Constraint Satisfaction

and

Graph Theory

Plan

- How much is lost by focusing on graphs
- How graphs help intuition
- How graph theory gets used

Graph Colouring

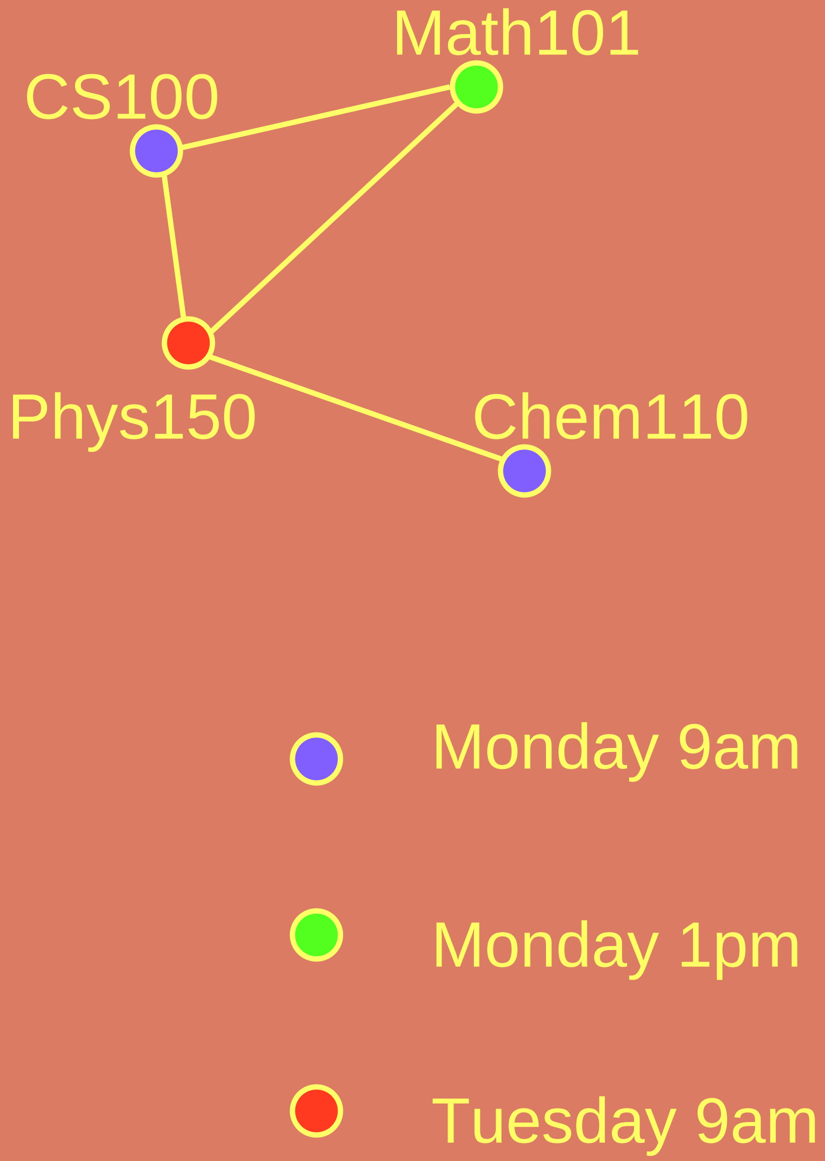
- One of the most common illustrations of constrain satisfaction problems
- All variables have the same domain, consisting of *k* colours
- One binary constraint requires **adjacent** pairs of variables to obtain **different** colours

Exam Scheduling via Graph Colouring

G = courses and their conflicts

Colours = examination times

A colouring of *G* = a schedule of exam periods in which conflicting courses are scheduled at different times


Math101
CS100
Phys150
Chem110
Monday 9am
Monday 1pm
Tuesday 9am

More realistic models

are possible with the full power of constraint satisfaction...

More realistic models

are possible with the full power of constraint satisfaction...

BUT

graph theory is not just colouring

Graph Homomorphisms

Suppose G and H are graphs.

A **homomorphism** of G to H is a mapping $f : V(G) \to V(H)$ such that $uv \in G$ implies $f(u)f(v) \in H$

Adjacent vertices have adjacent images

Exam Scheduling

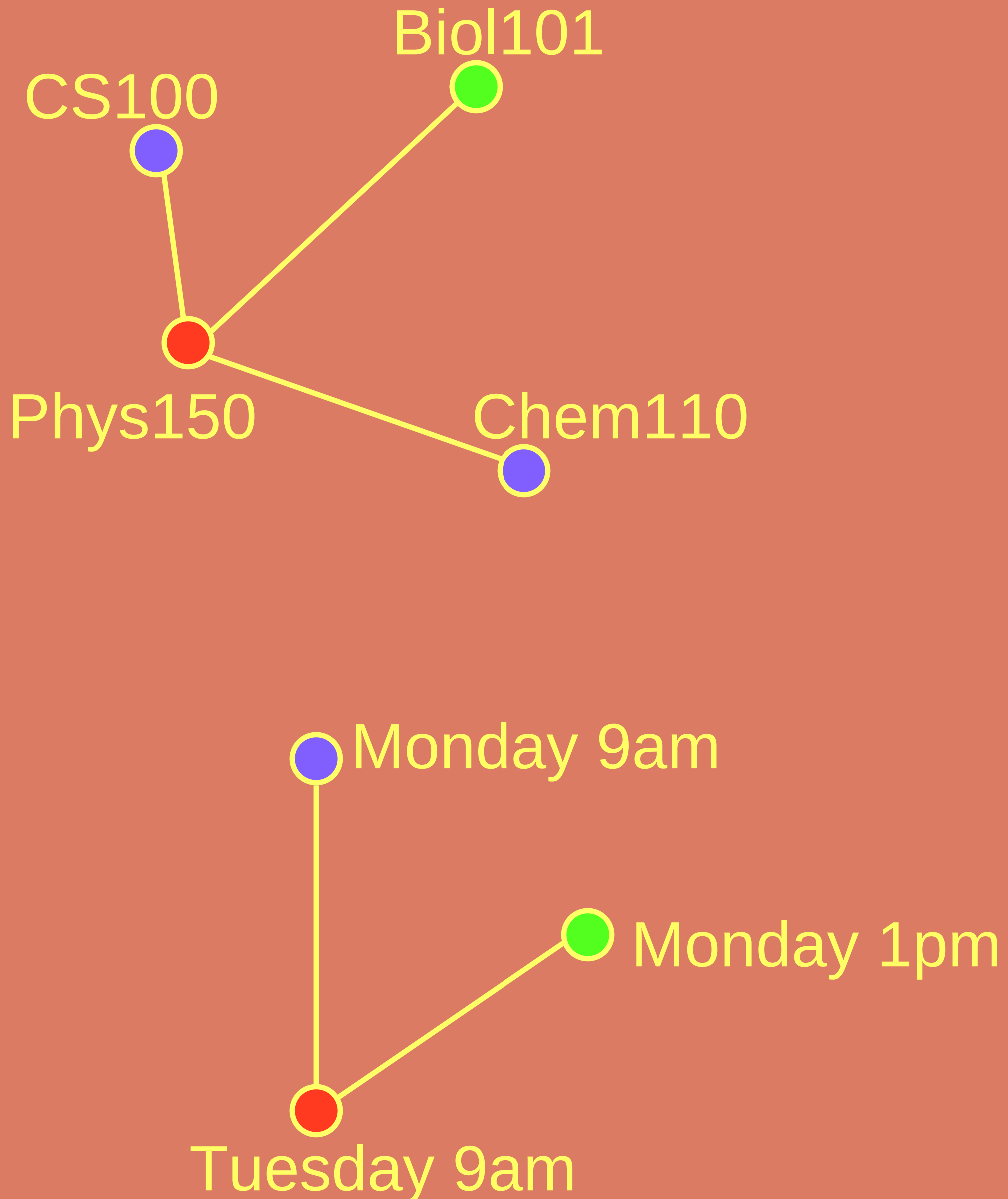
Via Graph Homomorphisms

G = courses and their conflicts

H = periods and their compatibility

A homomorphism of *G* to *H* =

a schedule of exam periods so that conflicting courses are scheduled at compatible times



Biol101
CS100
Phys150
Chem110
Monday 9am
Monday 1pm
Tuesday 9am

General CSP's

- Given a **vocabulary** (finite number of relation symbols, each of finite arity)
- and two **structures** *G* and *H* over the vocabulary, with ground sets *V(G)* and *V(H)*, and interpretations of all the relation symbols, as relations over the set of the stated arities

Homomorphism of G to H

A mapping $f : V(G) \to V(H)$ such that

$(v_1,\ldots,v_r) \in R(G)$ implies $(f(v_1),\ldots,f(v_r)) \in R(H)$,

for all relation symbols R of the vocabulary

Graph Theory in CSP

Take the vocabulary to consist of

one binary relation symbol *E*

(possibly with restricted interpretations, to reflexive, symmetric, etc, relations)

V(G) = courses *V(H)* = times

- Conflicting courses at compatible times
  (*R(G)* and *R(H)* binary relations)

- At most three chemistry exams in one day
  (*R(G)* and *R(H)* quaternary relations)

- All physics exams on the first ten days
  (*R(G)* and *R(H)* unary relations)

- *G* changes every semester
- *H* tends to stay constant for a while

The Constraint Satisfaction Problem

for a fixed a structure *H*:

$$\mathrm{CSP}(H)$$

- Given a structure *G* (same vocabulary as *H*)
- Is there a homomorphism of G to H ?

Dichotomy Conjecture [FV]

For each H, the problem CSP(H) is in P or is NP-complete

When H is a graph

(one binary relation symbol E)

CSP(H) is called the ***H*-colouring problem**

- graphs ... $E(G)$, $E(H)$ symmetric
- digraphs ... unrestricted interpretations

Also denoted COL(H) and HOM(H)

CSP(K_n)

(one binary relation, `distinct')

CSP(K_n) is the ***n*-colouring problem**

DICHOTOMY

The ***n*-colouring problem** is

- in P for $n = 1, 2$
- NP-complete for all other n

The graph ***H*-colouring problem** is

- in P for H bipartite or with a loop
- NP-complete for all other H

DICHOTOMY

The ***H*-colouring problem** is [HN]

- in P for H bipartite or with a loop
- NP-complete for all other H

H-colouring

- May assume G is connected

- May assume H is connected

- May assume H is a core

(there is no homomorphism to a proper subgraph)

Polynomial cases

- H has a loop (some $vv \in E(H)$)

  All G admit a homomorphism to H

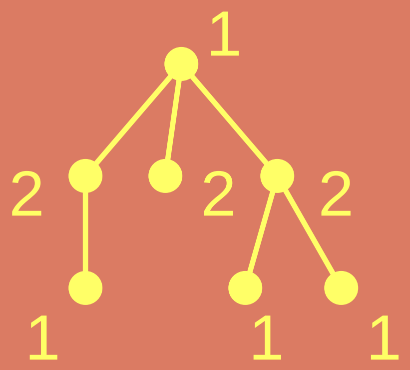
- H is bipartite

  ($V(H)$ = two independent sets)

  Thus we may assume $H = K_2$ (core)

Deciding if a 2-colouring exists

Obvious algorithm



Deciding if a 2-colouring exists

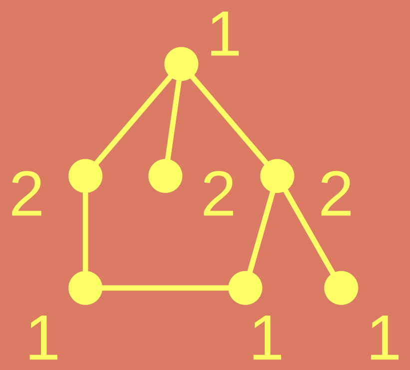
Algorithm succeeds

No odd cycles

2-colouring exists

Deciding if a 2-colouring exists

Obvious algorithm



Deciding if a 2-colouring exists

Algorithm succeeds

No odd cycles

2-colouring exists

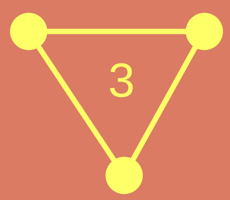
Deciding if a 2-colouring exists

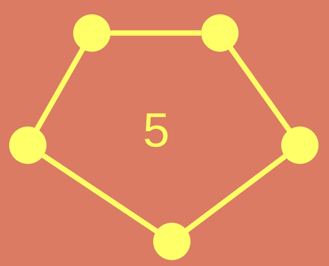
Algorithm succeeds

No odd cycles $\Leftrightarrow$ 2-colouring exists

G has a 2-colouring (is **bipartite**)

if and only if it contains no induced

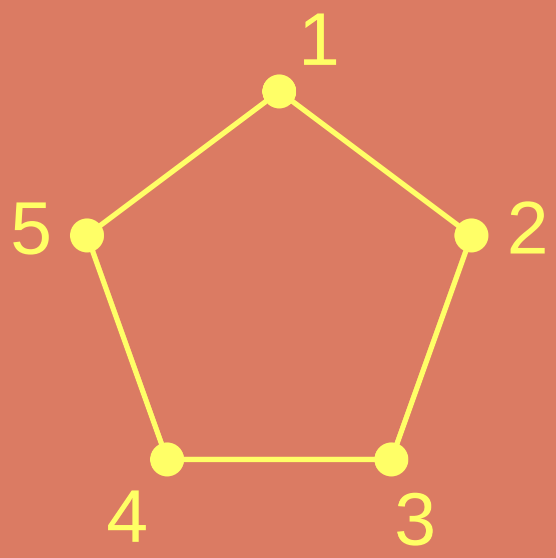




7 . . .

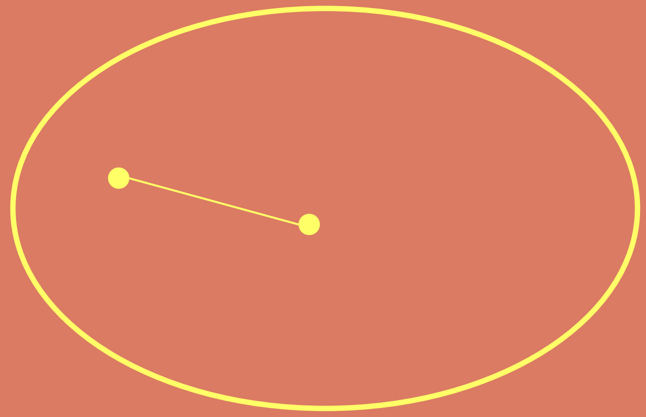
NP-complete cases

- Cliques K_n with $n \geq 3$
- Pentagon C_5



C_5 - colouring is NP-complete

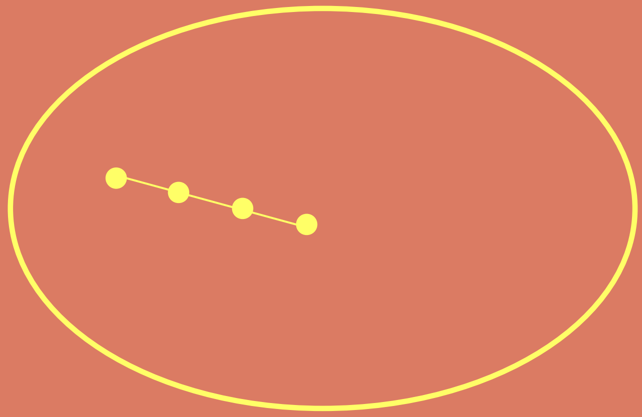
Reduce from 5-colourability:



G

C_5 - colouring is NP-complete

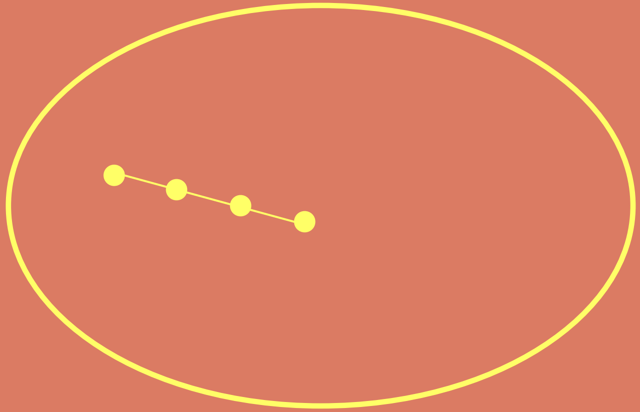
Reduce from 5-colourability:



G^*

C_5 - colouring is NP-complete

Reduce from 5-colourability:



G^*

G is 5-colourable

if and only if

G^* is C_5-colourable

C_5 - colouring is NP-complete

Reduce from K_5-colourability:

G is K_5-colourable

if and only if

G^* is C_5-colourable

From K_5 to C_5

C_5 - colouring is NP-complete

Reduce from K_5-colourability:

G is K_5-colourable

if and only if

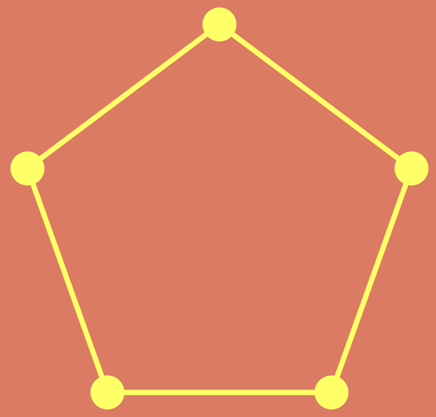
G^* is C_5-colourable

From K_5 to C_5

From denser to sparser...

W_5 - colouring is NP-complete

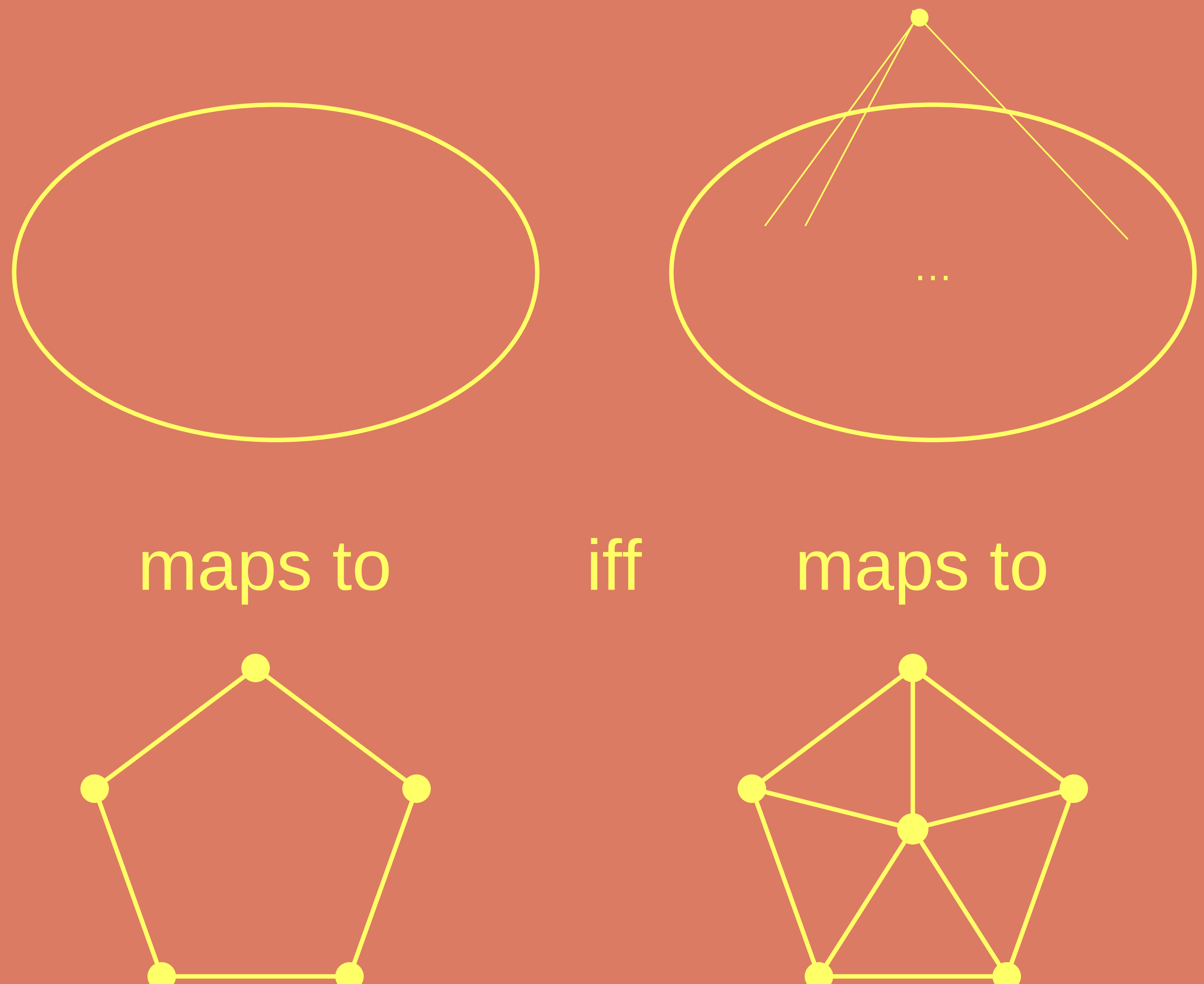
From to



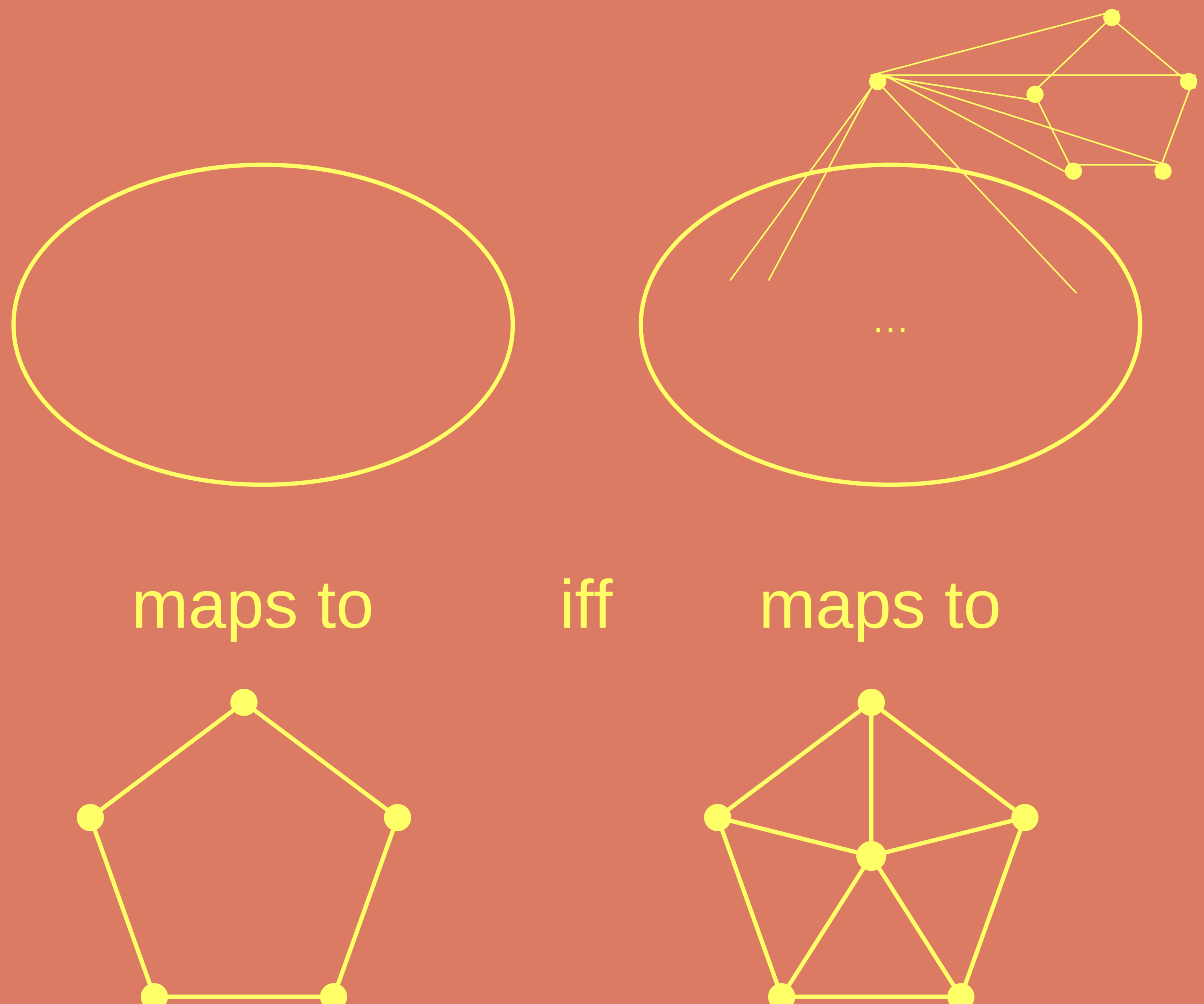
C_5

W_5

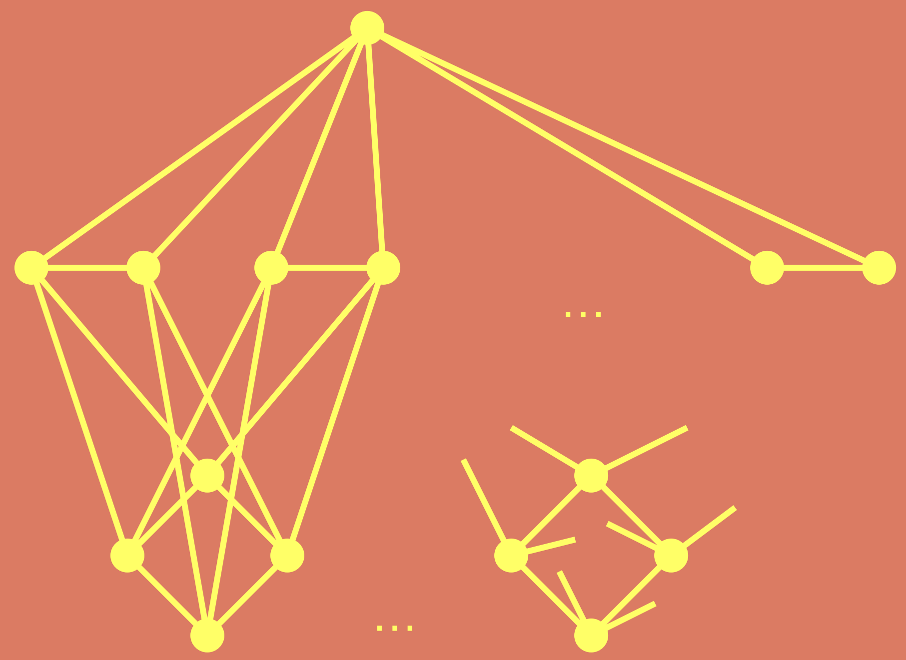
From smaller to larger...



...
maps to
iff
maps to


...
maps to
iff
maps to

Enough to prove for



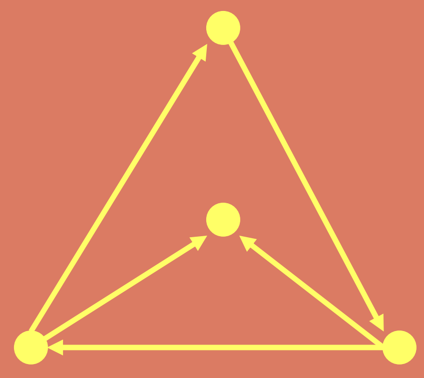
[HN] 1988
[B] 2005

(Monotonicity)

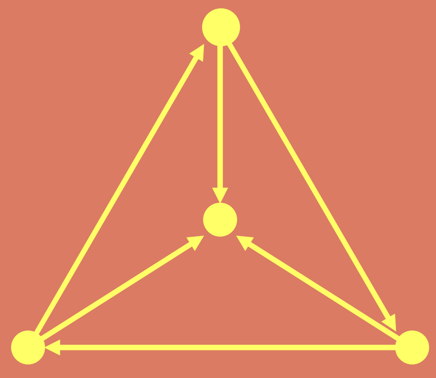
Digraph *H*-colouring

- Dichotomy not known
- No classification in terms of digraph properties proposed
- Each CSP(*H*) is polynomially equivalent to some digraph *H*'-colouring problem [FV]
- Monotonicity fails

Monotonicity fails



NP-complete



In P

[GWW]

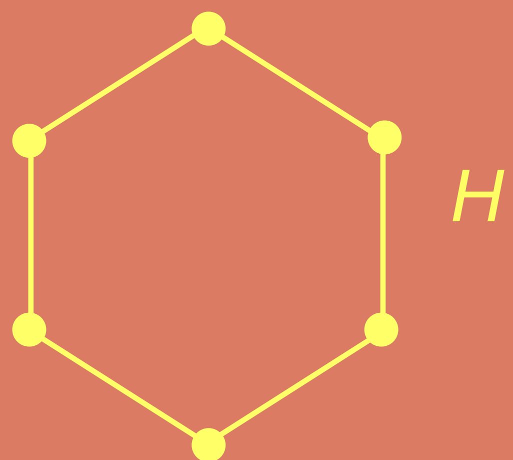
CSP(H) as H'-COL

- We may assume that H is a core

- We may replace CSP(H) by RET(H)

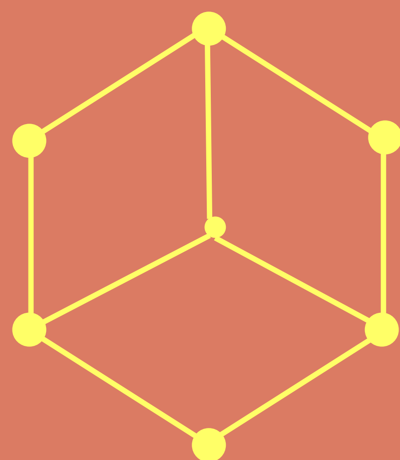
  (H is a substructure of G and we seek a homomorphism of G to H that keeps vertices of H fixed)

RET(H)



RET(H)

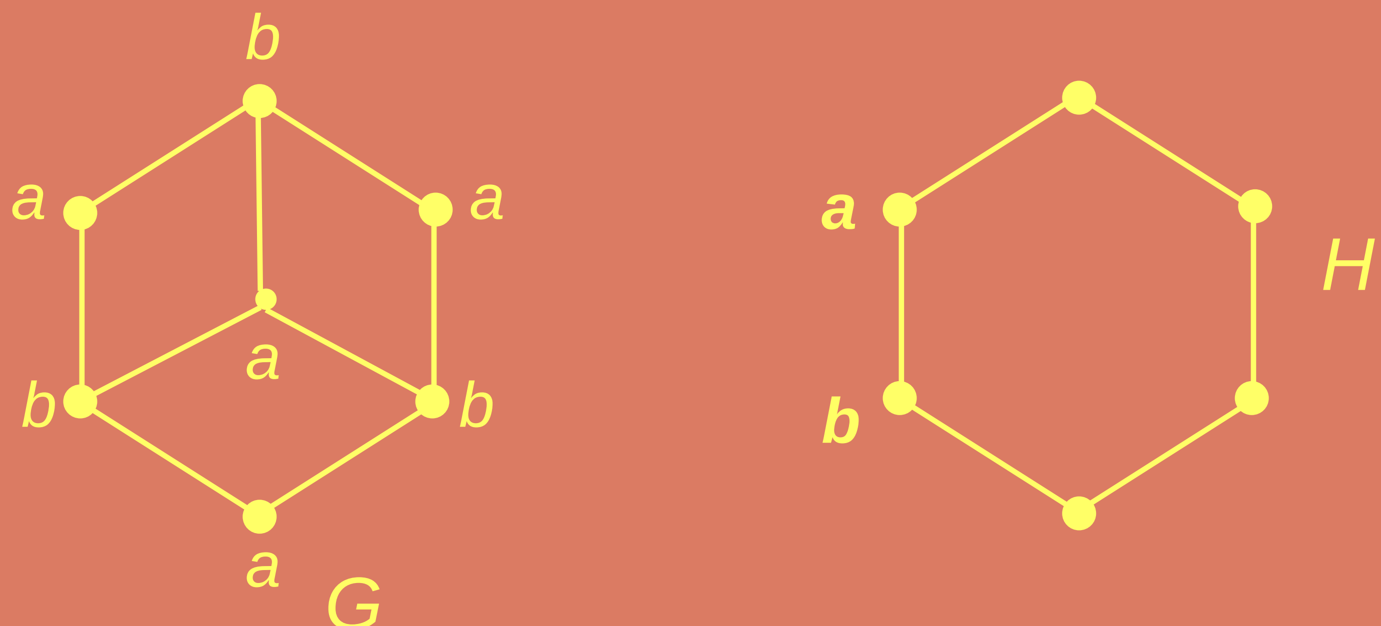
RET(H)



G

Retraction impossible

RET(H)



Retraction impossible, but homomorphism exists

RET(H)

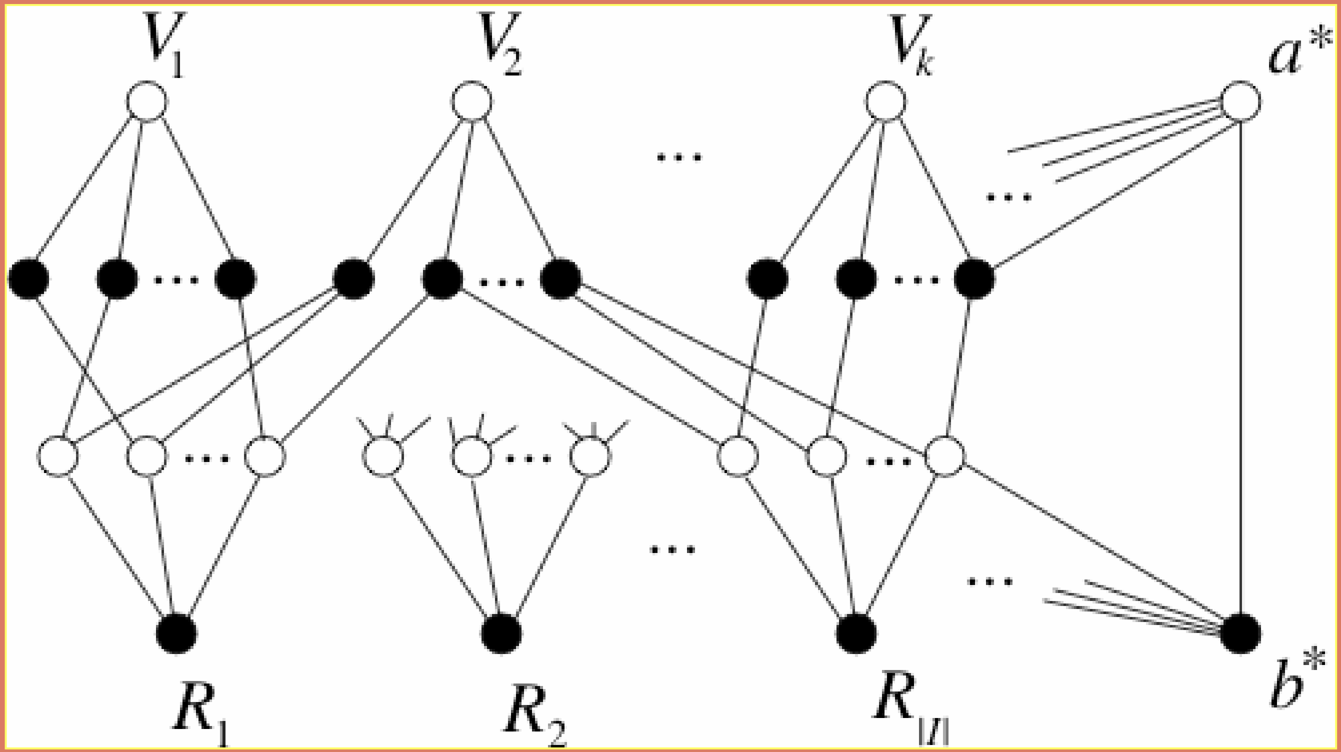
If H is a core then CSP(H) and RET(H) are polynomially equivalent

Each CSP(*H*) is polynomially equivalent to some digraph *H'*-colouring problem

- Each RET(*H*) is polynomially equivalent to some bipartite graph retraction problem
- Each bipartite graph retraction problem is polynomially equivalent to some digraph *H'*-colouring problem

RET(H) as RET(H')

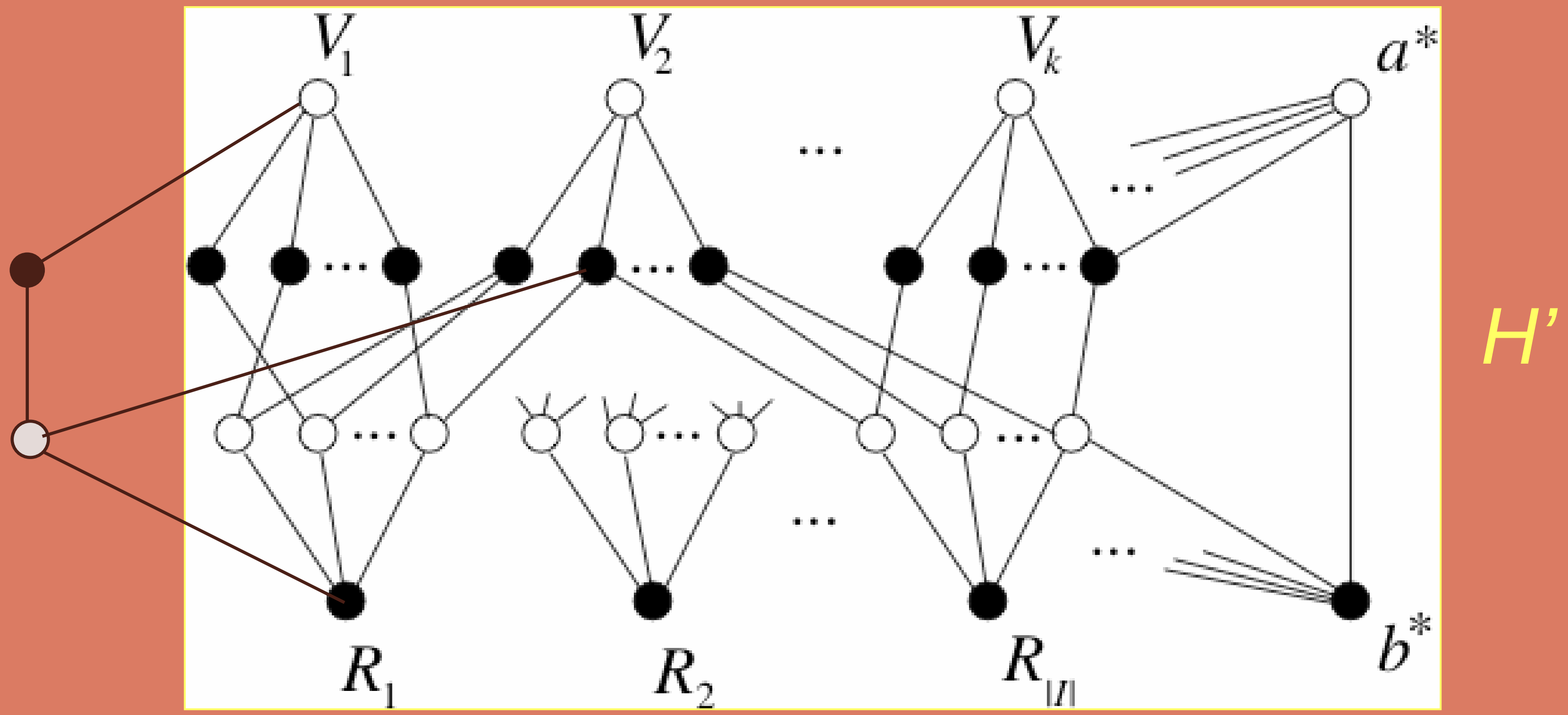
(H is an arbitrary structure, H' is a bip graph)



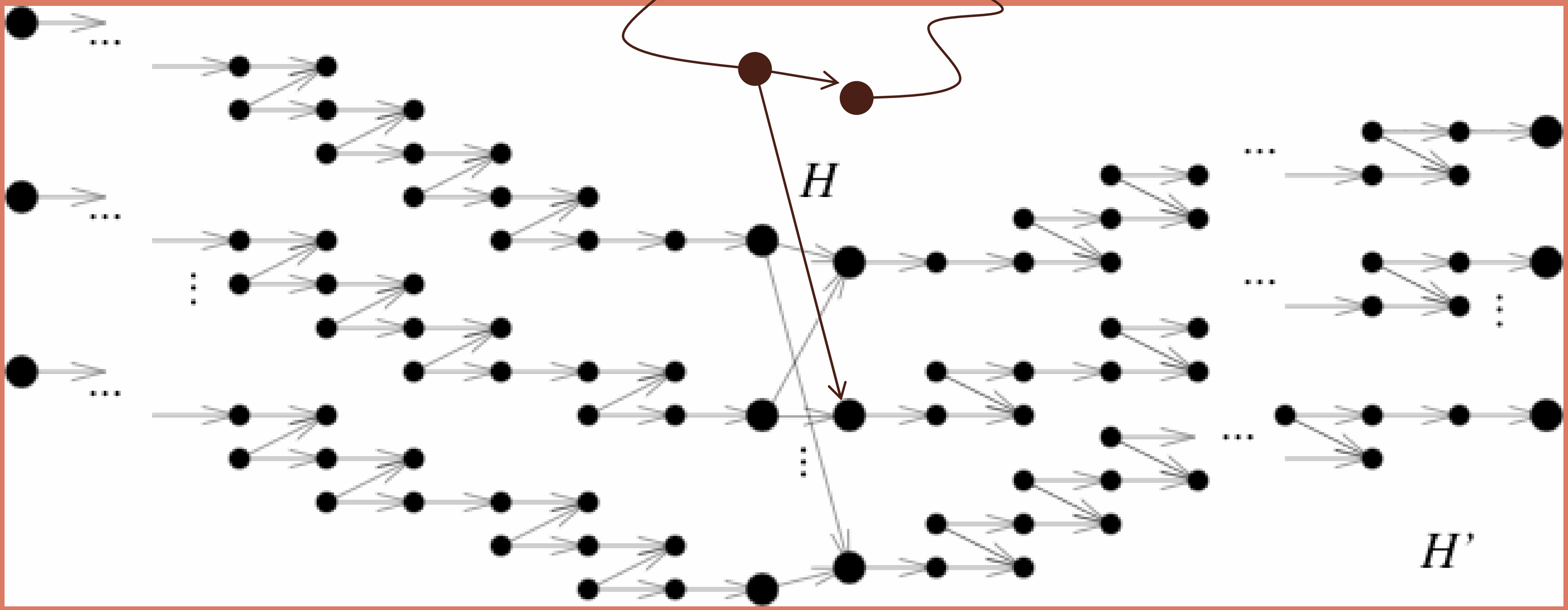
H'

RET(*H*) as RET(*H'*)

(*H* is an arbitrary structure, *H'* is a bip graph)



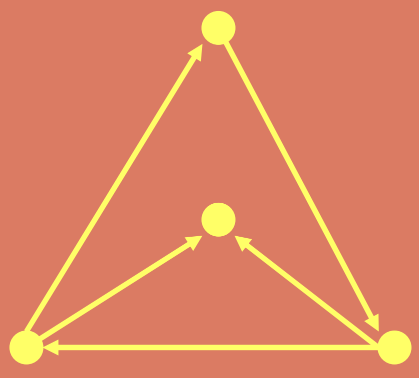
From bip RET(H) to digraph H'-colouring



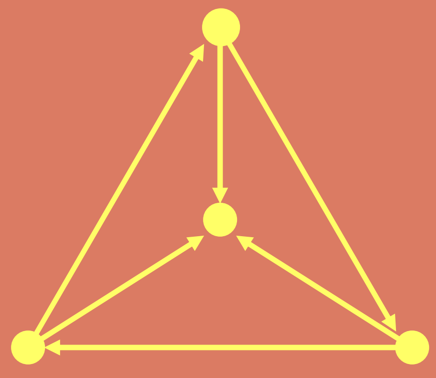
CSP(H) is polynomially equivalent

- to CSP(H') for some digraph H'
- to RET(H') for some digraph H'
- to RET(H') for some bipartite graph H'
- to RET(H') for some reflexive graph H'
- to RET(H') for some partial order H'

Monotonicity fails



NP-complete

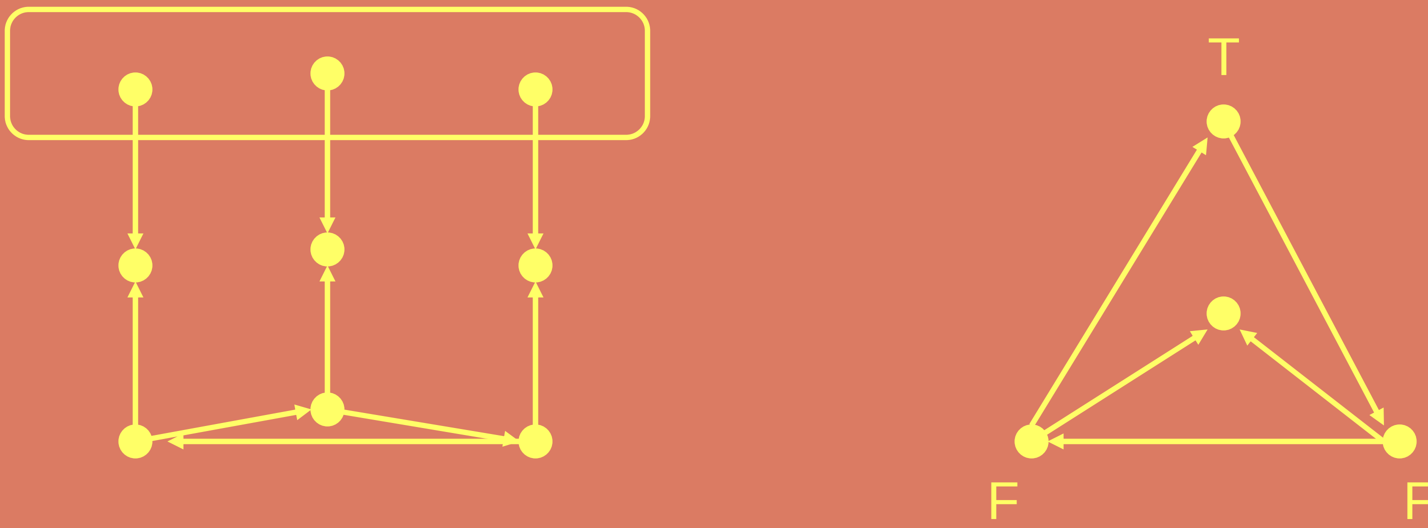


In P

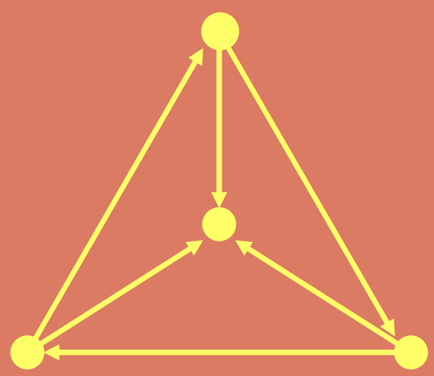
[GWW]

NP-completeness

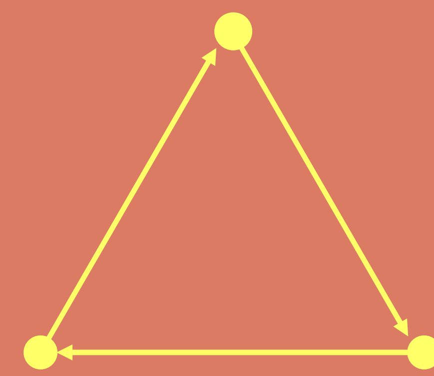
Reduce ONE-IN-THREE-SAT



Polynomial algorithm



is in *P* iff

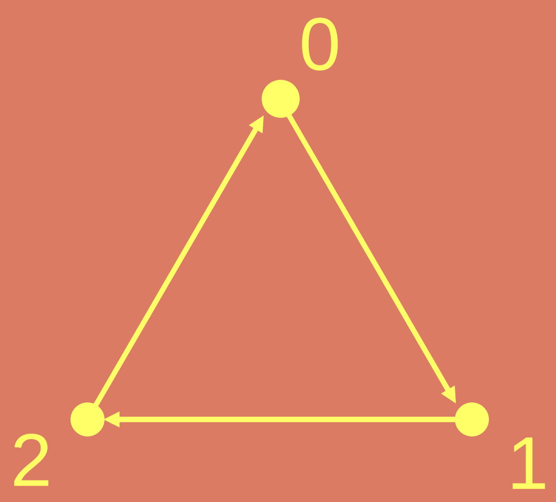


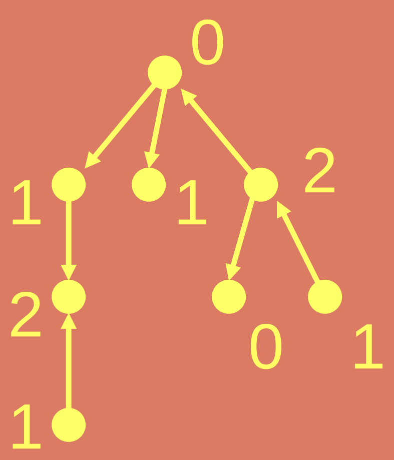
is

(and it **is**)

Deciding if a $\vec{C}_3$-colouring exists

Obvious algorithm





Deciding if a $\vec{C}_3$-colouring exists

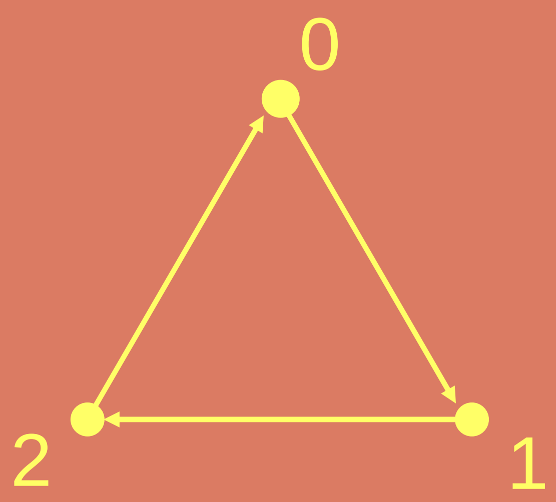
Algorithm succeeds

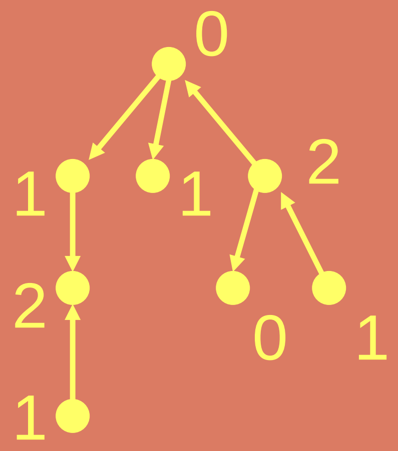
No **bad** cycles

C_3-colouring exists

Deciding if a $\vec{C}_3$-colouring exists

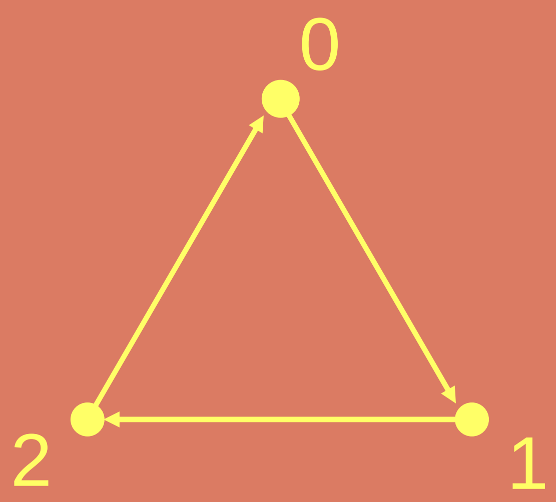
Obvious algorithm

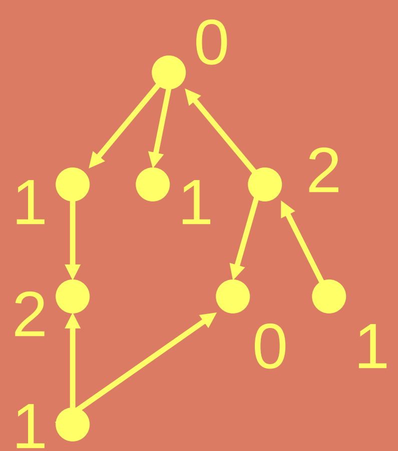




Deciding if a $\overrightarrow{C}_3$-colouring exists

Obvious algorithm





Deciding if a $\overrightarrow{C}_3$-colouring exists

Algorithm succeeds

No **bad** cycles

C_3-colouring exists

A cycle is **good** if it has net length $3k$

Deciding if a $\vec{C}_3$-colouring exists

Algorithm succeeds

$\Updownarrow$

No bad cycles

C_3-colouring exists

Deciding if a $\overrightarrow{C}_3$-colouring exists

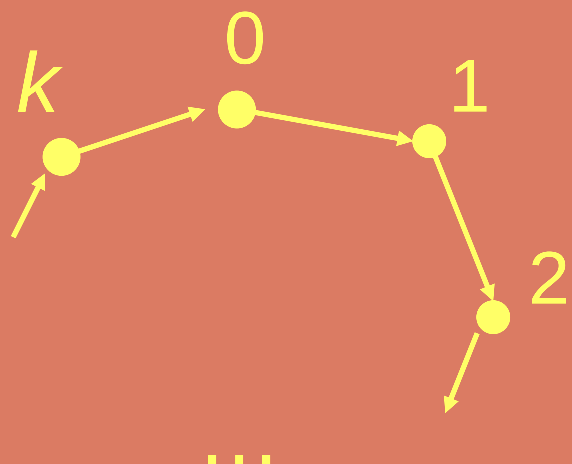
Algorithm succeeds

No bad cycles $\Longleftrightarrow$ C_3-colouring exists

G has a $\overrightarrow{C}_k$-colouring

if and only if

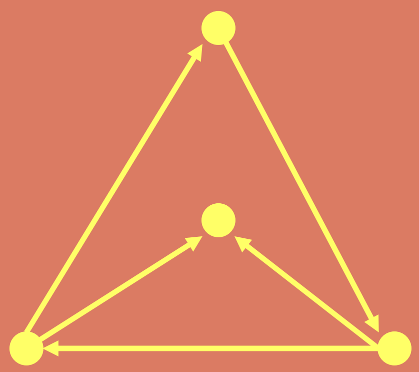
G contains no cycle of net length $\neq 0 \pmod{k}$



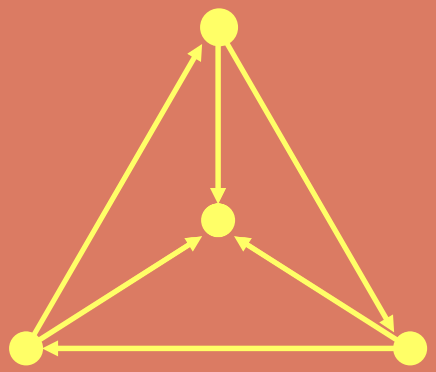
$\overrightarrow{C}_k$

[BB]

Monotonicity fails



NP-complete



In P

Not **hereditarily hard**

(Polynomial extension)

(irreflexive)

Another perspective on the undirected dichotomy

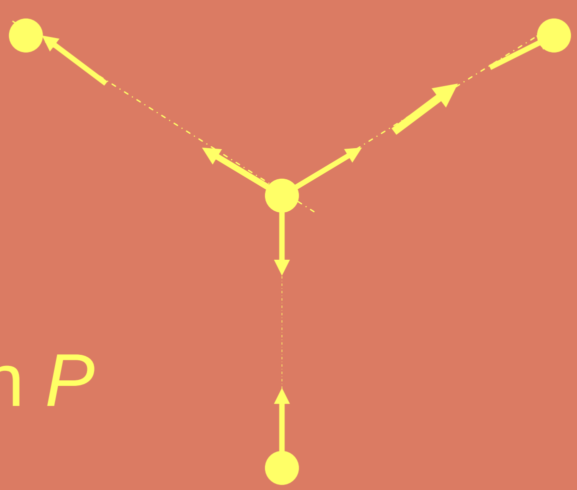
- C_{odd} is hereditarily hard

  enough to show

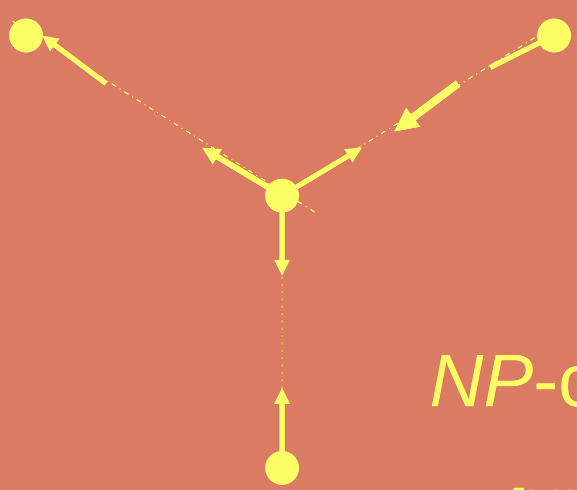
- C_3 is hereditarily hard

Two kinds of difficulty

- `Few' directed cycles
  - monotonicity fails (oscilation)
  - unclear distinctions



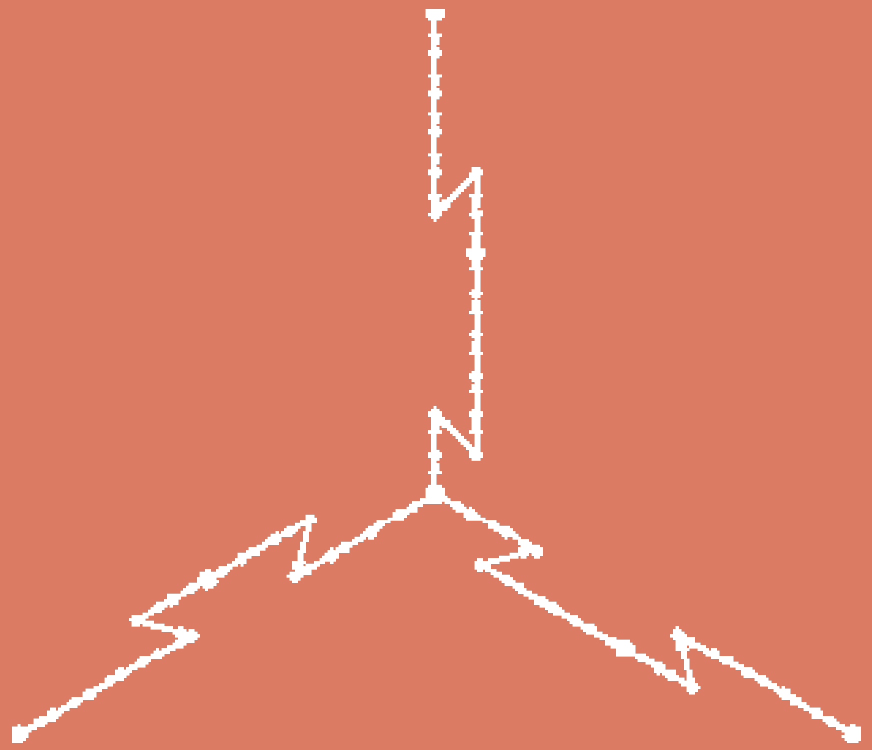
In *P*



NP-complete

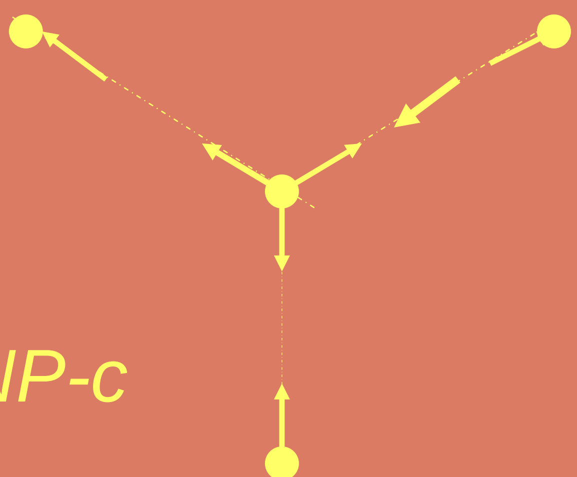
[HNZ] [GWW]

NP-complete case

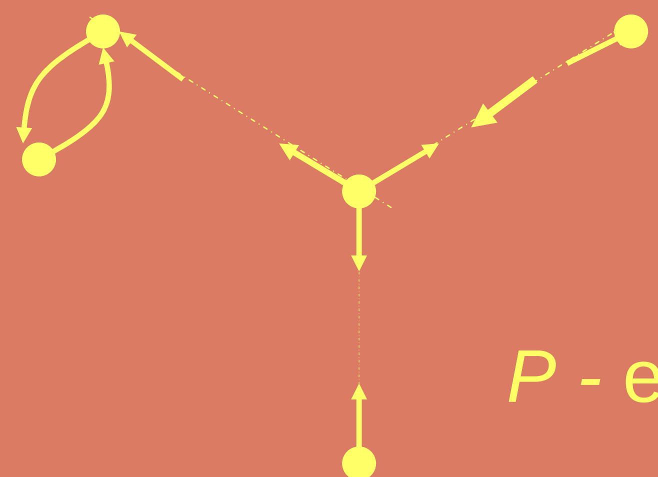


Two kinds of difficulty

- `Few’ directed cycles
  - monotonicity fails (oscilation)
  - unclear distinctions
  - BUT



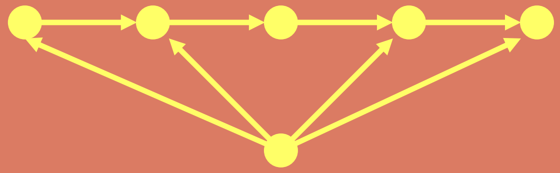
NP-c



P - extension

Two kinds of difficulty

- `Few' directed cycles

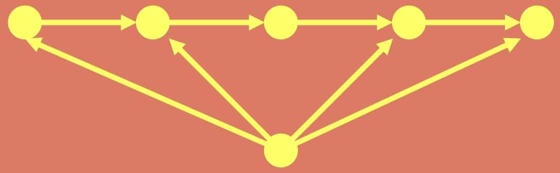


NP-complete

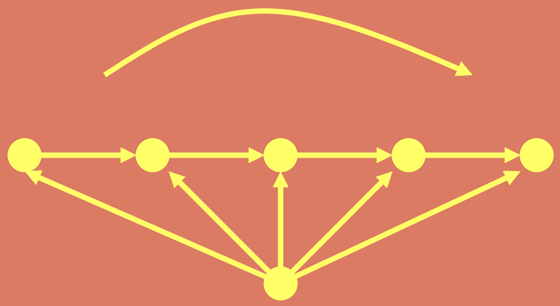
[BM] [GWW]

Two kinds of difficulty

- `Few' directed cycles
  - BUT



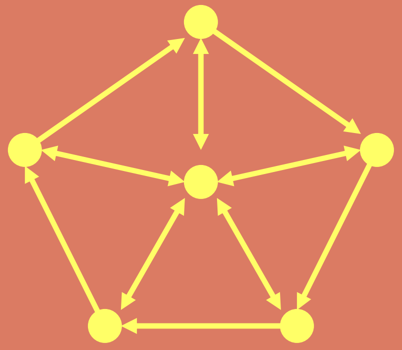
NP-complete



P - extension

Two kinds of difficulty

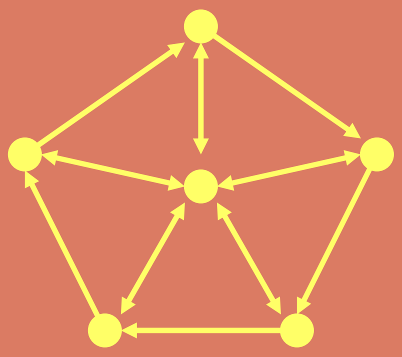
- `Many' directed cycles



H_0

Two kinds of difficulty

- `Many' directed cycles



H_0

Hereditarily hard

(if H contains H_0 and H has no loops, then CSP(H) is NP-complete)

[BHM]

Classification Conjecture [BHM]

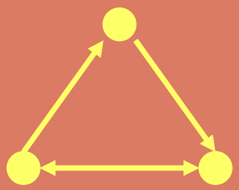
- If H^* (H with sources and sinks recursively removed) admits a homomorphism to some $\overrightarrow{C}_k$ ($k>1$) then H has a polynomial extension

  (`few cycles')

- Otherwise, H is hereditarily hard

  (`many cycles')

Status of the conjecture

- True for several graph families
- Open for 

Status of the conjecture

- True for several graph families
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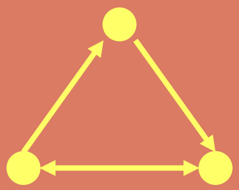
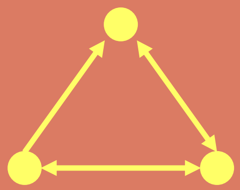
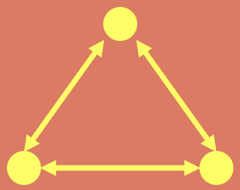
Status of the conjecture

- True for several graph families
- Open for



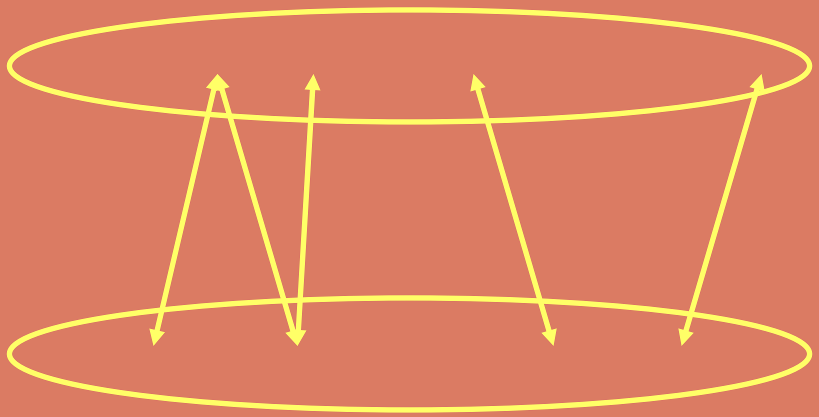


- True for



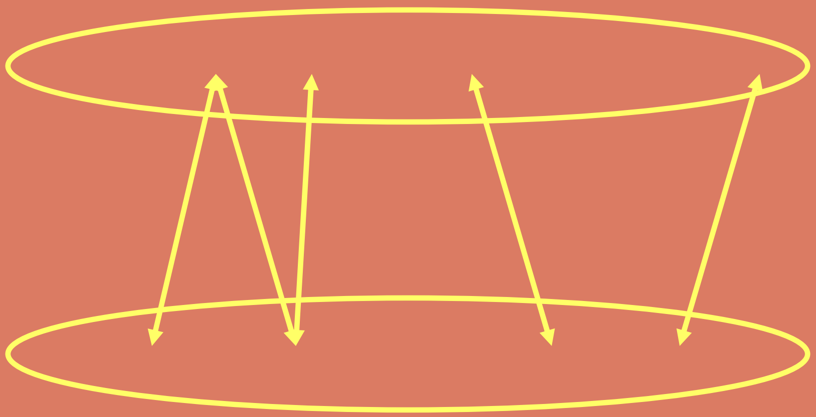
True for **partitionable** graphs

If the bi-directed edges of H form a nonbipartite graph, then CSP(H) is NPc



True for **partitionable** graphs

- Each v incident with a bi-directed edge
- Bi-directed edges between parts
- Uni-directed edges within parts



[BHM]

True for **partitionable** graphs

- Each v incident with a bi-directed edge
- Bi-directed edges between parts
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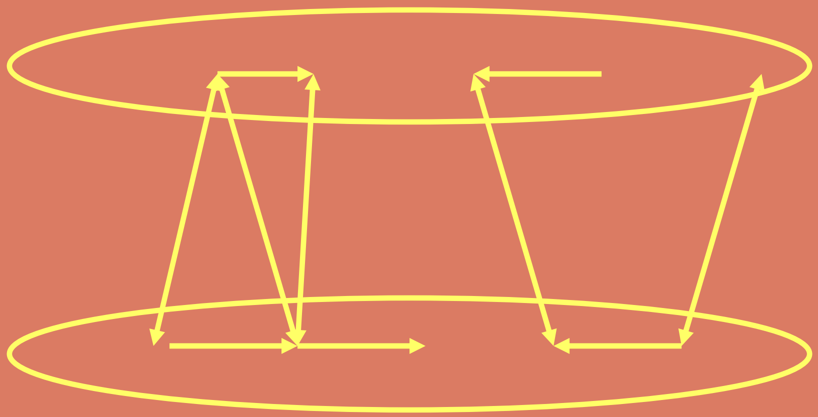

True for **partitionable** graphs

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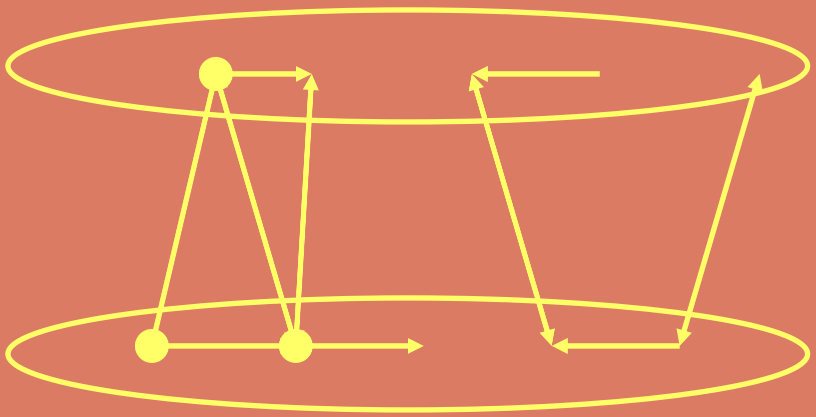
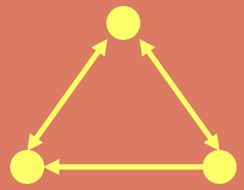
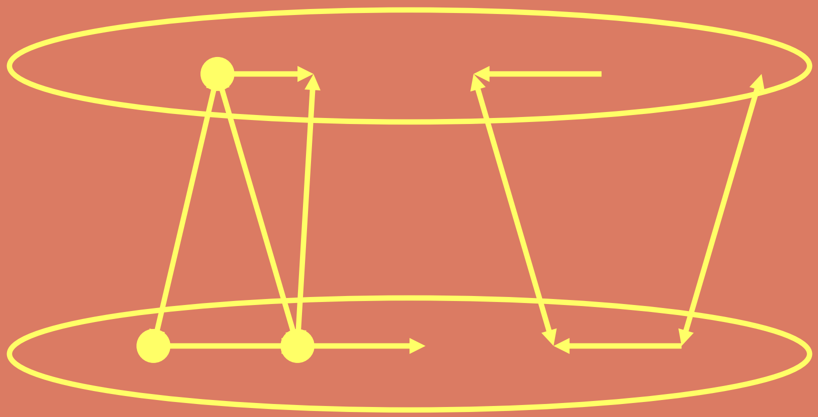
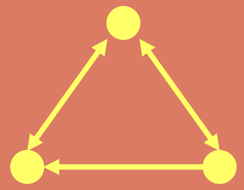




True for **partitionable** graphs

- Each v incident with a bi-directed edge
- Bi-directed edges between parts
- Uni-directed edges within parts ← **?**



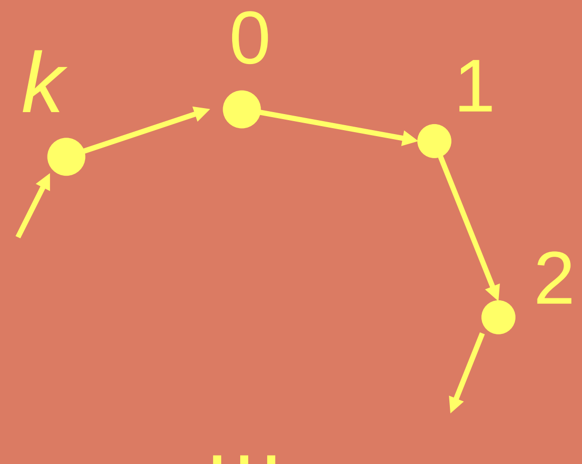


What algorithms?

G has a $\overrightarrow{C}_k$-colouring

if and only if

G contains no cycle of net length $\neq 0 \pmod{k}$

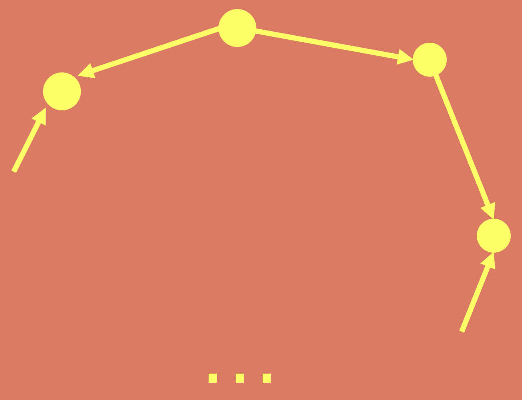


$\overrightarrow{C}_k$

G has a $\vec{C}_k$-colouring

if and only if

there is no homomorphism **to** *G* **from** a cycle of net length $\neq 0 \pmod{k}$



(obstacles are oriented cycles)

G has a $\overrightarrow{P}_k$-colouring

if and only if

there is no homomorphism **to** *G* **from** a path of net length > k [BB]

$$\bullet \longrightarrow \bullet \longrightarrow \bullet \rightarrow \;\ldots\; \rightarrow \bullet \qquad \overrightarrow{P}_k$$

(obstacles are oriented paths)

G has a *P*-colouring

if and only if

there is no homomorphism **to** *G* **from** a path *P'* which is bad

(does not admit a homomorphism to P)

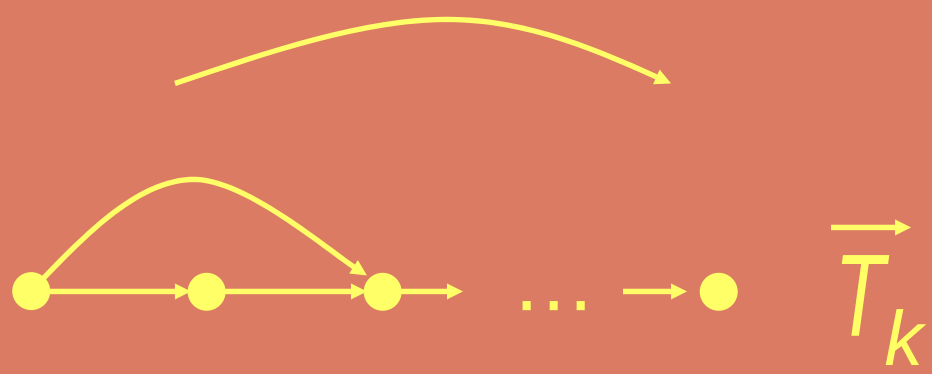
[HZ] (cf also [GWW])

(obstacles are oriented paths)

G has a $\overrightarrow{T_k}$-colouring

if and only if

there is no homomorphism **to** G **from** $\overrightarrow{P_k}$



(obstacles are oriented paths)

What algorithms?

- *H* has tree duality

  $\exists$ a family $\mathfrak{T}_H$ of oriented trees such that G admits an *H*-colouring iff there is no homomorphism **from** a $T \in \mathfrak{T}_H$ **to** G

  (obstacles are oriented trees)

What algorithms?

- *H* has treewidth *k* duality

  $\exists$ a family $\mathfrak{I}_H$ of digraphs of treewidth *k* such that G admits an *H*-colouring iff there is no homomorphism **from** a $T \in \mathfrak{I}_H$ **to** *G*

  (obstacles are digraphs of small treewidth)

Bounded treewidth duality

$\exists\, k$ such that obstacles are graphs of treewidth k

If H has bounded treewidth duality, then CSP(H) is in P

[HNZ]
[FV] width, datalog

What algorithms?

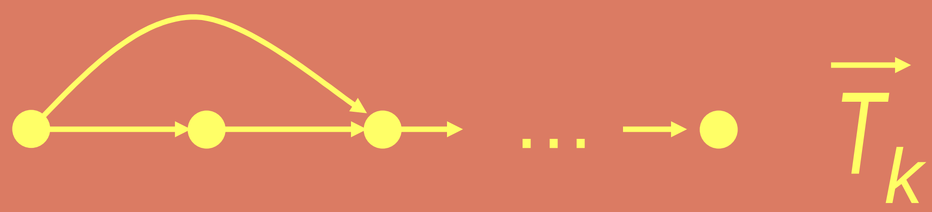
- Tree duality (obstacles are oriented trees)
- Treewidth two duality
  (obstacles are graphs of treewidth 2)
- Bounded treewidth duality

- There exist H without bounded treewidth duality but with CSP(H) in P [A]

G has a $\vec{T}_k$-colouring

if and only if

there is no homomorphism **to** G **from** $\vec{P}_k$



(obstacles are oriented paths)

Finitary dualities

- If H has finitary duality, then H has tree duality [NT]

- This happens if and only if
  H-colourability is first-order definable [A]

  Also follows from [R]

Add unary relations

(still graph theory)

Fix a graph H with k vertices.

Vocabulary has one binary relation name E and a set of unary relation names $U_1, U_2, \ldots, U_{2^k-1}$

Interpret H with $U_i(H)$ the subsets of $V(H)$

(**conservative** structure H)

List homomorphism problem

LHOM(H) CSP(H^*)

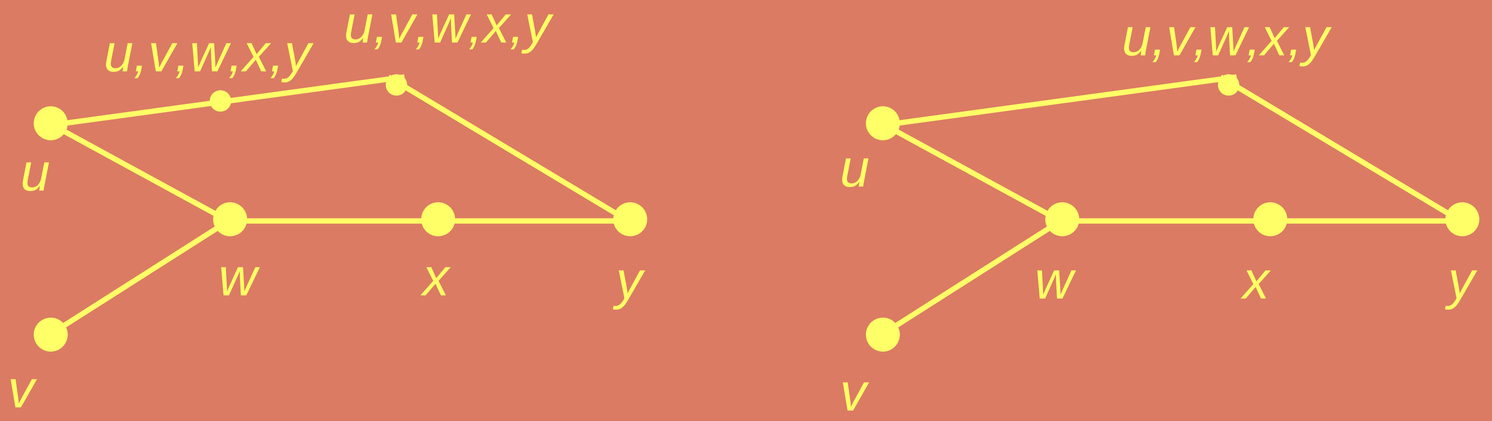
Fixed graph H

Given an input graph G, with lists
$L(v) \subseteq V(H)$, $v \in V(G)$

Is there a homomorphism f of G to H
with all $f(v) \in L(v)$?

RET(H)

LHOM(H) restricted to inputs G containing H with $L(v)=\{v\}$ for all vertices v of H



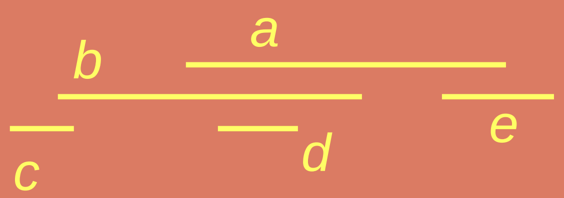
List homomorphism problem

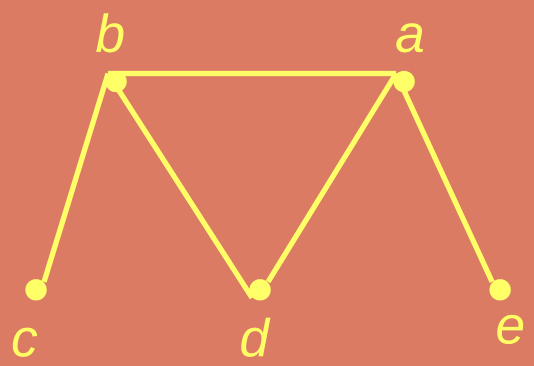
Assume *E(H)* is reflexive

- If H is an **interval** graph, then LHOM(H) is in P
- Otherwise, LHOM(H) is NP-complete

[FH]

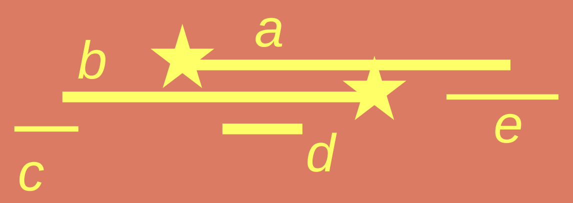
Interval graphs

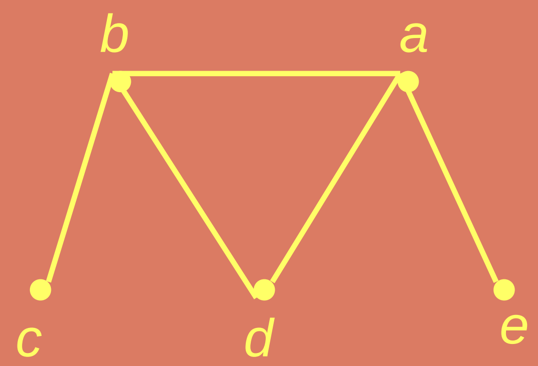




(reflexive)

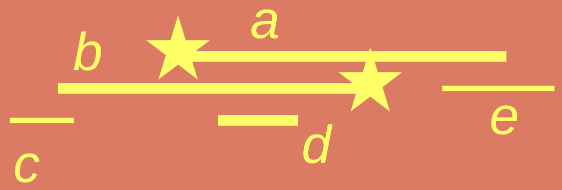
Interval graphs

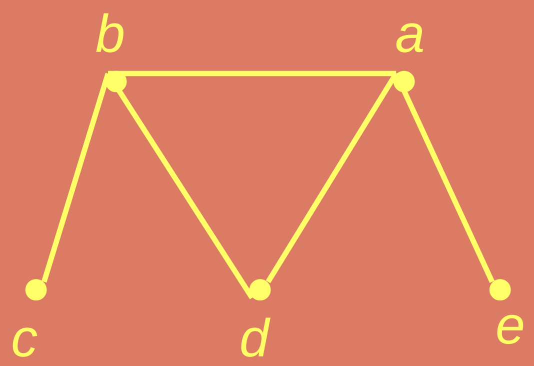




have conservative majority polymorphisms

Interval graphs



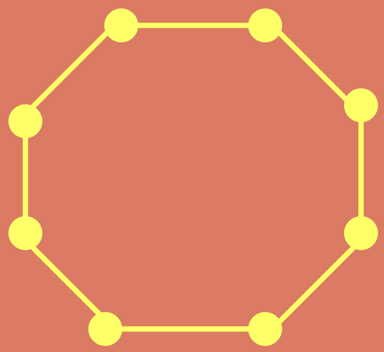


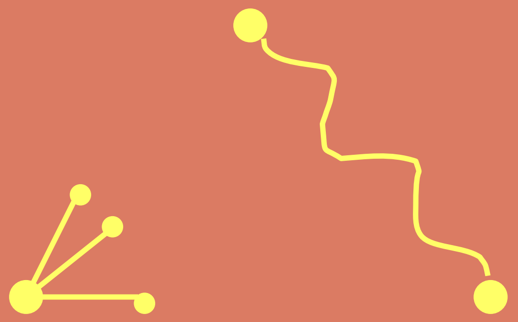
conservative majority polymorphisms

$m(a,b,c) = a$, b, or c

Structural characterization

- H is an interval graph if and only if it does not have an induced cycle of length >3, or an asteroidal triple of vertices [LB]





List homomorphism problem

Assume *E(H)* is reflexive

- If H is an **interval** graph, then LHOM(H) is in P
- Otherwise, LHOM(H) is NP-complete

[FH]

Conservative majority polymorphisms

- Reflexive interval graphs have them
- They imply polynomiality of LHOM(H)
- LHOM(H) is NP-complete for other reflexive graphs

Predicted theorem

A reflexive graph H has a conservative majority function iff it is an interval graph

[BFHHM]

Predicted theorem

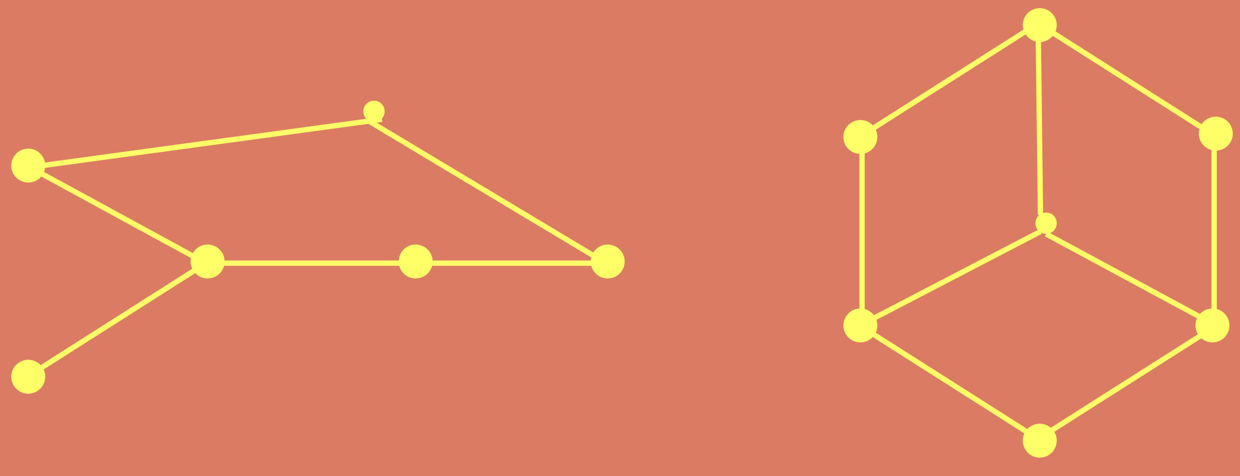
A reflexive graph H has a conservative majority function iff it is an interval graph

[BFHHM]

A reflexive graph H has a majority function if and only if it is a retract of a product of interval graphs [JMP] [HR]

Absolute retracts

- A reflexive graph H is a retract of every G which contains H as an isometric subgraph **iff** H has a majority polymorphism [JMP] [HR]



Absolute retracts

- A reflexive graph *H* is a retract of every *G* which contains *H* as an isometric subgraph **iff** *H* has a majority polymorphism [JMP] [HR]

- But RET(*H*) is polynomial in other cases

List homomorphism problem

In general

- If H is a **bi-arc** graph, then LHOM(H) is in P
- Otherwise, LHOM(H) is NP-complete

[FHH]

G has cons. majority **iff** it is a bi-arc graph

[BFHHM]

List homomorphism problem

In general

- If H is a **bi-arc** graph, then LHOM(H) is in P
- Otherwise, LHOM(H) is NP-complete

[FHH]

G has cons. near un. **iff** it is a bi-arc graph

[BFHHM]

Dichotomy for conservative H

(not graph theory)

CSP(H) is in P or is NP-complete [B]

Digraphs H

- Conjecture 1 [FH]
  - For reflexive digraphs, LHOM(H) $\in P$ when H has the X-underbar property and is NP-complete otherwise

- Conjecture 2 [FH]
  - For irreflexive digraphs, LHOM(H) $\in P$ when H has a majority function, and is NP-complete otherwise

Restricted lists

Suppose each list induces a connected subgraph of H

- If H is chordal CLHOM(H) $\in P$
- Otherwise CLHOM(H) is NP-complete

[FH]

Chordal graph

- Intersection graph of subtrees in a tree
- No induced cycle of length > 3
- Perfect elimination ordering of vertices
- Have near unanimity polymorphisms [BFHHM]

Minimum Cost Homomorphisms

- Let $c_v(w)$ = the cost of mapping
  $v \in V(G)$ to $w \in V(H)$

- Compute $\min_f \sum_v c_v(f(v))$

MCHOM(H) (H is fixed, input is G and c)

Dichotomy

- If each component of H is a reflexive interval graph or an irreflexive interval bigraph, then MCHOM(H) $\in P$

[GHRY,CCJK]

- Otherwise, MCHOM(H) is NP-complete

[GHRY]

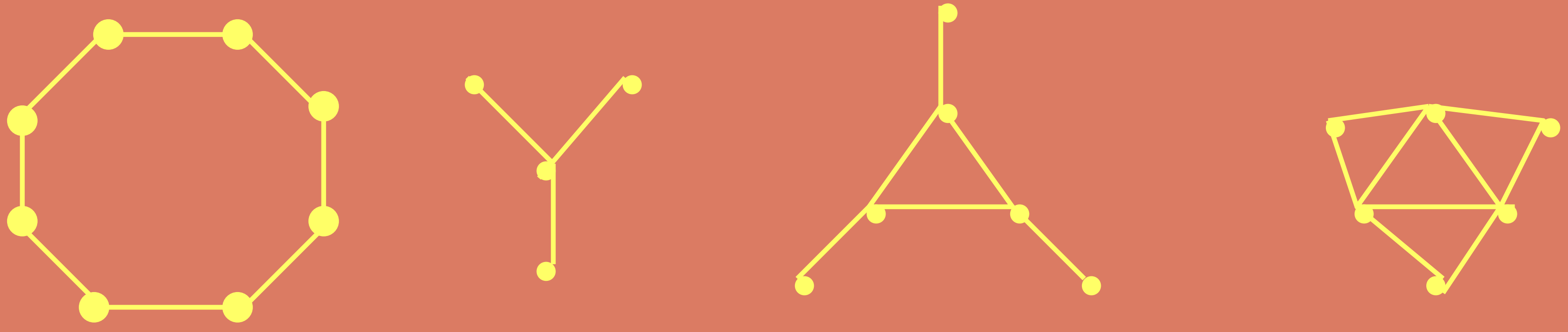
Relation to Soft Constraints

- Unary constraints (lists) are soft
- Binary constraint (edges) are crisp

Proper interval graphs

- Representable by an inclusion-free family of intervals

- A reflexive graph *G* is a proper interval graph **iff** it has no induced



Proper interval graphs

- Representable by an inclusion-free family of intervals

- A reflexive graph *G* is a proper interval graph **iff** it has a Min-Max polymorphism

Reflexive graphs H

- HOM(H) (=CSP(H)) always easy
- RET(H) dichotomy open
- LHOM(H) easy just for interval graphs
- MCHOM(H) easy just for proper interval graphs

Review

- How much is lost by focusing on graphs
- How graphs help intuition
- How graph theory gets used

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