Realizability in Cyclic Proof

Extracting Ordering Information for Infinite Descent

Reuben N. S. Rowe [1] James Brotherston [2]

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Motivation: Program Termination

```
struct ll { int data; ll *next; }

void rev(ll *x)  { /* reverses list */ }
void shuffle(ll *x)           {
   if ( x != NULL ) {

     ll *y = x -> next;

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$\texttt{void shuffle(ll *x)}\ \{\, \mathsf{list}_\alpha(\texttt{x}) \,\}\ \texttt{\{}$

$\quad \texttt{if (x != NULL) \{}$

$\qquad \{\, \mathsf{list}_\beta(\texttt{x->next}) \wedge \beta < \alpha \,\}$

$\qquad \texttt{ll *y = x -> next;}$

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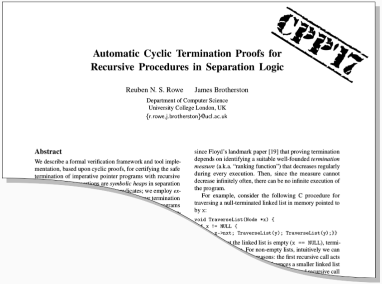
CPP'17

Automatic Cyclic Termination Proofs for Recursive Procedures in Separation Logic

Reuben N. S. Rowe James Brotherston

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Abstract

We describe a formal verification framework and tool implementation, based upon cyclic proofs, for certifying the safe termination of imperative pointer programs with recursive

since Floyd's landmark paper [19] that proving termination depends on identifying a suitable well-founded *termination measure* (a.k.a. "ranking function") that decreases regularly during every execution. Then, since the measure cannot decrease infinitely often, there can be no infinite execution of the program.

For example, consider the following C procedure for traversing a null-terminated linked list in memory pointed to by x:

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void TraverseList(Node *x) {
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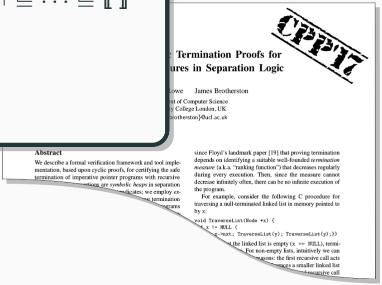
$$[\![\cdot]\!] : \texttt{Formulas} \rightarrow \wp(\texttt{Models})$$

$$[\![\cdot]\!]_\perp \sqsubseteq [\![\cdot]\!]_1 \sqsubseteq \ldots [\![\cdot]\!]_\alpha \sqsubseteq [\![\cdot]\!]_{\alpha+1} \sqsubseteq \ldots \sqsubseteq [\![\cdot]\!]$$



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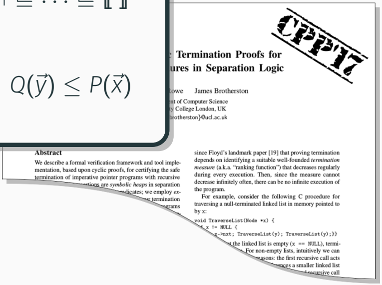
$\forall \alpha \,.\, [\![P(\vec{x})]\!]_\alpha \subseteq [\![Q(\vec{y})]\!]_\alpha \quad \equiv \quad Q(\vec{y}) \leq P(\vec{x})$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Termination Proofs for
ures in Separation Logic

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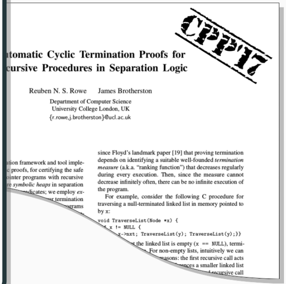
Intra-procedural analysis produces verification conditions, in the form of *entailments*, e.g.

$$x \neq \texttt{NULL} \wedge y = x\texttt{->next} \wedge \mathrm{list}(y) \vdash \mathrm{list}(x)$$



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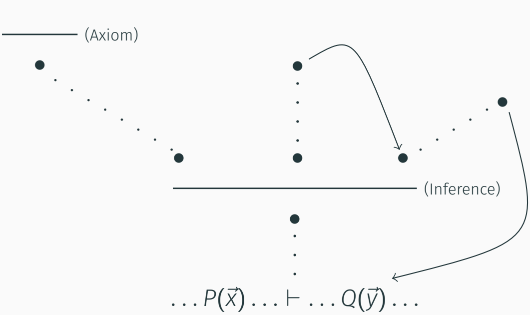

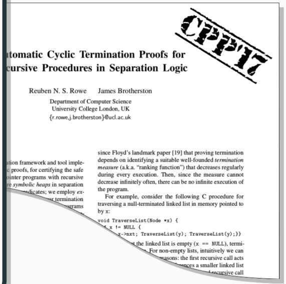

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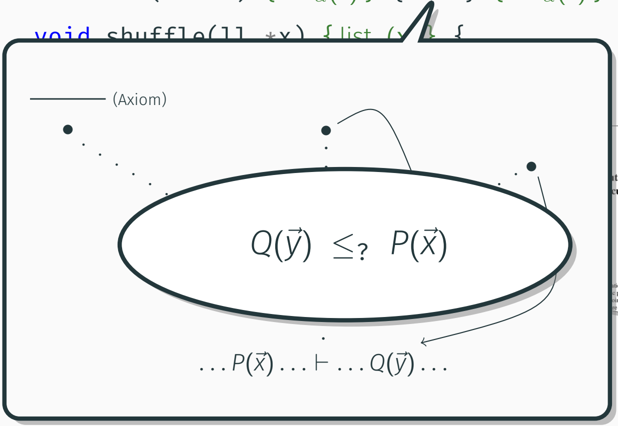

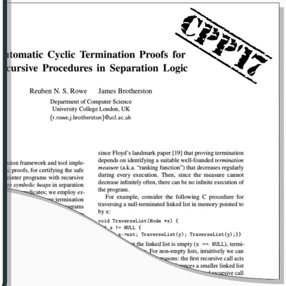
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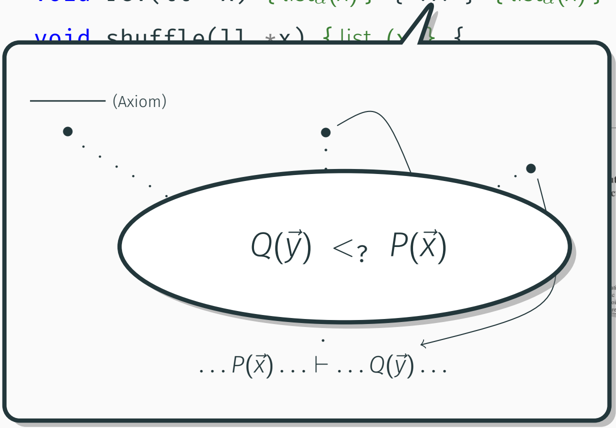

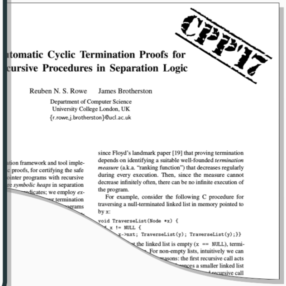

Overview of Results

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  - These inclusions hold when the proof graph satisfies a structural (realizability) condition that we define
- The realizability condition is equivalent to a containment between two weighted automata that can be constructed from the proof graph
  - Under certain extra structural conditions, this containment falls within existing decidability results

A Cyclic Proof in LK Sequent Calculus with Equality

$$\Rightarrow\ N\,0$$

$$N\,x \Rightarrow\ N\,sx$$

$$\Rightarrow\ E\,0$$

$$O\,x \Rightarrow\ E\,sx$$

$$E\,x \Rightarrow\ O\,sx$$

$$\dfrac{\dfrac{\dfrac{}{\vdash\ N\,0}\,(N\,R_1)}{x = 0 \vdash N\,x}\,(=L) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{E\,x \vdash N\,x}{E\,z \vdash\ N\,z}\,(\text{Subst})}{E\,z \vdash\ N\,sz}\,(N\,R_2)}{y = sz,\ E\,z \vdash N\,y}\,(=L)}{O\,y \vdash\ N\,y}\,(\text{Case O})}{O\,y \vdash\ N\,sy}\,(N\,R_2)}{x = sy,\ O\,y \vdash N\,x}\,(=L)}{E\,x \vdash N\,x}\,(\text{Case E})$$

A Cyclic Proof in LK Sequent Calculus with Equality

Left unfolding rule

$$\Rightarrow \mathrm{N}\,0$$
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$$\Rightarrow \mathrm{E}\,0$$
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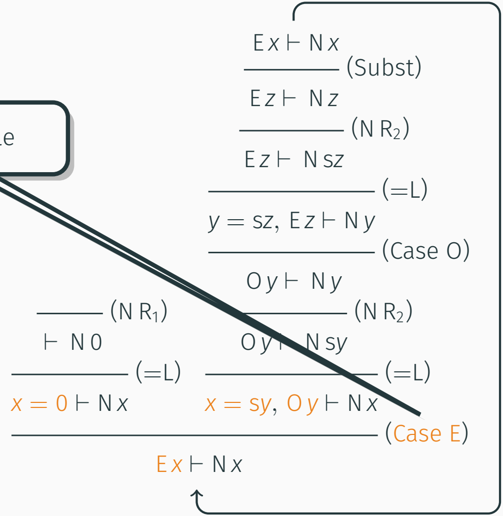

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Right unfolding rule

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Right unfolding rule

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A Cyclic Proof in LK Sequent Calculus with Equality

> A cyclic proof graph is globally sound when every infinite path (going from conclusion to premise) is eventually followed by a trace of predicate formulas (on the left-hand side of sequents) which progresses (through a case-split) infinitely often

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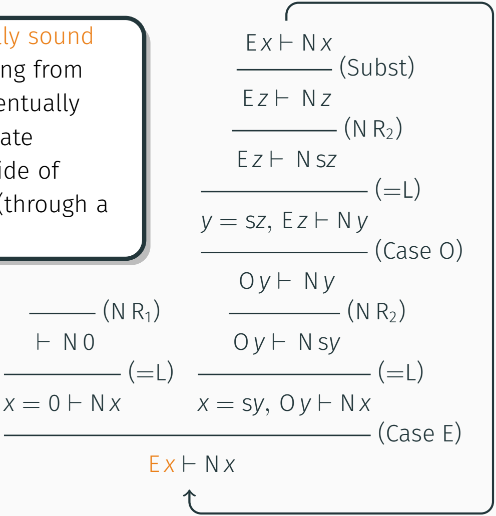

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$$\dfrac{\dfrac{\dfrac{}{\vdash \mathsf{N}\,0}\,(\mathsf{N}\,R_1)}{x = 0 \vdash \mathsf{N}\,x}\,(=L) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\mathsf{E}\,x \vdash \mathsf{N}\,x}{\mathsf{E}\,z \vdash \mathsf{N}\,z}\,(\mathrm{Subst})}{\mathsf{E}\,z \vdash \mathsf{N}\,\mathsf{s}z}\,(\mathsf{N}\,R_2)}{y = \mathsf{s}z,\ \mathsf{E}\,z \vdash \mathsf{N}\,y}\,(=L)}{\mathsf{O}\,y \vdash \mathsf{N}\,y}\,(\mathrm{Case}\ \mathsf{O})}{\mathsf{O}\,y \vdash \mathsf{N}\,\mathsf{s}y}\,(\mathsf{N}\,R_2)}{x = \mathsf{s}y,\ \mathsf{O}\,y \vdash \mathsf{N}\,x}\,(=L)}{\mathsf{E}\,x \vdash \mathsf{N}\,x}\,(\mathrm{Case}\ \mathsf{E})$$

A Cyclic Proof in LK Sequent Calculus with Equality

A cyclic proof graph is globally sound when every infinite path (going from conclusion to premise) is eventually followed by a trace of predicate formulas (on the left-hand side of sequents) which progresses (through a case-split) infinitely often

$$\Rightarrow \mathsf{N}\,0$$
$$\mathsf{N}\,x \Rightarrow \mathsf{N}\,\mathsf{s}x$$
$$\Rightarrow \mathsf{E}\,0$$
$$\mathsf{O}\,x \Rightarrow \mathsf{E}\,\mathsf{s}x$$
$$\mathsf{E}\,x \Rightarrow \mathsf{O}\,\mathsf{s}x$$

$$\dfrac{\mathsf{E}\,x \vdash \mathsf{N}\,x}{\mathsf{E}\,z \vdash \mathsf{N}\,z}\,(\text{Subst})$$
$$\dfrac{\mathsf{E}\,z \vdash \mathsf{N}\,z}{\mathsf{E}\,z \vdash \mathsf{N}\,\mathsf{s}z}\,(\mathsf{N}\,\mathsf{R}_2)$$
$$\dfrac{\mathsf{E}\,z \vdash \mathsf{N}\,\mathsf{s}z}{y = \mathsf{s}z,\ \mathsf{E}\,z \vdash \mathsf{N}\,y}\,(=\text{L})$$
$$\dfrac{y = \mathsf{s}z,\ \mathsf{E}\,z \vdash \mathsf{N}\,y}{\mathsf{O}\,y \vdash \mathsf{N}\,y}\,(\text{Case O})$$
$$\dfrac{\mathsf{O}\,y \vdash \mathsf{N}\,y}{\mathsf{O}\,y \vdash \mathsf{N}\,\mathsf{s}y}\,(\mathsf{N}\,\mathsf{R}_2)$$
$$\dfrac{}{\vdash \mathsf{N}\,0}\,(\mathsf{N}\,\mathsf{R}_1)$$
$$\dfrac{\vdash \mathsf{N}\,0}{x = 0 \vdash \mathsf{N}\,x}\,(=\text{L}) \qquad \dfrac{\mathsf{O}\,y \vdash \mathsf{N}\,\mathsf{s}y}{x = \mathsf{s}y,\ \mathsf{O}\,y \vdash \mathsf{N}\,x}\,(=\text{L})$$
$$\dfrac{x = 0 \vdash \mathsf{N}\,x \qquad x = \mathsf{s}y,\ \mathsf{O}\,y \vdash \mathsf{N}\,x}{\mathsf{E}\,x \vdash \mathsf{N}\,x}\,(\text{Case E})$$

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$$\Rightarrow \mathsf{E}\,0$$
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$$\dfrac{\dfrac{\dfrac{}{\vdash \mathsf{N}\,0}\,(\mathsf{N}\,R_1)}{x = 0 \vdash \mathsf{N}\,x}\,(=L) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\mathsf{E}\,x \vdash \mathsf{N}\,x}{\mathsf{E}\,z \vdash \mathsf{N}\,z}\,(\text{Subst})}{\mathsf{E}\,z \vdash \mathsf{N}\,sz}\,(\mathsf{N}\,R_2)}{y = sz,\ \mathsf{E}\,z \vdash \mathsf{N}\,y}\,(=L)}{\mathsf{O}\,y \vdash \mathsf{N}\,y}\,(\text{Case O})}{\mathsf{O}\,y \vdash \mathsf{N}\,sy}\,(\mathsf{N}\,R_2)}{x = sy,\ \mathsf{O}\,y \vdash \mathsf{N}\,x}\,(=L)}{\mathsf{E}\,x \vdash \mathsf{N}\,x}\,(\text{Case E})$$

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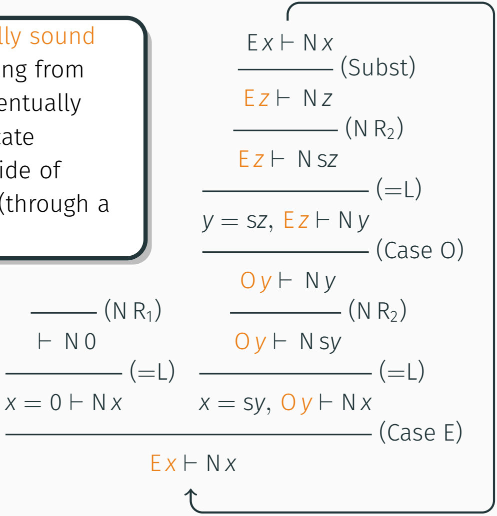

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Inductive Predicate Definitions and their Semantics

Definition (Inductive Definition Set)

An *inductive definition set* contains productions $\mathrm{P}_1\, \vec{t_1}, \ldots, \mathrm{P}_j\, \vec{t_j} \Rightarrow \mathrm{P}_0\, \vec{t_0}$

Definition (Characteristic Operators)

Inductive definition sets Φ induce *characteristic operators* φ_Φ on predicate interpretations X (functions from predicate formulas to sets of models):

$$\varphi_\Phi(X)(\mathrm{P}\, \vec{t}\theta) = \{m \mid \mathrm{P}_1\, \vec{t_1}, \ldots, \mathrm{P}_j\, \vec{t_j} \Rightarrow \mathrm{P}\, \vec{t} \in \Phi,\ m \in X(\mathrm{P}_i\, \vec{t_i}\theta) \text{ for all } 1 \leq i \leq j\}$$

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Characteristic operators φ_Φ are monotone wrt $\sqsubseteq$

We interpret predicates using the least fixed point, $[\![\cdot]\!]_\Phi \stackrel{def}{=} \mu X.\varphi_\Phi(X)$

$$X_\perp \sqsubseteq \varphi_\Phi(X_\perp) \sqsubseteq \varphi_\Phi(\varphi_\Phi(X_\perp)) \sqsubseteq \ldots \sqsubseteq \varphi_\Phi^\alpha(X_\perp) \sqsubseteq \ldots \sqsubseteq \mu X.\varphi_\Phi(X)$$

Inductive Predicate Definitions and their Semantics

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Characteristic operators φ_Φ are monotone wrt $\sqsubseteq$

We interpret predicates using the least fixed point, $[\![\cdot]\!]_\Phi \stackrel{def}{=} \mu X.\varphi_\Phi(X)$

$$[\![\cdot]\!]^\Phi_0 \sqsubseteq [\![\cdot]\!]^\Phi_1 \sqsubseteq [\![\cdot]\!]^\Phi_2 \sqsubseteq \ldots \sqsubseteq [\![\cdot]\!]^\Phi_\alpha \sqsubseteq \ldots [\![\cdot]\!]^\Phi$$

Cyclic Proof Formalises Infinite Descent

- Suppose, for contradiction, that the conclusion of the proof is not valid
    - That is, there is a counter-model of the sequent

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- By local soundness of the inference rules, we obtain an infinite sequence of counter-models for some infinite path in the proof
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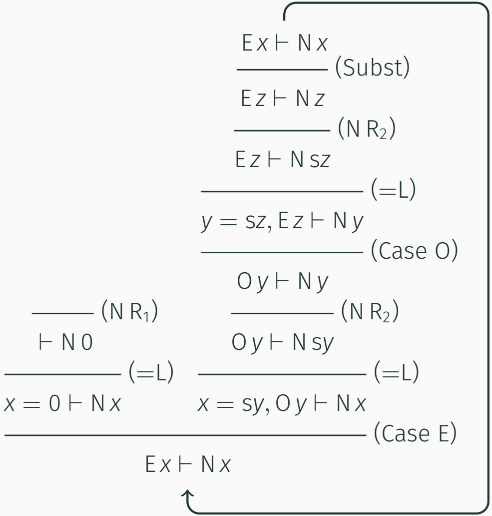
  - Each model can be mapped to an ever smaller approximation $[\![\mathsf{P}\,\vec{t}]\!]^{\Phi}_{\alpha}$ in which it appears
  - These strictly decrease over a case-split
- By global soundness of the proof, this gives an infinitely descending chain in $(\mathcal{X}, \sqsubseteq)$
  - But $(\mathcal{X}, \sqsubseteq)$ is a well-ordered set $\Rightarrow$ contradiction!

$$\Rightarrow \mathsf{N}\,0$$
$$\mathsf{N}\,x \Rightarrow \mathsf{N}\,\mathsf{s}x$$
$$\Rightarrow \mathsf{E}\,0$$
$$\mathsf{O}\,x \Rightarrow \mathsf{E}\,\mathsf{s}x$$
$$\mathsf{E}\,x \Rightarrow \mathsf{O}\,\mathsf{s}x$$



Extracting Semantic Orderings from Cyclic Proofs

The inductive definitions/semantics give immediately, e.g.

$$\forall m, \alpha : m \in [\![\mathsf{O}\, \mathsf{s}x]\!]_\alpha \Rightarrow m \in [\![\mathsf{E}\, x]\!]_\alpha$$

$\Rightarrow \mathsf{N}\,0$

$\mathsf{N}\,x \Rightarrow \mathsf{N}\,\mathsf{s}x$

$\Rightarrow \mathsf{E}\,0$

$\mathsf{O}\,x \Rightarrow \mathsf{E}\,\mathsf{s}x$

$\mathsf{E}\,x \Rightarrow \mathsf{O}\,\mathsf{s}x$

$$\dfrac{\dfrac{\dfrac{}{\vdash \mathsf{N}\,0}(\mathsf{N}\,\mathsf{R}_1)}{x = 0 \vdash \mathsf{N}\,x}(=\mathsf{L}) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\mathsf{E}\,x \vdash \mathsf{N}\,x}{\mathsf{E}\,z \vdash \mathsf{N}\,z}(\text{Subst})}{\mathsf{E}\,z \vdash \mathsf{N}\,\mathsf{s}z}(\mathsf{N}\,\mathsf{R}_2)}{y = \mathsf{s}z, \mathsf{E}\,z \vdash \mathsf{N}\,y}(=\mathsf{L})}{\mathsf{O}\,y \vdash \mathsf{N}\,y}(\text{Case O})}{\mathsf{O}\,y \vdash \mathsf{N}\,\mathsf{s}y}(\mathsf{N}\,\mathsf{R}_2)}{x = \mathsf{s}y, \mathsf{O}\,y \vdash \mathsf{N}\,x}(=\mathsf{L})}{\mathsf{E}\,x \vdash \mathsf{N}\,x}(\text{Case E})$$

The inductive definitions/semantics give immediately, e.g.

$$\forall m, \alpha : m \in [\![\mathsf{O}\,\mathsf{s}x]\!]_\alpha \Rightarrow m \in [\![\mathsf{E}\,x]\!]_\alpha$$

and even

$$\forall m, \alpha : m \in [\![\mathsf{O}\,\mathsf{s}x]\!]_\alpha \Rightarrow \exists \beta < \alpha . m \in [\![\mathsf{E}\,x]\!]_\beta$$

$\Rightarrow \mathsf{N}\,0$

$\mathsf{N}\,x \Rightarrow \mathsf{N}\,\mathsf{s}x$

$\Rightarrow \mathsf{E}\,0$

$\mathsf{O}\,x \Rightarrow \mathsf{E}\,\mathsf{s}x$

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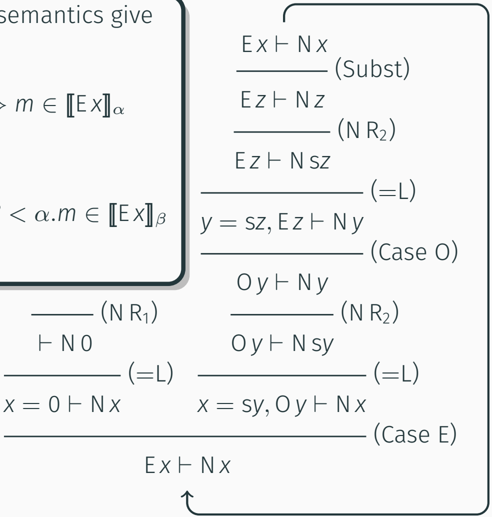

Extracting Semantic Orderings from Cyclic Proofs

The global soundness already gives

$$\forall m : m \in [\![\mathsf{E}\,x]\!] \Rightarrow m \in [\![\mathsf{N}\,x]\!]$$

$\Rightarrow \mathsf{N}\,0$

$\mathsf{N}\,x \Rightarrow \mathsf{N}\,\mathsf{s}x$

$\Rightarrow \mathsf{E}\,0$

$\mathsf{O}\,x \Rightarrow \mathsf{E}\,\mathsf{s}x$

$\mathsf{E}\,x \Rightarrow \mathsf{O}\,\mathsf{s}x$

$$\dfrac{\dfrac{\dfrac{}{\vdash \mathsf{N}\,0}\,(\mathsf{N}\,\mathsf{R}_1)}{x = 0 \vdash \mathsf{N}\,x}\,(=\mathsf{L}) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\mathsf{E}\,x \vdash \mathsf{N}\,x}{\mathsf{E}\,z \vdash \mathsf{N}\,z}\,(\mathsf{Subst})}{\mathsf{E}\,z \vdash \mathsf{N}\,\mathsf{s}z}\,(\mathsf{N}\,\mathsf{R}_2)}{y = \mathsf{s}z, \mathsf{E}\,z \vdash \mathsf{N}\,y}\,(=\mathsf{L})}{\mathsf{O}\,y \vdash \mathsf{N}\,y}\,(\mathsf{Case}\ \mathsf{O})}{\mathsf{O}\,y \vdash \mathsf{N}\,\mathsf{s}y}\,(\mathsf{N}\,\mathsf{R}_2)}{x = \mathsf{s}y, \mathsf{O}\,y \vdash \mathsf{N}\,x}\,(=\mathsf{L})}{\mathsf{E}\,x \vdash \mathsf{N}\,x}\,(\mathsf{Case}\ \mathsf{E})$$

Extracting Semantic Orderings from Cyclic Proofs

The global soundness already gives

$$\forall m : m \in [\![\mathrm{E}\,x]\!] \Rightarrow m \in [\![\mathrm{N}\,x]\!]$$

but we would also like to know whether

$$\forall \alpha \forall m : m \in [\![\mathrm{E}\,x]\!]_\alpha \Rightarrow m \in [\![\mathrm{N}\,x]\!]_\alpha$$

$\Rightarrow \mathrm{N}\,0$

$\mathrm{N}\,x \Rightarrow \mathrm{N}\,\mathrm{s}x$

$\Rightarrow \mathrm{E}\,0$

$\mathrm{O}\,x \Rightarrow \mathrm{E}\,\mathrm{s}x$

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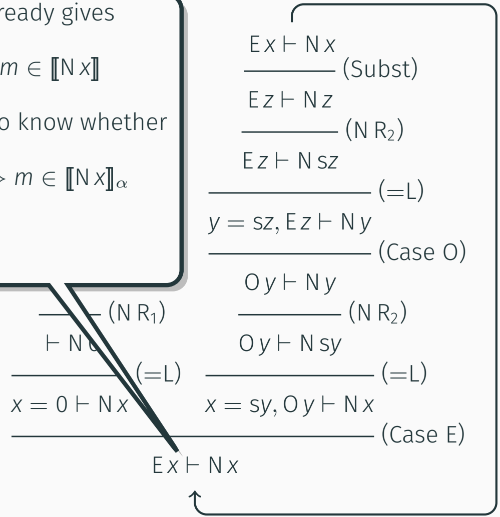

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$$\forall m : m \in [\![\mathsf{E}\,x]\!] \Rightarrow m \in [\![\mathsf{N}\,x]\!]$$

but we would also like to know whether

$$\forall \alpha \forall m : m \in [\![\mathsf{E}\,x]\!]_\alpha \Rightarrow m \in [\![\mathsf{N}\,x]\!]_\alpha$$

i.e. $\mathsf{N}\,x \leq \mathsf{E}\,x$

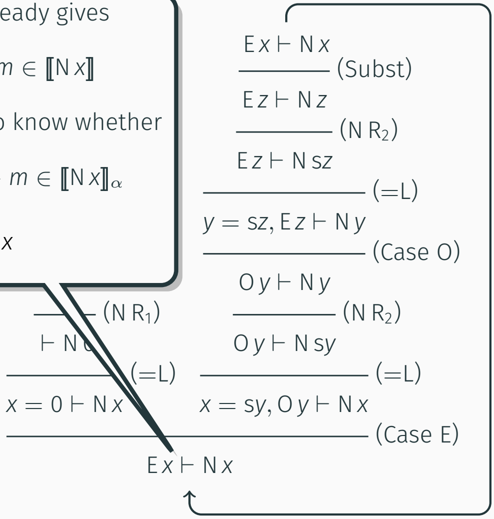
$\Rightarrow \mathsf{N}\,0$

$\mathsf{N}\,x \Rightarrow \mathsf{N}\,\mathsf{s}x$

$\Rightarrow \mathsf{E}\,0$

$\mathsf{O}\,x \Rightarrow \mathsf{E}\,\mathsf{s}x$

$\mathsf{E}\,x \Rightarrow \mathsf{O}\,\mathsf{s}x$



Extracting Semantic Orderings: Basic Ideas

To extract these semantic relationships from cyclic proofs:

- We have to consider traces along the right-hand side of sequents, which are
  - maximally finite
  - matched by some left-hand trace along the same path
- We then count the number of times each trace progresses
  - the left-hand one must progress at least as often as the right-hand one

$$\Rightarrow \mathrm{N}\,0$$
$$\mathrm{N}\,x \Rightarrow \mathrm{N}\,\mathrm{s}x$$
$$\Rightarrow \mathrm{E}\,0$$
$$\mathrm{O}\,x \Rightarrow \mathrm{E}\,\mathrm{s}x$$
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Extracting Semantic Orderings: Example I

$\Rightarrow \mathsf{N}\,0$

$\mathsf{N}\,x \Rightarrow \mathsf{N}\,\mathsf{s}x$

$\Rightarrow \mathsf{E}\,0$

$\mathsf{O}\,x \Rightarrow \mathsf{E}\,\mathsf{s}x$

$\mathsf{E}\,x \Rightarrow \mathsf{O}\,\mathsf{s}x$

$\mathsf{N}\,\mathsf{ss}0 \Rightarrow \mathsf{O}\,\mathsf{sss}0$

$$\dfrac{\dfrac{\dfrac{}{\vdash \mathsf{N}\,0}\,(\mathsf{N}\,\mathsf{R}_1)}{x = 0 \vdash \mathsf{N}\,x}\,(=\mathsf{L}) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\mathsf{N}\,\mathsf{ss}0 \vdash \mathsf{N}\,\mathsf{ss}0}\,(\mathsf{Ax})}{\mathsf{N}\,\mathsf{ss}0 \vdash \mathsf{N}\,\mathsf{sss}0}\,(\mathsf{N}\,\mathsf{R}_2)}{y = \mathsf{sss}0, \mathsf{N}\,\mathsf{ss}0 \vdash \mathsf{N}\,y}\,(=\mathsf{L}) \qquad \dfrac{\dfrac{\dfrac{\mathsf{E}\,x \vdash \mathsf{N}\,x}{\mathsf{E}\,z \vdash \mathsf{N}\,z}\,(\mathsf{Subst})}{\mathsf{E}\,z \vdash \mathsf{N}\,\mathsf{s}z}\,(\mathsf{N}\,\mathsf{R}_2)}{y = \mathsf{s}z, \mathsf{E}\,z \vdash \mathsf{N}\,y}\,(=\mathsf{L})}{\mathsf{O}\,y \vdash \mathsf{N}\,y}\,(\mathsf{Case}\ \mathsf{O})}{\mathsf{O}\,y \vdash \mathsf{N}\,\mathsf{s}y}\,(\mathsf{N}\,\mathsf{R}_2)}{x = \mathsf{s}y, \mathsf{O}\,y \vdash \mathsf{N}\,x}\,(=\mathsf{L})}{\mathsf{E}\,x \vdash \mathsf{N}\,x}\,(\mathsf{Case}\ \mathsf{E})$$

Extracting Semantic Orderings: Example I

$$\Rightarrow N\,0$$
$$N\,x \Rightarrow N\,sx$$
$$\Rightarrow E\,0$$
$$O\,x \Rightarrow E\,sx$$
$$E\,x \Rightarrow O\,sx$$
$$N\,ss0 \Rightarrow O\,sss0$$

$$\dfrac{\dfrac{\dfrac{}{\vdash N\,0}\,(N\,R_1)}{x = 0 \vdash N\,x}\,(=L) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{N\,ss0 \vdash N\,ss0}\,(Ax)}{N\,ss0 \vdash N\,sss0}\,(N\,R_2)}{y = sss0, N\,ss0 \vdash N\,y}\,(=L) \qquad \dfrac{\dfrac{\dfrac{E\,x \vdash N\,x}{E\,z \vdash N\,z}\,(Subst)}{E\,z \vdash N\,sz}\,(N\,R_2)}{y = sz, E\,z \vdash N\,y}\,(=L)}{O\,y \vdash N\,y}\,(Case\ O)}{O\,y \vdash N\,sy}\,(N\,R_2)}{x = sy, O\,y \vdash N\,x}\,(=L)}{E\,x \vdash N\,x}\,(Case\ E)$$

Extracting Semantic Orderings: Example I

$$
\begin{aligned}
&\Rightarrow N\,0 \\
N\,x &\Rightarrow N\,sx \\
&\Rightarrow E\,0 \\
O\,x &\Rightarrow E\,sx \\
E\,x &\Rightarrow O\,sx \\
N\,ss0 &\Rightarrow O\,sss0
\end{aligned}
$$

$$
\dfrac{
\dfrac{\dfrac{}{\vdash N\,0}\,(N\,R_1)}{x = 0 \vdash N\,x}\,(=L)
\qquad
\dfrac{\dfrac{\dfrac{
\dfrac{\dfrac{\dfrac{}{N\,ss0 \vdash N\,ss0}\,(\mathrm{Ax})}{N\,ss0 \vdash N\,sss0}\,(N\,R_2)}{y = sss0, N\,ss0 \vdash N\,y}\,(=L)
\quad
\dfrac{\dfrac{\dfrac{E\,x \vdash N\,x}{E\,z \vdash N\,z}\,(\mathrm{Subst})}{E\,z \vdash N\,sz}\,(N\,R_2)}{y = sz, E\,z \vdash N\,y}\,(=L)
}{O\,y \vdash N\,y}\,(\mathrm{Case}\ O)}{O\,y \vdash N\,sy}\,(N\,R_2)}{x = sy, O\,y \vdash N\,x}\,(=L)
}{E\,x \vdash N\,x}\,(\mathrm{Case}\ E)
$$

$\Rightarrow \mathrm{N}\,0$

$\mathrm{N}\,x \Rightarrow \mathrm{N}\,sx$

$\Rightarrow \mathrm{E}\,0$

$\mathrm{O}\,x \Rightarrow \mathrm{E}\,sx$

$\mathrm{E}\,x \Rightarrow \mathrm{O}\,sx$

$\mathrm{N}\,ss0 \Rightarrow \mathrm{O}\,sss0$

This trace is

- fully maximal: the final predicate is introduced by its rule
- grounded: the final predicate is derived from a zero premise production (N.B. $\forall m : m \in [\![\mathrm{N}\,0]\!]_1$)

$x \vdash \mathrm{N}\,x$ (Subst)

$z \vdash \mathrm{N}\,z$ $(\mathrm{N}\,\mathrm{R}_2)$

$\vdash \mathrm{N}\,sz$ (=L)

$z, \mathrm{E}\,z \vdash \mathrm{N}\,y$ (Case O)

$\mathrm{O}\,y \vdash \mathrm{N}\,y$ $(\mathrm{N}\,\mathrm{R}_2)$

$\mathrm{O}\,y \vdash \mathrm{N}\,sy$ (=L)

$x = sy, \mathrm{O}\,y \vdash \mathrm{N}\,x$

$(\mathrm{N}\,\mathrm{R}_1)$

$\vdash \mathrm{N}\,0$ (=L)

$x = 0 \vdash \mathrm{N}\,x$

(Case E)

$\mathrm{E}\,x \vdash \mathrm{N}\,x$

Extracting Semantic Orderings: Example I

$$\Rightarrow \mathsf{N}\,0$$
$$\mathsf{N}\,x \Rightarrow \mathsf{N}\,sx$$
$$\Rightarrow \mathsf{E}\,0$$
$$\mathsf{O}\,x \Rightarrow \mathsf{E}\,sx$$
$$\mathsf{E}\,x \Rightarrow \mathsf{O}\,sx$$
$$\mathsf{N}\,ss0 \Rightarrow \mathsf{O}\,sss0$$

$$\dfrac{}{\mathsf{N}\,ss0 \vdash \mathsf{N}\,ss0}\ (\mathrm{Ax}) \qquad \dfrac{\mathsf{E}\,x \vdash \mathsf{N}\,x}{\mathsf{E}\,z \vdash \mathsf{N}\,z}\ (\mathrm{Subst})$$

$$\dfrac{\mathsf{N}\,ss0 \vdash \mathsf{N}\,ss0}{\mathsf{N}\,ss0 \vdash \mathsf{N}\,sss0}\ (\mathsf{N}\,\mathrm{R}_2) \qquad \dfrac{\mathsf{E}\,z \vdash \mathsf{N}\,z}{\mathsf{E}\,z \vdash \mathsf{N}\,sz}\ (\mathsf{N}\,\mathrm{R}_2)$$

$$\dfrac{\mathsf{N}\,ss0 \vdash \mathsf{N}\,sss0}{y = sss0, \mathsf{N}\,ss0 \vdash \mathsf{N}\,y}\ (=\mathrm{L}) \qquad \dfrac{\mathsf{E}\,z \vdash \mathsf{N}\,sz}{y = sz, \mathsf{E}\,z \vdash \mathsf{N}\,y}\ (=\mathrm{L})$$

$$\dfrac{y = sss0, \mathsf{N}\,ss0 \vdash \mathsf{N}\,y \qquad y = sz, \mathsf{E}\,z \vdash \mathsf{N}\,y}{\mathsf{O}\,y \vdash \mathsf{N}\,y}\ (\mathrm{Case}\ \mathsf{O})$$

$$\dfrac{\mathsf{O}\,y \vdash \mathsf{N}\,y}{\mathsf{O}\,y \vdash \mathsf{N}\,sy}\ (\mathsf{N}\,\mathrm{R}_2)$$

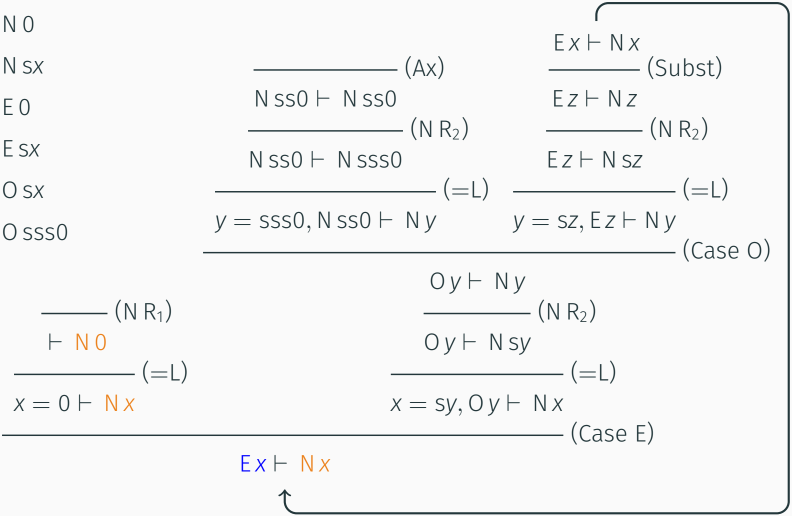
$$\dfrac{}{x = 0 \vdash \mathsf{N}\,x}\ (=\mathrm{L}) \qquad \dfrac{\mathsf{O}\,y \vdash \mathsf{N}\,sy}{x = sy, \mathsf{O}\,y \vdash \mathsf{N}\,x}\ (=\mathrm{L})$$

$$\dfrac{x = 0 \vdash \mathsf{N}\,x \qquad x = sy, \mathsf{O}\,y \vdash \mathsf{N}\,x}{\mathsf{E}\,x \vdash \mathsf{N}\,x}\ (\mathrm{Case}\ \mathsf{E})$$

This trace is partially maximal: the final predicate is the active formula of an axiom

Extracting Semantic Orderings: Example I

$$\Rightarrow \mathsf{N}\,0$$
$$\mathsf{N}\,x \Rightarrow \mathsf{N}\,\mathsf{s}x$$
$$\Rightarrow \mathsf{E}\,0$$
$$\mathsf{O}\,x \Rightarrow \mathsf{E}\,\mathsf{s}x$$
$$\mathsf{E}\,x \Rightarrow \mathsf{O}\,\mathsf{s}x$$
$$\mathsf{N}\,\mathsf{ss}0 \Rightarrow \mathsf{O}\,\mathsf{sss}0$$



Extracting Semantic Orderings: Example I

$$\Rightarrow \mathsf{N}\,0$$

$$\mathsf{N}\,x \Rightarrow \mathsf{N}\,sx$$

$$\Rightarrow \mathsf{E}\,0$$

$$\mathsf{O}\,x \Rightarrow \mathsf{E}\,sx$$

$$\mathsf{E}\,x \Rightarrow \mathsf{O}\,sx$$

$$\mathsf{N}\,ss0 \Rightarrow \mathsf{O}\,sss0$$

$$\dfrac{}{\mathsf{N}\,ss0 \vdash \mathsf{N}\,ss0}\,(\mathrm{Ax})$$

$$\dfrac{\mathsf{N}\,ss0 \vdash \mathsf{N}\,ss0}{\mathsf{N}\,ss0 \vdash \mathsf{N}\,sss0}\,(\mathsf{N}\,\mathrm{R}_2)$$

$$\dfrac{\mathsf{N}\,ss0 \vdash \mathsf{N}\,sss0}{y = sss0, \mathsf{N}\,ss0 \vdash \mathsf{N}\,y}\,(=\mathrm{L})$$

$$\dfrac{\mathsf{E}\,x \vdash \mathsf{N}\,x}{\mathsf{E}\,z \vdash \mathsf{N}\,z}\,(\mathrm{Subst})$$

$$\dfrac{\mathsf{E}\,z \vdash \mathsf{N}\,z}{\mathsf{E}\,z \vdash \mathsf{N}\,sz}\,(\mathsf{N}\,\mathrm{R}_2)$$

$$\dfrac{\mathsf{E}\,z \vdash \mathsf{N}\,sz}{y = sz, \mathsf{E}\,z \vdash \mathsf{N}\,y}\,(=\mathrm{L})$$

$$\dfrac{y = sss0, \mathsf{N}\,ss0 \vdash \mathsf{N}\,y \qquad y = sz, \mathsf{E}\,z \vdash \mathsf{N}\,y}{\mathsf{O}\,y \vdash \mathsf{N}\,y}\,(\mathrm{Case}\ \mathsf{O})$$

$$\dfrac{\mathsf{O}\,y \vdash \mathsf{N}\,y}{\mathsf{O}\,y \vdash \mathsf{N}\,sy}\,(\mathsf{N}\,\mathrm{R}_2)$$

$$\dfrac{\mathsf{O}\,y \vdash \mathsf{N}\,sy}{x = sy, \mathsf{O}\,y \vdash \mathsf{N}\,x}\,(=\mathrm{L})$$

$$\dfrac{}{\vdash \mathsf{N}\,0}\,(\mathsf{N}\,\mathrm{R}_1)$$

$$\dfrac{\vdash \mathsf{N}\,0}{x = 0 \vdash \mathsf{N}\,x}\,(=\mathrm{L})$$

$$\dfrac{x = 0 \vdash \mathsf{N}\,x \qquad x = sy, \mathsf{O}\,y \vdash \mathsf{N}\,x}{\mathsf{E}\,x \vdash \mathsf{N}\,x}\,(\mathrm{Case}\ \mathsf{E})$$

Extracting Semantic Orderings: Example II

$$\dfrac{\dfrac{\dfrac{}{\vdash \mathsf{E}\,0, \mathsf{O}\,0}\,(\mathsf{E}\,\mathsf{R}_1)}{x = 0 \vdash \mathsf{E}\,x,\ \mathsf{O}\,x}\,(=\mathsf{L}) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\mathsf{N}\,x \vdash \mathsf{E}\,x, \mathsf{O}\,x}{\mathsf{N}\,y \vdash \mathsf{E}\,y, \mathsf{O}\,y}\,(\mathsf{Subst})}{\mathsf{N}\,y \vdash \mathsf{O}\,y, \mathsf{E}\,y}\,(\mathsf{PR})}{\mathsf{N}\,y \vdash \mathsf{O}\,y, \mathsf{O}\,\mathsf{s}y}\,(\mathsf{O}\,\mathsf{R})}{\mathsf{N}\,y \vdash \mathsf{E}\,\mathsf{s}y, \mathsf{O}\,\mathsf{s}y}\,(\mathsf{E}\,\mathsf{R}_2)}{x = \mathsf{s}y, \mathsf{N}\,y \vdash \mathsf{E}\,x, \mathsf{O}\,x}\,(=\mathsf{L})}{\mathsf{N}\,x \vdash \mathsf{E}\,x,\ \mathsf{O}\,x}\,(\mathsf{Case}\ \mathsf{N})$$

Extracting Semantic Orderings: Example II

$$\cfrac{\cfrac{\cfrac{}{\vdash \mathsf{E}\,0, \mathsf{O}\,0}\ (\mathsf{E}\,\mathrm{R}_1)}{x = 0 \vdash \mathsf{E}\,x,\ \mathsf{O}\,x}\ (=\mathrm{L}) \qquad \cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\mathsf{N}\,x \vdash \mathsf{E}\,x, \mathsf{O}\,x}{\mathsf{N}\,y \vdash \mathsf{E}\,y, \mathsf{O}\,y}\ (\mathrm{Subst})}{\mathsf{N}\,y \vdash \mathsf{O}\,y, \mathsf{E}\,y}\ (\mathrm{PR})}{\mathsf{N}\,y \vdash \mathsf{O}\,y, \mathsf{O}\,\mathsf{s}y}\ (\mathsf{O}\,\mathrm{R})}{\mathsf{N}\,y \vdash \mathsf{E}\,\mathsf{s}y, \mathsf{O}\,\mathsf{s}y}\ (\mathsf{E}\,\mathrm{R}_2)}{x = \mathsf{s}y, \mathsf{N}\,y \vdash \mathsf{E}\,x, \mathsf{O}\,x}\ (=\mathrm{L})}{\mathsf{N}\,x \vdash \mathsf{E}\,x,\ \mathsf{O}\,x}\ (\mathrm{Case}\ \mathsf{N})$$

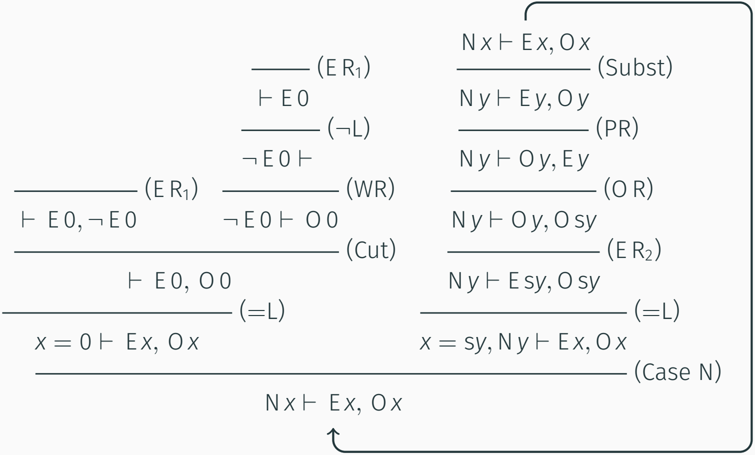
$$
\dfrac{
\dfrac{\dfrac{}{\vdash\ \mathrm{E}\,0,\ \mathrm{O}\,0}\,(\mathrm{E\,R}_1)}{x = 0 \vdash\ \mathrm{E}\,x,\ \mathrm{O}\,x}\,(=\mathrm{L})
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\mathrm{N}\,x \vdash \mathrm{E}\,x, \mathrm{O}\,x}{\mathrm{N}\,y \vdash \mathrm{E}\,y, \mathrm{O}\,y}\,(\mathrm{Subst})}{\mathrm{N}\,y \vdash \mathrm{O}\,y, \mathrm{E}\,y}\,(\mathrm{PR})}{\mathrm{N}\,y \vdash \mathrm{O}\,y, \mathrm{O}\,sy}\,(\mathrm{O\,R})}{\mathrm{N}\,y \vdash \mathrm{E}\,sy, \mathrm{O}\,sy}\,(\mathrm{E\,R}_2)}{x = sy, \mathrm{N}\,y \vdash \mathrm{E}\,x, \mathrm{O}\,x}\,(=\mathrm{L})
}{\mathrm{N}\,x \vdash\ \mathrm{E}\,x,\ \mathrm{O}\,x}\,(\mathrm{Case\ N})
$$

Extracting Semantic Orderings: Example II

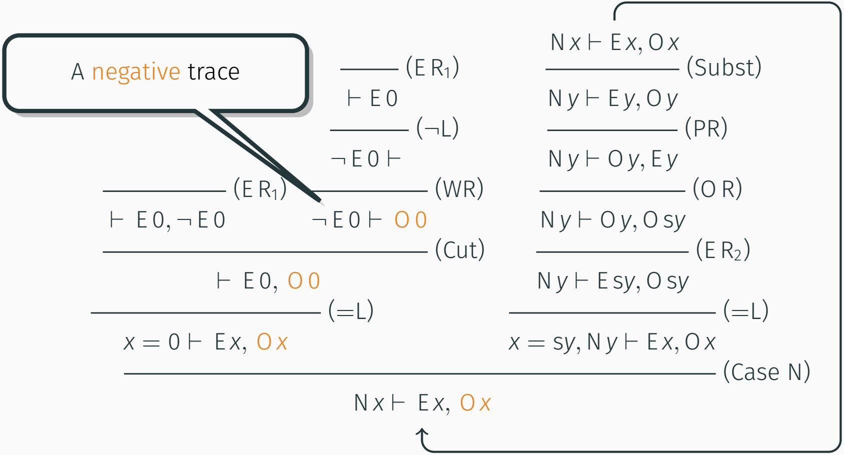
Not ground!

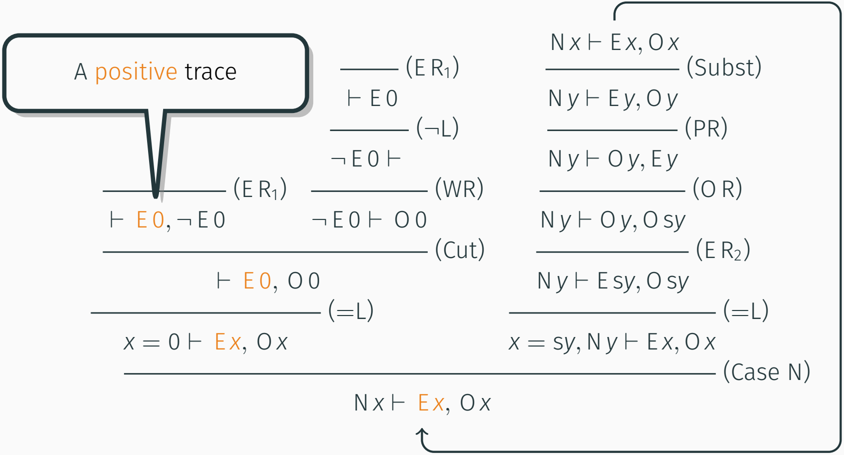
$$\dfrac{\dfrac{\dfrac{}{\vdash E\,0,\ O\,0}\,(E\,R_1)}{x = 0 \vdash E\,x,\ O\,x}\,(=L) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{N\,x \vdash E\,x,\ O\,x}{N\,y \vdash E\,y,\ O\,y}\,(\text{Subst})}{N\,y \vdash O\,y,\ E\,y}\,(\text{PR})}{N\,y \vdash O\,y,\ O\,sy}\,(O\,R)}{N\,y \vdash E\,sy,\ O\,sy}\,(E\,R_2)}{x = sy,\ N\,y \vdash E\,x,\ O\,x}\,(=L)}{N\,x \vdash E\,x,\ O\,x}\,(\text{Case } N)$$

Extracting Semantic Orderings: Example II



Extracting Semantic Orderings: Example II




A positive trace
(E R1)
⊢ E 0
(¬L)
¬ E 0 ⊢
(E R1)
(WR)
⊢ E 0, ¬ E 0
¬ E 0 ⊢ O 0
(Cut)
⊢ E 0, O 0
(=L)
x = 0 ⊢ E x, O x
N x ⊢ E x, O x
(Subst)
N y ⊢ E y, O y
(PR)
N y ⊢ O y, E y
(O R)
N y ⊢ O y, O sy
(E R2)
N y ⊢ E sy, O sy
(=L)
x = sy, N y ⊢ E x, O x
(Case N)
N x ⊢ E x, O x

Extracting Semantic Orderings: A Realizability Condition

Definition (Realizability Condition)

For every positive maximal right-hand trace, there must exist a left-hand trace following some prefix of the same path such that:

- either the right-hand trace is grounded, or it is partially maximal with the left-hand trace matching in the length and final predicate
- right unfoldings $\leq$ left unfoldings

Soundness of the Realizability Condition

Theorem

Suppose $\mathcal{P}$ is a cyclic proof of $\mathsf{P}\,\vec{x} \vdash \mathsf{Q}\,\vec{y}$ satisfying the realizability condition, then $[\![\mathsf{P}\,\vec{x}]\!]_\alpha \subseteq [\![\mathsf{Q}\,\vec{y}]\!]_\alpha$, for all α (i.e. $\mathsf{Q}\,\vec{y} \leq \mathsf{P}\,\vec{x}$)

Proof.

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Proof.

Pick a model $m \in [\![\mathsf{P}\,\vec{x}]\!]_\alpha$ (i.e. $\exists\,\beta \leq \alpha : m \in [\![\mathsf{P}\,\vec{x}]\!]_\beta$)

- m corresponds to a positive maximal right-hand trace in $\mathcal{P}$
- Since $\mathcal{P}$ is a proof $\mathsf{P}\,\vec{x} \vdash \mathsf{Q}\,\vec{y}$ is valid, in particular $m \in [\![\mathsf{Q}\,\vec{y}]\!]$
- The number of unfoldings in this right-hand trace is an upper bound on the least approximation $[\![\mathsf{Q}\,\vec{y}]\!]_\gamma$ containing m
- The number of unfoldings in any left-hand trace following the same path is a lower bound on the least approximation $[\![\mathsf{P}\,\vec{x}]\!]_\delta$ containing m
- From the realizability condition, we have that $\delta \geq \gamma$

Deciding the Realizability Condition

- We use weighted automata to decide whether the realizability condition holds
- We construct weighted automata that count the progression points in left and right-hand traces
- The realizability condition corresponds to an inclusion of the right-hand trace automaton within the left-hand one

Weighted Automata

Definition (Weighted Automata)

Let Σ be an alphabet, and $(V, \oplus, \otimes)$ a semiring of weights. A weighted automaton $\mathscr{A}$ is a tuple (Q, q_I, F, γ) consisting of a set Q of states containing an initial state $q_I \in Q$, a set $F \subseteq Q$ of final states, and a weighted transition function $\gamma : (Q \times \Sigma \times Q) \to V$.

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1. The value of a run of $\mathscr{A}$ is the semiring product of all its transitions
2. The value of a word is the semiring sum of all runs accepting that word
3. The quantitative language $\mathcal{L}_{\mathscr{A}}$ is the function $\Sigma^* \rightharpoonup V$ computed by $\mathscr{A}$

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Definition (Weighted Inclusion)

$\mathcal{L}_1 \leq \mathcal{L}_2$ if and only if for every word w such that $\mathcal{L}_1(w)$ is defined, $\mathcal{L}_2(w)$ is also defined and $\mathcal{L}_1(w) \leq \mathcal{L}_2(w)$

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Sum automata are weighted automata over $(\mathbb{N}, +, \max)$

Weighted Automata: Existing Results

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Theorem (Krob '94, Almagor Et Al. '11)

Given two quantitative languages (weighted automata) $\mathcal{L}_1$ and $\mathcal{L}_2$, it is undecidable whether $\mathcal{L}_1 \leq \mathcal{L}_2$

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Definition

A weighted automaton is called finite-valued if there exists a bound on the number of distinct values of accepting runs on any given word

Theorem (Filiot, Gentilini & Raskin '14)

Given two finite-valued weighted automata $\mathscr{A}$ and $\mathscr{B}$, it is decidable whether $\mathcal{L}_{\mathscr{A}} \leq \mathcal{L}_{\mathscr{B}}$

Weighted Automata from Cyclic Entailment Proofs

Given a cyclic entailment proof $\mathcal{P}$, we can construct two kinds of finite-valued sum automata, $\mathscr{A}_{\mathcal{P}}[n]$ ($n \in \mathbb{N}$) and $\mathscr{C}_{\mathcal{P}}$, which count the unfoldings in left- and right-hand traces, respectively:

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    - $\mathscr{C}_{\mathcal{P}}$ only counts traces following the full path
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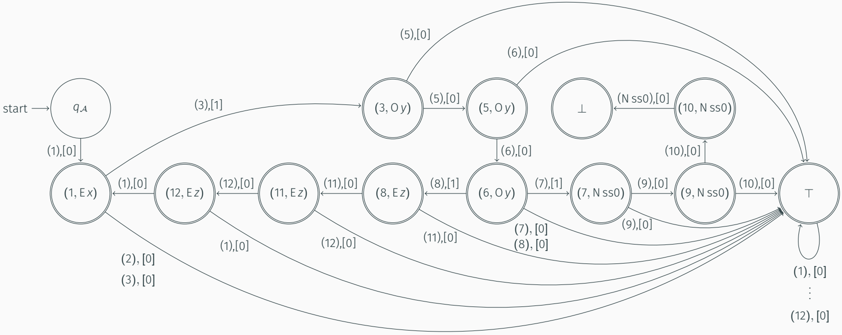
- Each $\mathscr{A}_{\mathcal{P}}[n]$ considers only a subset of the paths in the proof
  - A complete automaton can be constructed but is not, in general, finite-valued
- $\mathscr{C}_{\mathcal{P}}$ is grounded when all final states correspond to ground predicate instances

Weighted Automata from Cyclic Entailment Proofs

The full left-hand automaton for the example proof of $\mathsf{E}\,x \vdash \mathsf{N}\,x$



An Equivalence between Realizability and Weighted Inclusion

The construction of the weighted automata admits the following result:

Theorem

Let $\mathcal{P}$ be a cyclic entailment proof which is dynamic and balanced; then $\mathcal{P}$ satisfies the realizability condition if and only if $\mathscr{C}_{\mathcal{P}} \leq \mathscr{A}_{\mathcal{P}}[N]$ and $\mathscr{C}_{\mathcal{P}}$ is grounded (where N is a function of $\mathcal{P}$)

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The proof is:

- balanced when every (reachable) basic trace cycle has a non-zero number of progression points
- dynamic when (reachable) basic binary trace cycles has equal numbers of left and right-hand progression points
    - a binary cycle is a pair of left and right-hand trace cycles following the same path

The bound N is a function of other graph-theoretic quantities of $\mathcal{P}$

Corollary: Bootstrapping Cyclic Entailment Systems

Suppose we deduce $\mathrm{Q}\,\vec{u} \leq \mathrm{P}\,\vec{t}$ from a proof of $\Gamma, \mathrm{P}\,\vec{t} \vdash \Sigma, \mathrm{Q}\,\vec{u}$

Then we can safely trace across an active cut formula

$$\frac{\Gamma, \mathrm{P}\,\vec{t} \vdash \Sigma, \mathrm{Q}\,\vec{u} \qquad \mathrm{Q}\,\vec{u}, \Pi \vdash \Delta}{\Gamma, \mathrm{P}\,\vec{t}, \Pi \vdash \Sigma, \Delta}\,(\mathrm{Cut})$$

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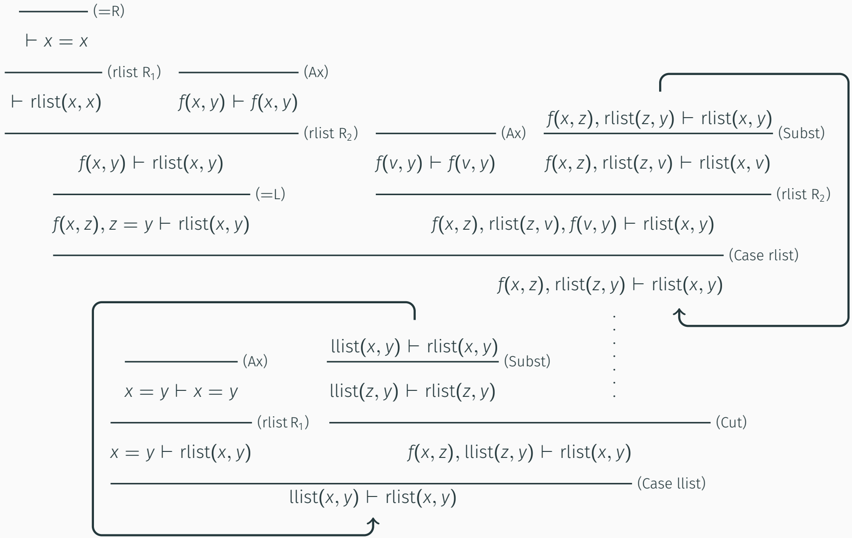
This is explicitly forbidden in existing cyclic proof systems, precisely because there is no way to ensure in general that there is an inclusion between $[\![\mathrm{P}\,\vec{t}]\!]_\alpha$ and $[\![\mathrm{Q}\,\vec{u}]\!]_\alpha$

Limitations: Problems with Cuts



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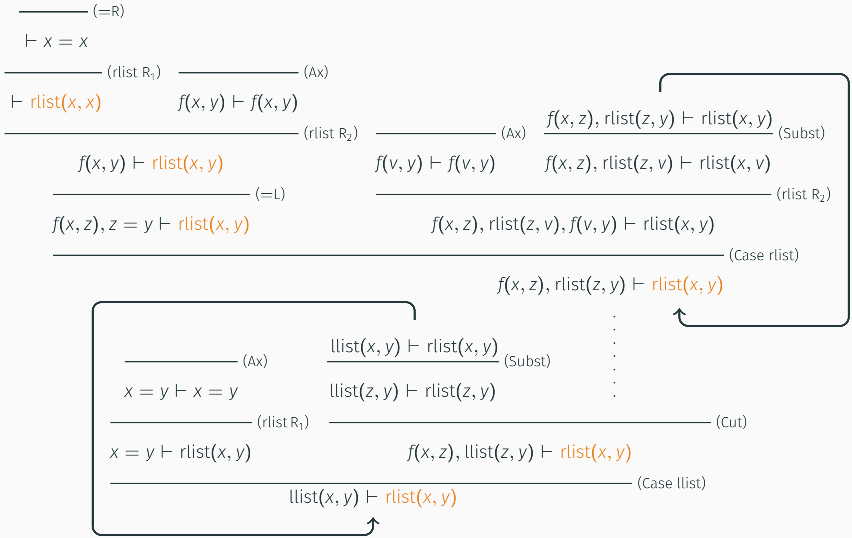

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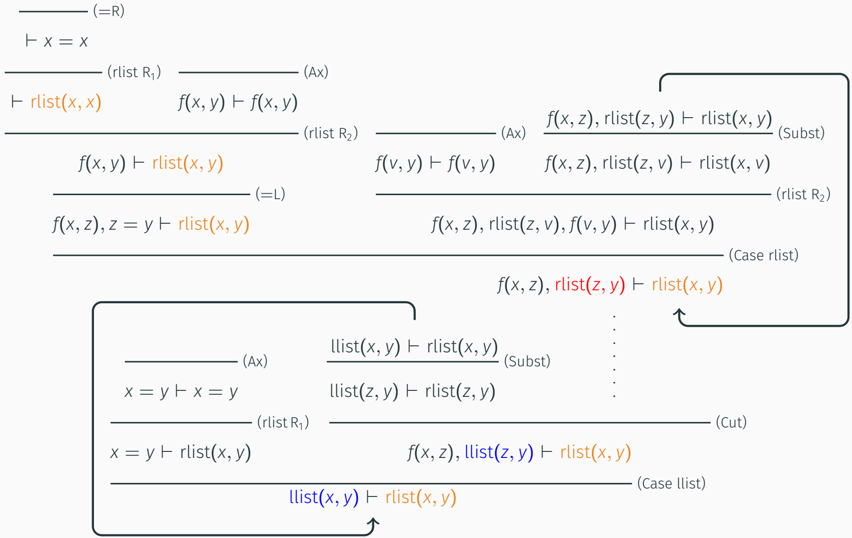

Conclusions

- We have shown that information about inclusions between the semantics of inductive predicates can be extracted from cyclic proofs of entailments
- This information can be used to construct ranking functions for programs
- Our results are formulated abstractly, and so hold for any cyclic proof system whose rules satisfy certain properties (e.g. separation logic)
- We use the term realizability because we extract semantic information from the proofs

Future Work

- Implement the decision procedure within the cyclic proof-based verification framework CYCLIST
- Evaluate to what extent entailments found 'in the wild' satisfy the realizability condition
- Extend the results to better handle cuts in proofs
- Investigate further theoretical questions:
    - are there weaker structural properties of proofs that still admit completeness with the approximate automata
    - If the semantic inclusion $[\![\mathsf{P}\,\vec{x}]\!]_\alpha \subseteq [\![\mathsf{Q}\,\vec{y}]\!]_\alpha$ holds, is there a cyclic proof of $\mathsf{P}\,\vec{x} \vdash \mathsf{Q}\,\vec{y}$ satisfying the realizability condition?