A Non-wellfounded, Labelled Proof System for Propositional Dynamic Logic

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What is Dynamic Logic?

Dynamic Logic was introduced by Pratt (1976)

- Reasoning about program executions (i.e. their dynamics)
- A modal logic (programs are modal operators)

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Intuitively, for a program p and assertion φ:

$[p]\varphi$ means φ holds after *all* (terminating) executions of p

$\langle p \rangle \varphi$ means there is *some* execution of p after which φ holds

The Language of Programs

Programs are constructed from:

- A set of basic programs (e.g. $x := x + 1$)
- Sequential composition $p \,;\, q$
- Non-deterministic choice $p \cup q$
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- Various extensions: converse p^-, intersection $p \cap q$, etc.

Relational (Kripke) Semantics of Dynamic Logic

Basic programs are accessibility relations on (memory) states $s \in \mathcal{S}$

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But tests introduce a mutual recursion: $[\![\varphi?]\!] = \{(s, s) \mid s \in [\![\varphi]\!]\}$

The Influence of Dynamic Logic

Lots of variants and extensions:

- Games (Parikh, ’83)
- Natural language (Groenendijk & Stokhof, ’91)
- Knowledge representation (De Giacomo & Lenzarini, ’94)
- XML (Afanasiev Et Al, 2005)
- Cyber-physical systems (Platzer, 2008)
- Epistemic reasoning for agents (Patrick Girard Et Al, 2012)
- etc.

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$$\texttt{if}\ \varphi\ \texttt{then}\ \alpha\ \texttt{else}\ \beta \stackrel{\text{def}}{=} (\varphi?\,;\alpha) \cup (\neg\varphi?\,;\beta)$$

$$\texttt{while}\ \varphi\ \texttt{do}\ \alpha \stackrel{\text{def}}{=} (\varphi?\,;\alpha)^*\,;\neg\varphi?$$

PDL: Main Properties and Results

- Small model property
- Satisfiability **EXPTIME**-complete
- Finitely axiomatisable

(K)	$\vdash [\alpha](\varphi \to \psi) \to ([\alpha]\varphi \to [\alpha]\psi)$	(Test)	$\vdash [\psi?]\varphi \leftrightarrow (\psi \to \varphi)$
(Distributivity)	$\vdash [\alpha](\varphi \land \psi) \leftrightarrow ([\alpha]\varphi \land [\alpha]\psi)$	(Fixed Point)	$\vdash \varphi \land [\alpha][\alpha^*]\varphi \leftrightarrow [\alpha^*]\varphi$
(Choice)	$\vdash [\alpha \cup \beta]\varphi \leftrightarrow [\alpha]\varphi \land [\beta]\varphi$	(Induction)	$\vdash \varphi \land [\alpha^*](\varphi \to [\alpha]\varphi) \to [\alpha^*]\varphi$
(Composition)	$\vdash [\alpha ; \beta]\varphi \leftrightarrow [\alpha][\beta]\varphi$	(Necessitation)	from $\vdash \varphi$ infer $\vdash [\alpha]\varphi$

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Dual axioms for $\langle\alpha\rangle$ (if taken as a primitive)

- But not compact $\quad \{\neg\varphi, [\alpha]\neg\varphi, [\alpha\,;\alpha]\neg\varphi, [\alpha\,;\alpha\,;\alpha]\neg\varphi, \ldots\} \not\models \langle\alpha^*\rangle\varphi$

Proof Systems for PDL

Tableaux-based systems:

- De Giacomo & Massacci, 2000
- Goré & Widmann, 2009

Sequent-based with ω-rules/infinite contexts:

- Renardel de Lavalette Et Al, 2008
- Hill & Poggiolesi, 2010
- Fritella Et Al, 2014

Our Goal: A Satisfactory Proof Theory

A robust, structural proof theory for PDL and PDL-type logics

- Analytic and finitary (i.e. automatable!)
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We combine two methodologies

- Labelled sequent calculus
- Non-wellfounded proof theory

Why Labelled Sequent Calculus?

Modularly capture a range of modal logics (Negri, 2005) using:

- Labelled formulas $x : \varphi$ and relational statements $x\ R\ y$
- Proof rules expressing the meaning of modalities

$$\frac{y : \varphi, x : \Box\varphi, x\ R\ y, \Gamma \Rightarrow \Delta}{x : \Box\varphi, x\ R\ y, \Gamma \Rightarrow \Delta} \qquad \frac{x\ R\ y, \Gamma \Rightarrow \Delta, y : \varphi}{\Gamma \Rightarrow \Delta, x : \Box\varphi}\ (y \text{ fresh})$$

- Proof rules characterising different (geometric) frame properties, e.g.

$$\text{(symm):}\ \frac{y\ R\ x, x\ R\ y, \Gamma \Rightarrow \Delta}{x\ R\ y, \Gamma \Rightarrow \Delta} \qquad \text{(trans):}\ \frac{x\ R\ z, x\ R\ y, y\ R\ z, \Gamma \Rightarrow \Delta}{x\ R\ y, y\ R\ z, \Gamma \Rightarrow \Delta}$$

- Even possible to capture some non-modally definable frame properties

Why Non-wellfounded Proofs?

They allow us to tame (inductive) infinitary behaviour

- Allow derivations to be infinitely tall (vs. wide) — not generally sound!
- Distinguish 'good' derivations with a global trace condition
- Restrict to (finitely representable) cyclic proofs

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Examples of non-wellfounded proof theories include:

- FOL + Inductive Definitions (Brotherston & Simpson)
- FOL over Herbrand models (Cohen, R, Zohar)
- Linear Logic with fixed points
  (Fortier & Santocanale, Baelde/Saurin/Doumane/Nollet/Tasson)
- Kleene/Action Algebra (Das & Pous)

Our Non-wellfounded, Labelled Sequent Calculus for PDL

- Relational statements $x\ R_a\ y$ refer to atomic programs a
- Rules for atomic modalities à la Negri

$$(\Box L): \frac{y : \varphi, \Gamma \Rightarrow \Delta}{x : [a]\varphi, x\ R_a\ y, \Gamma \Rightarrow \Delta} \qquad (\Box R): \frac{x\ R_a\ y, \Gamma \Rightarrow \Delta, y : \varphi}{\Gamma \Rightarrow \Delta, x : [a]\varphi}\ (y \text{ fresh})$$

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- Decompose non-atomic modalities as per semantics, e.g.

$$(\cup L):\ \frac{x : [\alpha]\varphi, x : [\beta]\varphi, \Gamma \Rightarrow \Delta}{x : [\alpha \cup \beta]\varphi, \Gamma \Rightarrow \Delta} \qquad (\cup R):\ \frac{\Gamma \Rightarrow \Delta, x : [\alpha]\varphi \quad \Gamma \Rightarrow \Delta, x : [\beta]\varphi}{\Gamma \Rightarrow \Delta, x : [\alpha \cup \beta]\varphi}$$

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- Rules for iteration express its nature as a fixed point

$$(*L): \frac{x : \varphi, x : [\alpha][\alpha^*]\varphi, \Gamma \Rightarrow \Delta}{x : [\alpha^*]\varphi, \Gamma \Rightarrow \Delta} \qquad (*R): \frac{\Gamma \Rightarrow \Delta, x : \varphi \quad \Gamma \Rightarrow \Delta, x : [\alpha][\alpha^*]\varphi}{\Gamma \Rightarrow \Delta, x : [\alpha^*]\varphi}$$

A ‘Bad’ Non-wellfounded Derivation

$$
\dfrac{
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\dfrac{\begin{array}{c}\vdots \\ \Rightarrow x : [\alpha^*]\varphi, x : [\alpha^*]\varphi\end{array}}{\Rightarrow x : [\alpha^*]\varphi}\,(\mathrm{CR})
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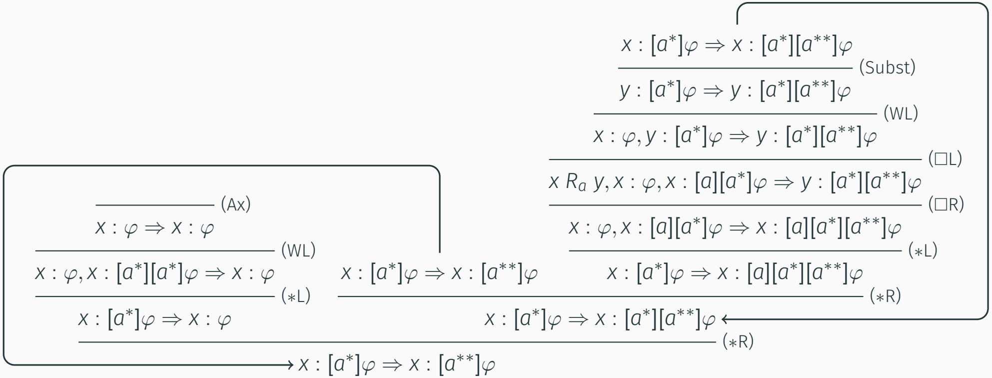
We trace (possibly nested) modalities on the right-hand side

- They must be unfolded infinitely often along infinite paths



'Good' Proofs: The Global Trace Condition

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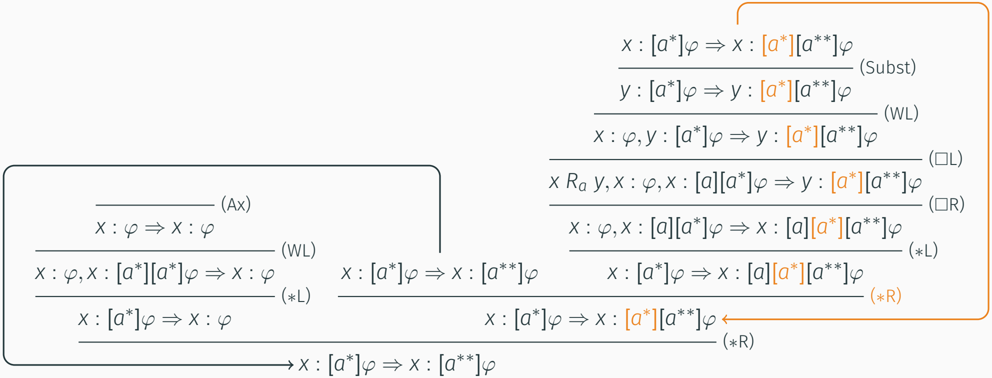

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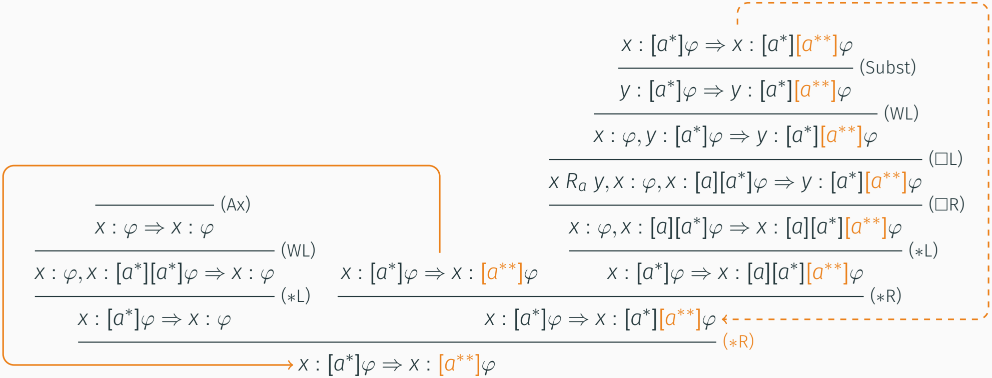
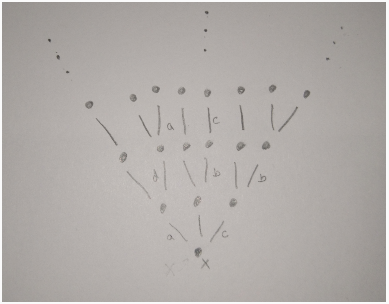


Soundness

Theorem

$\Gamma \Rightarrow \Delta$ is valid if there is a non-wellfounded proof deriving it

- Traced modalities $\Gamma \Rightarrow \Delta, x : [\alpha_1] \dots [\alpha_n][\beta^*]\varphi$ identify particular substructures in countermodels:



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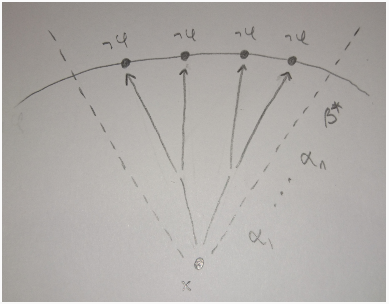

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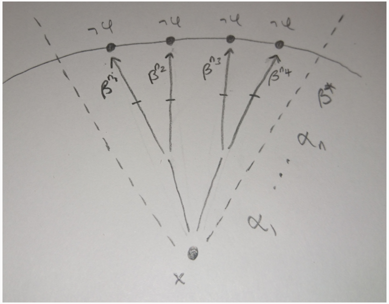

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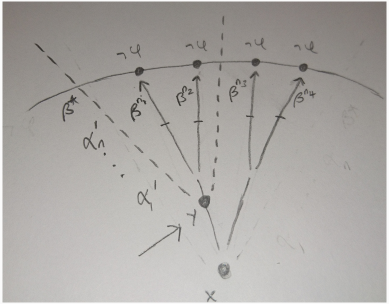

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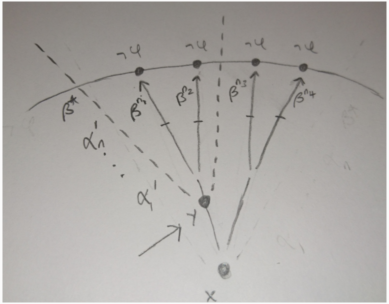

- Cyclic proofs capture an infinite-descent style proof by contradiction.

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The axioms characterising PDL have cyclic proofs

Lemma (Necessitation)

There is a cyclic derivation simulating the rule

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Theorem

If φ *is a PDL theorem, there is a cyclic proof deriving* $\Rightarrow x : \varphi$

Proof Search for Test-free sequents

We propose the following proof-search strategy:

- Apply (invertible) logical rules as much as possible
  - But do not allow traces to progress more than once
  - For test-free sequents, this terminates
- Close open leaves with axioms where possible
- Apply a series of validity-preserving weakenings
- Repeat process for any remaining open leaves

All formulas that appear are in the Fischer-Ladner closure of the end sequent

Conjecture

The number of distinct labels appearing in a sequent is bounded

Future Work

- Prove cut-free regular completeness results (also for tests?)
- Demonstrate capture of different frame conditions
- Incorporate additional constructs in the program algebra
    - Converse, Intersection
- Extend to capture other modal fixpoints (temporal, common knowledge)
- Derive interpolation results from the proof theory
    - cf. Cyclic system and Lyndon interpolation for for GL (Shamkanov, 2014)