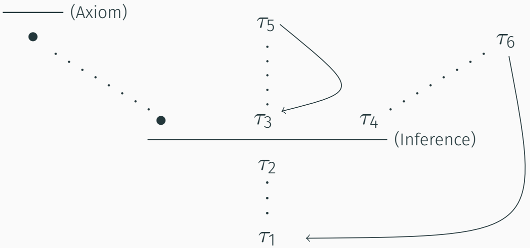
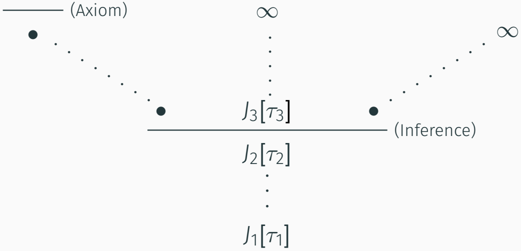


- We trace syntactic elements τ through judgements
  - At certain points, there is a notion of ‘progression’
- Each infinite path must admit some infinite descent
- This global trace condition is an ω-regular property
  - i.e. decidable using Büchi automata

Non-well-founded Proofs: Soundness via Infinite Descent



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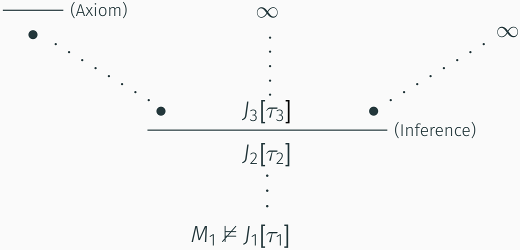

- Assume for contradiction that the conclusion is invalid

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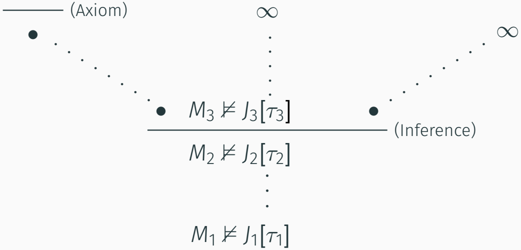

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  - Local soundness $\Rightarrow$ counter-models $M_1, M_2, M_3, \ldots$

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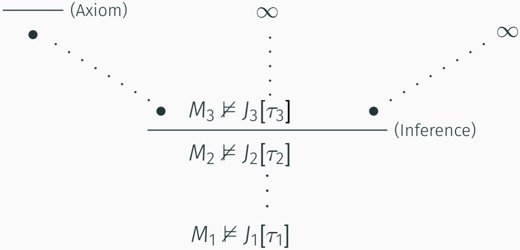

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- We demonstrate a mapping into well-founded $(D, <)$ s.t.
  - $[\![M_1]\!]_{J_1[\tau_1]} \leq [\![M_2]\!]_{J_2[\tau_2]} \leq [\![M_3]\!]_{J_3[\tau_3]} \leq \ldots$

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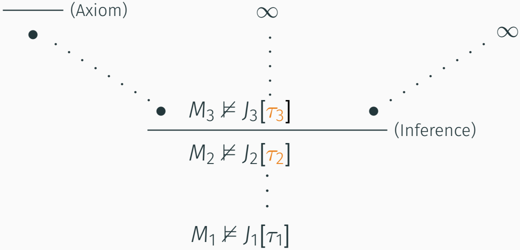

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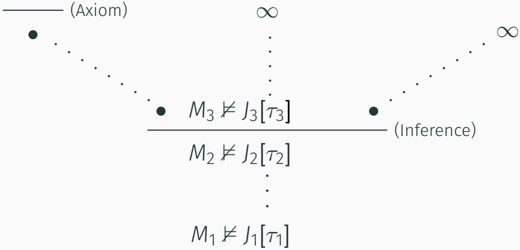

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- Global trace condition $\Rightarrow$ infinitely descending chain in D!

Why Study Non-well-founded Proof Theory?

Non-well-founded/cyclic proof theory allows to:

- Obtain (cut-free) completeness results

  μ-calculus: Fortier&Santocanale, Afshari&Leigh, Doumane Et Al.

  Kleene Algebra: Das&Pous

- Effectively search for proofs of inductive properties
- Automatically verify properties of programs

  [Brotherston, Bornat, Calcagno, Gorogiannis, Peterson, R, Tellez]

- Formally study explicit induction vs infinite descent

  μ-calculus: Santocanale, Sprenger&Dam, Baelde Et Al., Nollet Et Al.

  Ind. Defs: Brotherston&Simpson, Berardi&Tatsuta

  Arithmetic: Simpson, Das

Example: Martin-Löf-style Inductive Predicates in FOL

- We give productions for each 'inductive' predicate P_i

$$\frac{Q_1(\vec{s_1}) \quad \dots \quad Q_n(\vec{s_n})}{P_i(\vec{t})}$$

- We take the smallest interpretation closed under the rules

$$\frac{}{\mathsf{N}\,0} \qquad \frac{\mathsf{N}\,x}{\mathsf{N}\,\mathsf{s}x} \qquad \frac{}{\mathsf{E}\,0} \qquad \frac{\mathsf{O}\,x}{\mathsf{E}\,\mathsf{s}x} \qquad \frac{\mathsf{E}\,x}{\mathsf{O}\,\mathsf{s}x}$$

$$\begin{aligned}
[\![\mathsf{N}]\!] &= \{0, \mathsf{s}0, \mathsf{ss}0, \dots, \mathsf{s}^n\,0, \dots\} \\
[\![\mathsf{E}]\!] &= \{0, \mathsf{ss}0, \dots, \mathsf{s}^{2n}\,0, \dots\} \\
[\![\mathsf{O}]\!] &= \{\mathsf{s}0, \dots, \mathsf{s}^{2n+1}\,0, \dots\}
\end{aligned}$$

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$$[\![\mathsf{N}]\!]_0 = \{ \qquad\qquad\qquad\qquad \}$$

$$[\![\mathsf{E}]\!]_0 = \{ \qquad\qquad\qquad \}$$

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$$[\![\mathsf{N}]\!]_1 = \{0, \qquad\qquad\qquad\qquad \}$$
$$[\![\mathsf{E}]\!]_1 = \{0, \qquad\qquad\qquad \}$$
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$$\begin{aligned}
[\![\mathsf{N}]\!]_2 &= \{0, \mathsf{s}0, \qquad\qquad\qquad\qquad \} \\
[\![\mathsf{E}]\!]_2 &= \{0, \qquad\qquad\qquad\qquad \} \\
[\![\mathsf{O}]\!]_2 &= \{\mathsf{s}0, \qquad\qquad\qquad \}
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$$[\![\mathsf{N}]\!]_3 = \{0, \mathsf{s}0, \mathsf{ss}0, \qquad\qquad \}$$
$$[\![\mathsf{E}]\!]_3 = \{0, \mathsf{ss}0, \qquad\qquad \}$$
$$[\![\mathsf{O}]\!]_3 = \{\mathsf{s}0, \qquad\qquad \}$$

Example: Martin-Löf-style Inductive Predicates in FOL

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$$\llbracket \mathsf{N} \rrbracket_\omega = \{0, \mathsf{s}0, \mathsf{ss}0, \dots, \mathsf{s}^n\,0, \dots\}$$
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Example: A Cyclic Proof

$$
\begin{aligned}
&\Rightarrow \mathrm{N}\,\mathbf{0} \\
\mathrm{N}\,x &\Rightarrow \mathrm{N}\,\mathbf{s}x \\
&\Rightarrow \mathrm{E}\,\mathbf{0} \\
\mathrm{O}\,x &\Rightarrow \mathrm{E}\,\mathbf{s}x \\
\mathrm{E}\,x &\Rightarrow \mathrm{O}\,\mathbf{s}x
\end{aligned}
$$

$$
\dfrac{
 \dfrac{\dfrac{}{\vdash \mathrm{N}\,\mathbf{0}}\;(\mathrm{N\,R_1})}{x = \mathbf{0} \vdash \mathrm{N}\,x}\;(=\mathrm{L})
 \qquad
 \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\mathrm{E}\,x \vdash \mathrm{N}\,x}{\mathrm{E}\,z \vdash \mathrm{N}\,z}\;(\mathrm{Subst})}{\mathrm{E}\,z \vdash \mathrm{N}\,\mathbf{s}z}\;(\mathrm{N\,R_2})}{y = \mathbf{s}z, \mathrm{E}\,z \vdash \mathrm{N}\,y}\;(=\mathrm{L})}{\mathrm{O}\,y \vdash \mathrm{N}\,y}\;(\text{Case O})}{\mathrm{O}\,y \vdash \mathrm{N}\,\mathbf{s}y}\;(\mathrm{N\,R_2})}{x = \mathbf{s}y, \mathrm{O}\,y \vdash \mathrm{N}\,x}\;(=\mathrm{L})
}{\mathrm{E}\,x \vdash \mathrm{N}\,x}\;(\text{Case E})
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Example: A Cyclic Proof

Left unfolding rule

$$
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Example: A Cyclic Proof

Left unfolding rule

$$\Rightarrow \mathrm{N}\,\mathbf{0}$$

$$\mathrm{N}\,x \Rightarrow \mathrm{N}\,\mathbf{s}x$$

$$\Rightarrow \mathrm{E}\,\mathbf{0}$$

$$\mathrm{O}\,x \Rightarrow \mathrm{E}\,\mathbf{s}x$$

$$\mathrm{E}\,x \Rightarrow \mathrm{O}\,\mathbf{s}x$$

$$\dfrac{\dfrac{\dfrac{}{\vdash \mathrm{N}\,\mathbf{0}}\,(\mathrm{N\,R}_1)}{x = \mathbf{0} \vdash \mathrm{N}\,x}\,(=\mathrm{L}) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\mathrm{E}\,x \vdash \mathrm{N}\,x}{\mathrm{E}\,z \vdash \mathrm{N}\,z}\,(\text{Subst})}{\mathrm{E}\,z \vdash \mathrm{N}\,\mathbf{s}z}\,(\mathrm{N\,R}_2)}{y = \mathbf{s}z, \mathrm{E}\,z \vdash \mathrm{N}\,y}\,(=\mathrm{L})}{\mathrm{O}\,y \vdash \mathrm{N}\,y}\,(\text{Case O})}{\mathrm{O}\,y \vdash \mathrm{N}\,\mathbf{s}y}\,(\mathrm{N\,R}_2)}{x = \mathbf{s}y, \mathrm{O}\,y \vdash \mathrm{N}\,x}\,(=\mathrm{L})}{\mathrm{E}\,x \vdash \mathrm{N}\,x}\,(\text{Case E})$$

Right unfolding rule

$$\begin{aligned} &\Rightarrow \mathsf{N}\,\mathbf{0} \\ \mathsf{N}\,x &\Rightarrow \mathsf{N}\,\mathbf{s}x \\ &\Rightarrow \mathsf{E}\,\mathbf{0} \\ \mathsf{O}\,x &\Rightarrow \mathsf{E}\,\mathbf{s}x \\ \mathsf{E}\,x &\Rightarrow \mathsf{O}\,\mathbf{s}x \end{aligned}$$

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Example: A Cyclic Proof

Right unfolding rule

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\end{aligned}
$$

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{$(\mathrm{N}\,\mathrm{R}_1)$}
\UnaryInfC{$\vdash \mathrm{N}\,\mathbf{0}$}
\RightLabel{$(=\mathrm{L})$}
\UnaryInfC{$x = \mathbf{0} \vdash \mathrm{N}\,x$}
\AxiomC{$\mathrm{E}\,x \vdash \mathrm{N}\,x$}
\RightLabel{(Subst)}
\UnaryInfC{$\mathrm{E}\,z \vdash \mathrm{N}\,z$}
\RightLabel{$(\mathrm{N}\,\mathrm{R}_2)$}
\UnaryInfC{$\mathrm{E}\,z \vdash \mathrm{N}\,\mathbf{s}z$}
\RightLabel{$(=\mathrm{L})$}
\UnaryInfC{$y = \mathbf{s}z, \mathrm{E}\,z \vdash \mathrm{N}\,y$}
\RightLabel{(Case O)}
\UnaryInfC{$\mathrm{O}\,y \vdash \mathrm{N}\,y$}
\RightLabel{$(\mathrm{N}\,\mathrm{R}_2)$}
\UnaryInfC{$\mathrm{O}\,y \vdash \mathrm{N}\,\mathbf{s}y$}
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\BinaryInfC{$\mathrm{E}\,x \vdash \mathrm{N}\,x$}
\end{prooftree}
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$$

Cyclic Proof vs Explicit Induction

- To reason explicitly by induction is more complex, involving an induction formula F

$$\frac{\Gamma \vdash \mathrm{IND}_{Q_i}(F) \quad (\forall Q_i \text{ mutually recursive with } P) \quad \Gamma, F(\vec{t}) \vdash \Delta}{\Gamma, \mathrm{P}\,\vec{t} \vdash \Delta}$$

- E.g. the productions $\Rightarrow \mathrm{N}\,\mathbf{0}$ and $\mathrm{N}\,x \Rightarrow \mathrm{N}\,\mathbf{s}x$ give

$$\frac{\Gamma \vdash F(\mathbf{0}) \quad \Gamma, F(x) \vdash F(\mathbf{s}x) \quad \Gamma, F(t) \vdash \Delta}{\Gamma, \mathrm{N}\,t \vdash \Delta}$$

- Implicit induction using unfolding conceptually simpler
    - Induction schemes captured using cycles

Non-well-founded Proofs: Some Meta-theory

For FOL with Inductive Definitions:

- Non-well-founded proof system $LKID^{\omega}$ sound and cut-free complete for standard semantics
- Explicit induction system LKID sound and cut-free complete for a Henkin-style semantics
- Cyclic system $CLKID^{\omega}$ subsumes explicit induction
  [Brotherston & Simpson, LICS'07, JL&C'11]
- $CLKID^{\omega}$ and LKID equivalent under arithmetic
  [Berardi & Tatsuta, LICS'17]
  [Simpson, FoSSaCS'17]
- $CLKID^{\omega}$ and LKID not equivalent in general (2-Hydra counterexample)
  [Berardi & Tatsuta, FoSSaCS'17]

Transitive Closure Logic

Transitive Closure (**TC**) Logic extends FOL with formulas:

- $(RTC_{x,y}\,\varphi)(s,t)$
  - φ is a formula
  - x and y are distinct variables (which become bound in φ)
  - s and t are terms

whose intended meaning is an infinite disjunction

$$
\begin{aligned}
s = t \lor \varphi[s/x, t/y] \\
&\lor (\exists w_1 \,.\, \varphi[s/x, w_1/y] \land \varphi[w_1/x, t/y]) \\
&\lor (\exists w_1, w_2 \,.\, \varphi[s/x, w_1/y] \land \varphi[w_1/x, w_2/y] \land \varphi[w_2/x, t/y]) \\
&\lor \ \ldots
\end{aligned}
$$

The formal semantics:

- M is a (standard) first-order model with domain D
- v is a valuation of terms in M:

$$M, v \models (RTC_{x,y}\, \varphi)(s, t)$$

Transitive Closure Logic

The formal semantics:

- M is a (standard) first-order model with domain D
- v is a valuation of terms in M:

$$M, v \models (RTC_{x,y}\,\varphi)(s,t) \Leftrightarrow$$
$$\exists a_0, \ldots, a_n \in D$$

Transitive Closure Logic

The formal semantics:

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Transitive Closure Logic

The formal semantics:

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&\qquad \wedge M, v[x := a_i, y := a_{i+1}] \models \varphi \quad \text{for all } i < n
\end{aligned}
$$

$$
\boxed{v(s)} \xrightarrow{\varphi} a_1 \xrightarrow{\varphi} a_2 \overset{\varphi}{\dashrightarrow} a_{n-1} \xrightarrow{\varphi} \boxed{v(t)}
$$

Example: Arithmetic in TC

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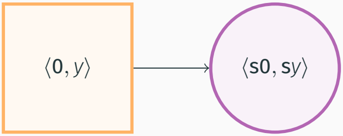

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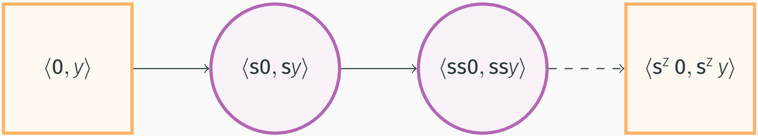

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- The following characterise natural numbers in TC:

$$\forall x \,.\, \mathsf{s}x \neq \mathsf{0}$$

$$\forall x, y \,.\, \mathsf{s}(x) = \mathsf{s}(y) \rightarrow x = y$$

$$\forall x \,.\, \mathsf{Nat}(x)$$

Why Study TC and its Non-well-founded Proof Theory?

- Provides a uniform way to express inductive definitions
    - Single framework for modelling many areas of CS
    - Better for automated reasoning?
- It is a minimal, yet expressive, extension of FOL

Theorem (Avron '03, Thm. 3)

All finitely inductively definable relations† *are definable in* TC.

A. Avron, *Transitive Closure and the Mechanization of Mathematics*.

- Alternative setting for studying cyclic vs explicit induction
    - No need to 'choose' predicates up-front
    - Uniformity makes meta-theory more straightforward
    - Displays some subtle but important differences with FOL+ID

† as formalised in: S. Feferman, *Finitary Inductively Presented Logics*, 1989

Implicit and Explicit Induction Rules for TC

reflexivity
$$\frac{}{\vdash (RTC_{x,y}\,\varphi)(t,t)}$$

step
$$\frac{\Gamma \vdash \Delta, (RTC_{x,y}\,\varphi)(s,r) \quad \Gamma \vdash \Delta, \varphi[r/x, t/y]}{\Gamma \vdash \Delta, (RTC_{x,y}\,\varphi)(s,t)}$$

case-split
$$\frac{\Gamma, s = t \vdash \Delta \quad \Gamma, (RTC_{x,y}\,\varphi)(s,z), \varphi[z/x, t/y] \vdash \Delta}{\Gamma, (RTC_{x,y}\,\varphi)(s,t) \vdash \Delta}\ (z \text{ fresh})$$

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induction
$$\frac{\Gamma \vdash \Delta, \psi[s/x] \quad \Gamma, \psi(x), \varphi(x,y) \vdash \Delta, \psi[y/x] \quad \Gamma, \psi[t/x] \vdash \Delta}{\Gamma, (RTC_{x,y}\,\varphi)(s,t) \vdash \Delta}$$

$x \notin \mathbf{fv}(\Gamma, \Delta)$ and $y \notin \mathbf{fv}(\Gamma, \Delta, \psi)$

Proof-theoretic Results for TC

- Non-well-founded system RTC_G^ω sound + cut-free complete for standard semantics
- Explicit induction system RTC_G sound + cut-free complete for a Henkin-style semantics
- Cyclic system subsumes explicit induction
$$\mathrm{RTC}_G \subseteq \mathrm{NCRTC}_G^\omega \text{ (non-overlapping cycles)} \subseteq \mathrm{CRTC}_G^\omega$$

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$$\boxed{\mathrm{RTC}_G{+}\mathrm{A}} \longleftrightarrow \boxed{\mathrm{CRTC}_G^\omega{+}\mathrm{A}}$$

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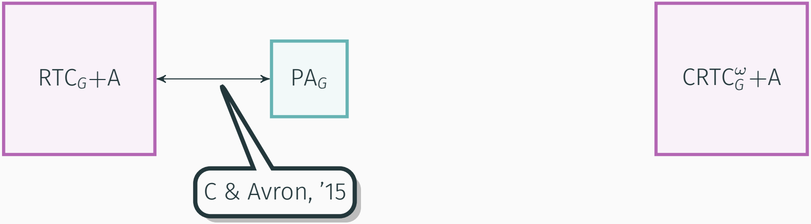

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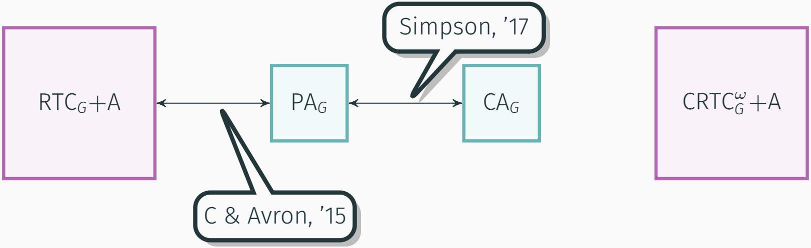

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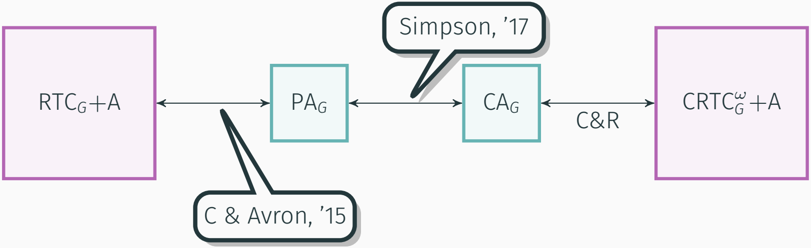

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  $$\mathsf{RTC}_G \subseteq \mathsf{NCRTC}_G^\omega \text{ (non-overlapping cycles)} \subseteq \mathsf{CRTC}_G^\omega$$
- Systems with arithmetic are equivalent
- 2-Hydra counterexample does not show $\mathsf{RTC}_G \subsetneq \mathsf{CRTC}_G^\omega$
  - Relies on not being able to express ordering on numbers
  - **TC** allows all inductive definitions 'at once'

Future Work

- open question of equivalence for RTC_G, NCRTC_G^ω and CRTC_G^ω
- Implementing CRTC_G^ω to support automated reasoning.
- Use TC to better study implicit vs explicit induction.
- Adapt TC for coinductive reasoning?

(Non-reflexive) transitive closure is a least fixed point

$$R^+ = \mu X.\Psi_R(X) \qquad\qquad \Psi_R(S) = R \cup (R \circ S)$$

The greatest fixed point gives the transitive co-closure

- Pairs (s, t) in $\nu X.\Psi_R(X)$ are those connected by a possibly infinite number of R-steps
- We can write $(RTC^{\mathrm{op}}_{x,y}\,\varphi)(s, t)$ to denote that (s, t) is in the reflexive, transitive co-closure of φ

We have the following standard semantics

$$M, v \models (RTC^{\mathrm{op}}_{x,y}\, \varphi)(s,t) \Leftrightarrow$$
$$\exists(\vec{a}_i)_{i \geq 0} \,.\, \forall i \geq 0 \,.\, a_i = v(t) \vee M, v[x := a_i, y := a_{i+1}] \models \varphi$$

E.g. The following formula defines possibly infinite lists

$$(RTC^{\mathrm{op}}_{x,y}\, \exists z \,.\, x = \mathsf{cons}(z, y))(v, [\,])$$