# Polynomial approximation algorithms for the TSP and the QAP with a factorial domination number

Gregory Gutin
Department of Mathematics and Statistics
Brunel, The University of West London
Uxbridge, Middlesex, UB8 3PH, UK
z.g.gutin@brunel.ac.uk

Anders Yeo
Department of Mathematics and Statistics
University of Victoria, P.O. Box 3045, Victoria B.C.
Canada V8W 3P4
yeo@Math.UVic.CA

Dedicated to the memory of the first author's father.

**Abstract**

Glover and Punnen (1997) asked whether there exists a polynomial time algorithm that always produces a tour which is not worse than at least $n!/p(n)$ tours for some polynomial $p(n)$ for every TSP instance on $n$ cities. They conjectured that, unless P=NP, the answer to this question is negative. We prove that the answer to this question is, in fact, positive. A generalization of the TSP, the quadratic assignment problem, is also considered with respect to the analogous question. Probabilistic, graph-theoretical, group-theoretical and number-theoretical methods and results are used.

*Key words:* Traveling salesman problem, quadratic assignment problem, approximation algorithm.

## 1 Introduction

The *domination number*, $dom(\mathcal{A}, n)$, of an approximation algorithm for the traveling salesman problem (TSP) is the maximum integer $k = k(n)$ such that, for every instance $\mathcal{I}$ of the TSP on $n$ cities, $\mathcal{A}$ produces a tour $T$ which is not worse than at least $k$ tours in $\mathcal{I}$ including $T$ itself. F. Glover and A.P. Punnen [12] asked whether there exists a polynomial time (in $n$) algorithm $\mathcal{A}$ with domination number $dom(\mathcal{A},n) \geq n!/p(n)$ for some polynomial $p(n)$. They conjectured that, unless P=NP, the answer to this question is negative. We prove that the answer to this question is, in fact, positive.

Polynomial algorithms with exponentially large domination number were suggested in a number of papers including [3, 5, 10, 12, 13, 14, 16, 21]. The strongest result was obtained in [14]: there is a polynomial algorithm $\mathcal{B}$ with $dom(\mathcal{B},n) = \Omega(n!/t^n)$ for every constant $t > 1.5$.

In [11], in a series of computational experiments with several families of instances, it was shown that a combination of an algorithm from [21] (with proven large domination number) and a modification of a traditional approach leads to a construction heuristic for the TSP which clearly outperforms well-known construction heuristics for the asymmetric TSP. A high potential of local search heuristics which use neighbourhoods of exponential cardinality (and, thus, polynomial algorithms of exponential domination number to search the neighbourhoods) was shown in [3, 7, 17].

In this paper, we introduce a polynomial approximation algorithm for a wide family of combinatorial optimization problems which is based on the derandomization method of conditional probabilities, see e.g. [1, 18]. We call this algorithm the *greedy-expectation algorithm* (GEA). We prove that the adaptation $\mathcal{G}$ of the GEA for the TSP has the domination number $dom(\mathcal{G},n) \geq (n-2)!$ for every $n \neq 6$. To establish this, both probabilistic and graph-theoretical approaches and results are used. (Preliminary computational experiments with the GEA for some Euclidean instances taken from the well-known TSPlib show that the GEA produces tours of quality superior to that of the best well-known construction heuristics, but the GEA is not as fast as most of them.)

We also consider a generalization of the TSP, the quadratic assignment problem (QAP), see [8, 9]. The domination number of an algorithm for the QAP can be defined similarly to that of an algorithm for the TSP. Let $\mathcal{A}$ be the GEA specialized for the QAP. We show that $dom(\mathcal{A},n) \geq (n-2)!$ for every prime power $n$. We also prove that, given $\beta > 1$, $dom(\mathcal{A},n) \geq n!/\beta^n$ for every sufficiently large $n$. Since no QAP neighbourhoods of (any) exponential cardinality are known so far (Deineko and Woeginger [9] conjecture that such neighbourhoods do not exist at all), our results are first of its kind for the QAP. To show these results, probabilistic, group-theoretical and number-theoretical approaches and results are applied.

## 2 Greedy-expectation algorithm

For integers $k \leq m$, let $[k,m]$ stand for the set $\{k, k+1, ..., m\}$. Let $m$ be a positive integer and let $D$ be a subset of $[1,m]^n = \{(x_1, x_2, ..., x_n) : \ x_i \in [1,m], i = 1,2,...,n\}$. Let $f$ be a mapping from $[1,m]^n$ into the set of reals. We consider the following optimization problem: find

$$\min\{f(x_1,...,x_n):\ (x_1,...,x_n)\in D\}. \tag{1}$$

Let $x^0_{i_1},...,x^0_{i_k}\in[1,m]$, where $1\le i_s\ne i_t\le n$, $1\le s<t\le k$. Then, $D(x^0_{i_1},...,x^0_{i_k})=\{(x_1,...,x_n)\in D:\ x_{i_1}=x^0_{i_1},...,x_{i_k}=x^0_{i_k}\}$. For $k=0$, let $D(x^0_{i_1},...,x^0_{i_k})=D$.

Consider $D$ as a probability space by assigning to each element $x^0=(x^0_1,...,x^0_n)$ of $D$ a non-negative weight (probability) $\mathbf{P}(x^0)$ such that $\sum_{x^0\in D}\mathbf{P}(x^0)=1$. Then, $f$ is a random variable; thus, we can deal with the expectation $\mathbf{E}f$ of $f$ as well as its conditional expectations. This interpretation of $D$ and $f$ allows us to apply the derandomization method of conditional probabilities (see, e.g., [1, 18]) to obtain Algorithm 2.1.

Assume that $m$ is bounded by a polynomial in $n$, and for every $(x^0_{i_1},...,x^0_{i_k})\in[1,m]^k$ $(0\le k\le n)$, the claim $D(x^0_{i_1},...,x^0_{i_k})\ne\emptyset$ can be verified in time polynomial in $n$. Suppose also that the conditional expectations (for non-empty $D(x^0_{i_1},...,x^0_{i_k})$) $\mathbf{E}(f|x_{i_1}=x^0_{i_1},...,x_{i_k}=x^0_{i_k})$ can be computed in time polynomial in $n$. Then, the following algorithm is of polynomial complexity (in $n$). We call it the *greedy-expectation algorithm* (GEA).

**Algorithm 2.1** *For $k:=1$ to $n$ find $x^0_{i_k}$ such that the following holds.*

$$\mathbf{E}(f|x_{i_1}=x^0_{i_1},...,x_{i_{k-1}}=x^0_{i_{k-1}},x_{i_k}=x^0_{i_k})=\min\{\mathbf{E}(f|x_{i_1}=x^0_{i_1},...,x_{i_{k-1}}=x^0_{i_{k-1}},x_p=j):(p,j)\in Q\},$$

*where $Q=\{(p,j):\ p\in[1,n]-\{i_1,...,i_{k-1}\},\ j\in[1,m]$ and $D(x^0_{i_1},...,x^0_{i_{k-1}},x_p=j)\ne\emptyset$.*

**Theorem 2.2** *Let $(x^0_1,...,x^0_n)$ be a solution obtained by the GEA. Then $f(x^0_1,...,x^0_n)\le\mathbf{E}f$.*

**Proof:** By the formula of total expectation (see formula (16) in [19]), for $k=1,2,...,n$,

$$\mathbf{E}(f|B_{k-1})=\sum\{\mathbf{E}(f|B_{k-1},x_{i_k}=j)\mathbf{P}(x_{i_k}=j|B_{k-1}):\ j\in[1,m],\ D(x^0_{i_1},...,x^0_{i_{k-1}},x^0_{i_k}=j)\ne\emptyset\},$$

where $B_{k-1}=(x_{i_1}=x^0_{i_1},...,x_{i_{k-1}}=x^0_{i_{k-1}})$ Thus, there exists an $x^0_{i_k}$ such that

$$\mathbf{E}(f|x_{i_1}=x^0_{i_1},...,x_{i_k}=x^0_{i_k})\le\mathbf{E}(f|x_{i_1}=x^0_{i_1},...,x_{i_{k-1}}=x^0_{i_{k-1}}).$$

Such an $x^0_{i_k}$ is found by the algorithm. Hence,

$$\mathbf{E}f\ge\mathbf{E}(f|x_{i_1}=x^0_{i_1})\ge...\ge\mathbf{E}(f|x_{i_1}=x^0_{i_1},...,x_{i_n}=x^0_{i_n})=f(x^0_1,...,x^0_n). \tag{2}$$

□

**Remarks.** 1. The main limitation in use of GEA is that one has to be able to compute the conditional expectations of $f$ in polynomial time. For the TSP and QAP, in order to be able to compute the conditional expectations, we will only consider uniform distributions, i.e. $\mathbf{P}(x) = 1/|D|$ for every $x \in D$.

2. Method of pessimistic estimators developed within the derandomization method of conditional probabilities can be used to relax the assumption that the conditional expectations are computed in polynomial time. Instead, it can be required that some upper bounds on the expectations are computed in polynomial time (see [1, 18] for more details).

## 3 GEA for the TSP

The (asymmetric) TSP is defined as follows. Let $K$ be a complete digraph with vertex set $V(K)$ and arc set $A(K)$ (if $x$ and $y$ are distinct vertices in $K$, then both $xy, yx \in A(K)$; $|V(K)| = n$). Every arc $xy$ in $K$ is assigned a real *cost* $c(xy) = c_K(xy)$. It is required to find a hamiltonian cycle (*tour*) $H$ of minimum cost in $K$. (The *cost* $c(G)$ of a subgraph $G$ of $K$ is the sum of the costs of arcs in $G$.)

It is easy to see that the TSP is a special problem of (1). Indeed, we can reformulate the TSP as follows. Let $D$ be the collection of sets with $n$ arcs from $A(K)$ such that the arcs in every set form a tour in $K$. Find $\min\{\sum_{i=1}^{n} c(a_i) : \{a_1, ..., a_n\} \in D\}$.

For an arc $a = xy$ in $K$, the *contraction* of $K$ at $a$, $K/a$ is a complete digraph with vertex set $V(K/a) = V(K) \cup \{v_a\} - \{x, y\}$, where $v_a \notin V(K)$, such that the cost $c_{K/a}(b)$ of an arc $b$ in $K/a$ is defined as follows: $c_{K/a}(uw) = c_K(uw), c_{K/a}(uv_a) = c_K(ux), c_{K/a}(v_a w) = c_K(yw)$, where $u, w \in V(K) - \{x, y\}$. We will omit the subscripts $K$ and $K/a$ when the costs are defined from the context. We assume that $c(xx) = 0$ for every vertex $x$ in $K$. It is easy to verify that

$$c(K/xy) = c(K) - c^+(x) - c^-(y) + c(xy) - c(yx), \tag{3}$$

where $c^+(x) = \sum_{u \in V(K)} c(xu)$ and $c^-(y) = \sum_{u \in V(K)} c(uy)$. The *total cost of all tours* in $K$ is denoted by $T(K)$. The *average cost of a tour* in $K$ is denoted by $\tau(K)$. As every arc of $K$ is contained in $(n-2)!$ tours, $\tau(K) = T(K)/(n-1)! = (n-2)!c(K)/(n-1)!$, hence, $\tau(K) = c(K)/(n-1)$. This formula can also be shown using linearity of expectation. Let $\tau_a(K)$ be the *average cost of a tour containing an arc* $a$. Clearly, $\tau_a(K) = \tau(K/a) + c(a)$.

The following algorithm is an adaptation of the GEA to the TSP.

**Algorithm 3.1** *Compute $c(K)$ and call the recursive procedure TSPGEA$(n, K, c(K))$, which returns a tour in $K$.*

*Procedure TSPGEA$(n, K, c(K))$:*

*1. If $n = 2$ return the tour of $K$.*

*2. Compute $c^+(x)$ and $c^-(x)$ for every $x \in V(K)$.*

3. *For every $b \in A(K)$ compute $c(K/b)$ using (3).*
4. *Find $a = xy$ in $K$ such that $\tau_a(K) = \min\{\tau_{uw}(K) : \ u \neq w \in V(K)\}$.*
5. *Compute $T :=$ TSPGEA$(n-1, K/a, c(K/a))$.*
6. *In $T$ substitute the vertex $v_a$ with the arc $a$.*
7. *Return $T$.*

It is straightforward to show that the complexity of Algorithm 3.1 is $O(n^3)$. To prove Theorem 3.3, we use the following result on decomposition of $A(K)$ into tours. A set $\{C_1, ..., C_{n-1}\}$ of $(n-1)$ tours in $K$ is a *decomposition* of $A(K)$ if $A(K) = \cup_{i=1}^{n-1} A(C_i)$. As $|A(K)| = n(n-1)$, $A(C_i) \cap A(C_j) = \emptyset$ for every pair of distinct $i$ and $j$.

**Lemma 3.2** *For every $n \geq 2$, $n \neq 4$, $n \neq 6$, there exists a decomposition of $A(K)$ into tours.*

While the assertion of this lemma for odd $n$ was already known to Rev Kirkman (see [4], p. 187), the even case result was only established in [20] as a solution to the corresponding conjecture by J.C. Bermond and V. Faber (who observed that the decomposition does not exist for $n = 4$ and $n = 6$).

**Theorem 3.3** *Let $H$ be a tour in $K$ such that $c(H) \leq \tau(K)$. If $n \neq 6$, then $H$ is not worse than at least $(n-2)!$ tours in $K$ (for any $K$).*

**Proof:** The result is trivial for $n = 2, 3$. If $n = 4$, the result follows from the simple fact that the most expensive tour $T$ in $K$ has cost $c(T) \geq c(H)$.

Assume that $n \geq 5$ and $n \neq 6$. Let $D_1 = \{C_1, ..., C_{n-1}\}$ be a decomposition of the arcs of $K$ into tours (such a decomposition exists by Lemma 3.2). Given a tour $H$ in $K$, clearly there is an automorphism of $K$ that maps $C_1$ into $H$. Therefore, if we consider $D_1$ together with the decompositions $(D_1, ..., D_{(n-1)!})$ of $K$ obtained from $D_1$ using all automorphisms of $K$ which map the vertex 1 into itself, we will have every tour of $K$ in one of $D_i$'s. Moreover, every tour is in exactly $n-1$ decompositions $D_i$'s (by mapping a tour $C_i$ into a tour $C_j$ $(i, j \in [1, n-1])$ we fix the automorphism).

Choose the most expensive tour in each of $D_i$ and form a set $\mathcal{E}$ from all distinct tours obtained in this manner. Clearly, $|\mathcal{E}| \geq (n-2)!$. As $\sum_{i=1}^{n-1} c(C_i) = c(K)$, every tour $T$ of $\mathcal{E}$ has cost $c(T) \geq \tau(K)$. Therefore, $c(H) \leq c(T)$ for every $T \in \mathcal{E}$. $\square$

To see that the assertion of Theorem 3.3 is best possible, choose a tour $H$ in $K$ and an arc $a$ not in $H$. Let every arc in $H$ be of cost one, let $c(a) = n(n-1)$ and let every arc not in $A(H) \cup \{a\}$ be of cost zero. Clearly the cost of $H$ is less than the average (which is $n^2/(n-1)$), but only tours using the arc $a$ have higher cost.

**Corollary 3.4** *Let $n \neq 6$. Then the domination number of Algorithm 3.1 is at least $(n-2)!$.*

We can show that the domination number of Algorithm 3.1 is less than $4(n-2)!$. Let $x, y, u, v$ be four distinct vertices in $K$. Let $c(xy) = c(uv) = 1, c(xv) = c(uy) = n$, and let the cost of an arc different from the above four be zero. As there are less than $4(n-2)!$ tours using arcs $xy$, $uv$, $uy$ and/or $xv$, there are less than $4(n-2)!$ tours with positive cost. Observe that $\tau_{xy}(K) = \tau_{uv}(K) = 1 + \frac{1}{n-2}$, $\tau_{zw}(K) > \frac{n}{n-2}$, where $zw$ is any arc other than $xy$ and $uv$ (after contraction of $zw$ at least one of the two arcs of cost $n$ will remain), and $1 + \frac{1}{n-2} < \frac{n}{n-2}$. Thus, Algorithm 3.1 starts by choosing one of the two arcs of cost one, hence it will return a tour of positive cost.

The proof of Theorem 3.3 shows another way of obtaining a tour of $K$ dominating at least $(n-2)!$ of others. If we had a decomposition of $A(K)$ into tours, then we could choose the cheapest tour, which would have cost at most $\tau(K)$. This approach would allow us to yield a tour of factorial domination number even faster (in $O(n^2)$ time). However, in this case, we need to know at least one decomposition rather than the fact of its existence. Such decompositions are non-trivial to obtain when $n$ is even (see the remark before Theorem 3.3). In practice, this approach would very likely give worse results than those of the GEA.

# 4 GEA for the QAP

The QAP can be stated as follows. Given a pair of $n \times n$ matrices of reals $A = [a_{ij}]$ and $B = [b_{ij}]$, find a permutation $\pi$ on $[1, n]$ that provides minimum to $\phi(\pi) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} b_{\pi(i)\pi(j)}$. Let $S_n$ denote the symmetric group of permutations on $[1, n]$. The QAP can be reformulated as a problem of type (1): find $\min\{\phi(\pi) = \phi(\pi(1), ..., \pi(n)) : \ \pi \in S_n\}$. Recall that it is assumed that $\mathbf{P}(\pi \in S_n) = 1/n!$.

To show that Algorithm 2.1 is polynomial, it suffices to prove that the conditional expectations for the QAP can be computed in polynomial time. Without loss of generality, we may restrict ourselves to $\mathbf{E}(\phi|\pi(1) = c(1), ..., \pi(k) = c(k))$, $0 \leq k \leq n$, where $c(1), ..., c(k)$ are distinct constants from $[1, n]$. Let $M = [1, n] - \{c(1), ..., c(k)\}$. By linearity of expectation,

$$\mathbf{E}(\phi|\pi(1) = c(1), ..., \pi(k) = c(k)) =$$

$$\sum_{i=1}^{k} \sum_{j=1}^{k} a_{ij} b_{c(i)c(j)} + \frac{1}{n-k} \sum_{i=1}^{k} \sum_{j=k+1}^{n} a_{ij} \sum_{m \in M} b_{c(i)m} +$$

$$\frac{1}{n-k} \sum_{i=k+1}^{n} \sum_{j=1}^{k} a_{ij} \sum_{m \in M} b_{mc(j)} + \frac{1}{n-k} \sum_{i=k+1}^{n} a_{ii} \sum_{m \in M} b_{mm} +$$

$$\frac{1}{(n-k)(n-k-1)}\sum\{a_{ij}:\ i\neq j;\ i,j\in[k+1,n]\}\sum\{b_{st}:\ s\neq t;\ s,t\in M\}.$$

It follows from the above formula that $\mathbf{E}(\phi|\pi(1)=c(1),...,\pi(k)=c(k))$ (and the other conditional expectations) can be computed in time $O(n^2)$. Thus, Algorithm 2.1 is of complexity $O(n^5)$ for the QAP.

A set of permutations $G\subseteq S_n$ is called *sharply 2-transitive* if for every two pairs $(i,j),(k,t)$ of distinct elements of $[1,n]$ there is one and only one permutation $\pi\in G$ such that $\pi(i)=k,\pi(j)=t$. Clearly, $|G|=n(n-1)$.

The proof of Theorem 4.2 uses the following lemma by H. Zassenhaus (published in 1935; see Theorem 20.3 in [15]).

**Lemma 4.1** *There is a sharply 2-transitive permutation group on* $[1,n]$ *if and only if* $n$ *is a prime power.*

**Theorem 4.2** *Let* $n$ *be a prime power and let* $\mu$ *be a permutation from* $S_n$ *such that* $\phi(\mu)\leq\mathbf{E}\phi$. *Then* $|\{\pi:\ \phi(\pi)\geq\phi(\mu)\}|\geq(n-2)!$.

**Proof:** The multiplication in $S_n$ is determined as follows: for $\pi,\nu\in S_n$ and $i\in[1,n]$, $\pi\nu(i)=\nu(\pi(i))$. Let $L=\{(i,j):\ i,j\in[1,n],i\neq j\}$.

By Lemma 4.1, there exists a sharply 2-transitive permutation group $H$ on $[1,n]$. Since $H$ is a subgroup of $S_n$ and $|H|=n(n-1)$, there is a decomposition of $S_n$ into $m=(n-2)!$ cosets of the form $H\tau$. Let $\tau_1,...,\tau_m$ be a collection of permutations such that $S_n=\cup_{s=1}^m H\tau_s$.

We prove that the set $H\tau_s$ is sharply 2-transitive for every $s\in[1,m]$. Let $(i,j),(k,t)\in L$ be arbitrary. As $H$ is 2-transitive and $\tau_s$ is a permutation, there exists $\pi\in H$ such that $\pi(i)=\tau_s^{-1}(k),\pi(j)=\tau_s^{-1}(t)$. Thus, $\pi\tau_s(i)=k,\pi\tau_s(j)=t$. This and the fact that $|H\tau_s|=n(n-1)$ imply that $H\tau_s$ is sharply 2-transitive.

It follows from the formula of the conditional expectations for the QAP that

$$\mathbf{E}\phi=\frac{1}{n(n-1)}\sum_{(i,j)\in L}a_{ij}\sum_{(i,j)\in L}b_{ij}+\frac{1}{n}\sum_{i=1}^n a_{ii}\sum_{j=1}^n b_{jj}.$$

We now prove that

$$\sum_{\pi\in H\tau_s}\phi(\pi)=n(n-1)\mathbf{E}\phi \tag{4}$$

for every $s\in[1,m]$. We can express the above sum as follows:

$$\sum_{\pi\in H\tau_s}\phi(\pi)=\sum_{(i,j)\in L}\sum_{\pi\in H\tau_s}a_{ij}b_{\pi(i)\pi(j)}+\sum_{i=1}^n\sum_{\pi\in H\tau_s}a_{ii}b_{\pi(i)\pi(i)}. \tag{5}$$

As $H\tau_s$ is sharply 2-transitive,

$$\sum_{(i,j)\in L}\sum_{\pi\in H\tau_s} a_{ij}b_{\pi(i)\pi(j)} = \sum_{(i,j)\in L} a_{ij} \sum_{(i,j)\in L} b_{ij}.$$

To complete the proof of (4), it suffices to show that

$$\sum_{\pi\in H\tau_s} b_{\pi(i)\pi(i)}(n-1) = \sum_{j=1}^{n} b_{jj}$$

for every fixed $i \in [1,n]$. To prove the last equality, it is sufficient to show that

$$|\{\pi \in H\tau_s : \ \pi(i) = k\}| = n-1$$

for fixed $i$ and $k$. This follows from

$$\{\pi \in H\tau_s : \ \pi(i) = k\} = \{\pi_t \in H\tau_s : \ t \in [1,n]-k, \pi_t(i) = k, \pi_t(j) = t\},$$

where $j$ is a fixed element of $[1,n]-i$.

By (4), we can choose a permutation $\nu_s \in H\tau_s$ such that $\phi(\nu_s) \geq \mathbf{E}\phi$. As for the permutation $\mu$, given in this theorem, $\phi(\mu) \leq \mathbf{E}\phi$, we conclude that $\phi(\mu) \leq \phi(\nu_s)$ for every $s \in [1,m]$. $\square$

We conjecture that the assertion of the above theorem is valid for every integer $n \geq 2$. Combining the last theorem with Theorem 2.2, we obtain the following:

**Corollary 4.3** *The domination number of Algorithm 2.1 applied to the QAP, $\mathcal{A}$, is $dom(\mathcal{A},n) \geq (n-2)!$ for every prime power $n$.*

A permutation group $G \subseteq S_n$ is called *2-transitive* if for every two pairs $(i,j),(k,t)$ of distinct elements of $[1,n]$ there are $s(i,j,k,t) > 0$ permutations $\pi \in G$ such that $\pi(i) = k, \pi(j) = t$. Every 2-transitive group has the property that the number of permutations carrying one pair of distinct elements to another pair is constant, i.e., $s(i,j,k,t)$ is a constant. (The set of such permutations is a coset of the subgroup fixing the first pair, and so this is just the fact that all such cosets contain the same number of elements.)

For almost all values of $n$, the only 2-transitive groups of degree $n$ are the symmetric and alternating groups (see [6]). The only two series of $n$ such that there exist 2-transitive permutations groups of degree $n$ and polynomial (in $n$) order are prime powers and numbers of the form $(q^d-1)/(q-1)$, where $q$ is a prime power and $d \geq 2$ (their order is roughly $n^{d+1}$, i.e., polynomial when $d$ is bounded) [6]. Using this result, we can readily obtain factorial bound for $|\{\pi : \ \phi(\pi) \geq \phi(\mu)\}|$ (see Theorem 4.2) when $n = (q^d-1)/(q-1)$ and $d$

is bounded. Still, this bound is valid for a small fraction of positive integers and unlikely to be sharp. Thus, we proceed by deriving a bound which is even weaker, but valid for all sufficiently large $n$.

The following number-theoretical assertion can be found in [2] (this was proved by R. Baker and G. Harman).

**Lemma 4.4** *Let $p_1, p_2, \ldots$ be the increasing sequence of all primes. Then $p_{k+1} - p_k \leq k^{\alpha+o(1)}$ for every $k \geq 2$, where $\alpha = 0.535$.*

**Theorem 4.5** *Let $\beta > 1$ be arbitrary. Then, for sufficiently large $n$, there are at least $n!/\beta^n$ permutations $\omega$ such that $\phi(\omega) \geq \mathbf{E}\phi$.*

**Proof:** Let $p$ be the largest prime number not exceeding $n$. Assume that $p < n$ (otherwise the proof is trivial).

By the formula of total expectation given in the proof of Theorem 2.2, there is a sequence $x^0_{n-p+1}, \ldots, x^0_n$ of distinct numbers in $[1,n]$ such that $\mathbf{E}\phi \leq \mathbf{E}(\phi(\pi)|\pi(i) = x^0_i,\ i \in [n-p+1,n])$. Let $S'$ be the set of permutations, $\pi$ in $S_n$, such that $\pi(i) = x^0_i$ for all $i \in [n-p+1,n]$. We want to prove that there are $(p-2)!$ distinct permutations, $\omega$, in $S'$ with $\phi(\omega) \geq \mathbf{E}(\phi(\pi)|\pi \in S')$. This will imply that there are $(p-2)!$ distinct permutations, $\omega$, in $S'$ with $\phi(\omega) \geq \mathbf{E}\phi$.

Let $y^0_1, y^0_2, \ldots, y^0_p$ be the sequence obtained from $1, 2, \ldots, n$ by deleting the integers $x^0_{n-p+1}, \ldots, x^0_n$. If $\pi$ is an element in $S_p$ then define the element $\pi' \in S_n$ as follows: $\pi'(i) = x^0_i$ for all $i \in [n-p+1,n]$ and $\pi'(i) = y^0_{\pi(i)}$ for all $i \in [1, n-p]$. Clearly the above mapping is a bijection from $S_p$ to $S'$.

Let $H$ be a sharply 2-transitive permutation group on $[1,p]$. Let $m = (p-2)!$ and let $\tau_1, \tau_2, \ldots, \tau_m$ be a collection of permutations such that $S_p = \cup^m_{s=1} H\tau_s$. Let $H'_s = (H\tau_s)'$ (i.e. we use the earlier mentioned bijection). Observe that $H'_1, H'_2, \ldots, H'_m$ partitions $S'$.

Analogously to the proof of Theorem 4.2, it is not difficult to show that the permutation of highest cost in each of $H'_1, H'_2, \ldots, H'_m$ are distinct and have cost at least $\mathbf{E}(\phi(\pi)|\pi \in S')$.

Now it is suffices to demonstrate that $(p-2)! \geq n!/\beta^n$ for $n$ large enough. By Lemma 4.4, the gap between $p$ and the next prime $p^+$, $p^+ - p$, does not exceed $k^{\alpha+o(1)}$, where $p$ is the $k$th prime and $\alpha = 0.535$. Therefore, the gap between $n$ and $p$ is at most $n^{\alpha+o(1)}$. Thus, for $n$ large enough, $\pi$ is not worse than $q = \lfloor n - n^{\alpha+o(1)} - 2 \rfloor! \geq \lceil n - n^{\gamma} \rceil!$ permutations, where $\gamma = 0.6$. However, this implies that $q \geq n!/n^{n^\gamma} \geq n!/\beta^n$ for every sufficiently large $n$. □

Combining the results of the last theorem and Theorem 2.2, we obtain the following:

**Corollary 4.6** *Let $\beta > 1$ be arbitrary. Then, the domination number of Algorithm 2.1 applied to the QAP, $\mathcal{A}$, is $dom(\mathcal{A}, n) \geq n!/\beta^n$ for every sufficiently large $n$.*

For the QAP, we conjecture that the domination number of Algorithm 2.1 is at least $(n-2)!$ for every $n \geq 2$. It would be very interesting to verify whether there exists a polynomial approximation algorithm $\mathcal{C}$ for the QAP such that $dom(\mathcal{C}, n) \geq \alpha n!$ for some positive real constant $\alpha < 1$.

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# References

[1] N. Alon and J. Spencer, *The probabilistic method*, Wiley, New York, 1992.

[2] E. Bach and J. Shallit, *Algorithmic number theory, Volume 1*, MIT Press, Cambridge, Ma., 1996.

[3] E. Balas and N. Simonetti, Linear time dynamic programming algorithms for some new classes of restricted TSP's. *Proc. IPCO V*, LNCS **1084** (1996) 316-329.

[4] C. Berge, *The Theory of Graphs*, Methuen, London, 1958.

[5] R.E. Burkard, V.G. Deineko and G.J. Woeginger, The travelling salesman problem and the PQ-tree. *Proc. IPCO V*, LNCS **1084** (1996) 490-504.

[6] P.G. Cameron, Permutation Groups. R. L. Grahaam, M. Grötschel, L. Lovász, eds., *Handbook of Combinatorics*, Elsevier, Amsterdam, 1995.

[7] J. Carlier and P. Villon, A new heuristic for the traveling salesman problem. *RAIRO* **24** (1990) 245-253.

[8] E. Çela, *The Quadratic Assignment Problem: Theory and Algorithms*, Kluwer, 1998.

[9] V. Deineko and G.J. Woeginger, A study of exponential neighbourhoods for the travelling salesman problem and for the quadratic assignment problem. TR Woe-05, TU Graz, Austria, July (1997).

[10] F. Glover, Ejection chains, reference structures, and alternating path algorithms for traveling salesman problem. Research Report, Univ. of Colorado-Boulder, Graduate School of Business (1992). A shortened version appeared in *Discrete Appl. Math.* **65** (1996) 223-253.

[11] F. Glover, G. Gutin, A. Yeo and A. Zverovich, Construction heuristics for the asymmetric traveling salesman problem, *European J. Oper. Res.*, to appear.

[12] F. Glover and A. Punnen, The traveling salesman problem: New solvable cases and linkages with the development of approximation algorithms. *J. Oper. Res. Soc.* **48** (1997) 502-510.

[13] G. Gutin, Exponential neighbourhood local search for the traveling salesman problem. Special Issue of *Computers and OR* on the TSP **26** (1999) 313-320.

[14] G. Gutin and A. Yeo, *TSP heuristics with large domination number*, Technical Report No. 12/98, Dept Maths and Stats, Brunel University.

[15] D.S. Passman, *Permutation Groups*, Benjamin Inc., New York, 1968.

[16] A.P. Punnen, The traveling salesman problem: new polynomial approximation algorithms and domination analysis, preprint, 1996.

[17] A.P. Punnen and F. Glover, *Implementing ejection chains with combinatorial leverage for the TSP*, Research Report, Univ. of Colorado-Boulder, 1997.

[18] P. Raghavan, Probabilistic construction of deterministic algorithms: approximating packing integer programs. *J. Computer and System Sci.* **37** (1988) 130-143.

[19] A.N. Shiryayev, *Probability*, Springer-Verlag, New York, 1984.

[20] T.W. Tillson, A hamiltonian decomposition of $K^*_{2m}$, $m \geq 8$. *J. Combinatorial Theory* **B 29** (1980) 68-74.

[21] A. Yeo, Large exponential neighbourhoods for the traveling salesman problem, Preprint no. 47, Department Maths and CS, Odense University, 1997.