Refining Schaefer's Theorem

Heribert Vollmer

Institut für Theoretische Informatik
Universität Hannover

Joint work with: E. Allender, M. Bauland, N. Immerman, H. Schnoor

Γ – a finite set of Boolean constraint relations

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Input: a propositional Γ-formula F in CNF

Question: Is F satisfiable?

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Basic Goal: Determine the computational complexity of CSP(Γ) as a function of Γ!

The Galois Connection

$\mathrm{Pol}(\Gamma)$ is the set of all polymorphisms of Γ, i.e., the set of all Boolean functions that preserve every relation in Γ.

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Inv(B) is the set of all invariants of B, i.e., the set of all Boolean relations that are preserved by every function in B.

- Inv(B) is a relational clone, i.e., a set of Boolean relations that contains the equality relation and is closed under primitive positive definitions, i.e., if ϕ is an Inv(B)-formula and $R(x_1, \ldots, x_n) \equiv \exists y_1 \ldots y_\ell \; \phi(x_1, \ldots, x_n, y_1, \ldots, y_\ell)$ then $R \in \mathrm{Inv}(B)$.

Schaefer's Theorem

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- If $\langle\Gamma\rangle = \langle\Gamma'\rangle$, then $\mathrm{CSP}(\Gamma) \equiv_m^{\log} \mathrm{CSP}(\Gamma')$, i.e., the complexity of $\mathrm{CSP}(\Gamma)$ depends only on $\mathrm{Pol}(\Gamma)$.

We only have to study co-clones in order to obtain a full classification.

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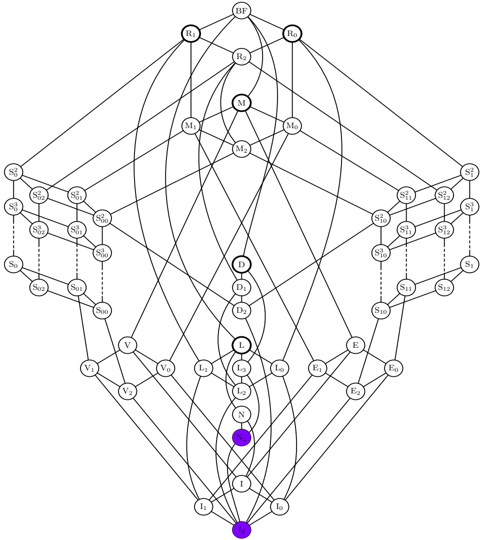
- If $\langle\Gamma\rangle \supseteq \mathrm{Inv}\big(\mathrm{N}_2\big)$ then $\mathrm{CSP}(\Gamma)$ is NP-complete, otherwise $\mathrm{CSP}(\Gamma)$ is in P [Schaefer].



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Towards a Finer Classification

Goal: Determine logspace-degree of CSP(Γ) for every constraint language Γ!

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Examples:

- If $\Gamma \subseteq \mathrm{Inv}(\mathrm{L}_2)$, then $\mathrm{CSP}(\Gamma) \in \oplus\mathrm{L}$.
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  ($\mathrm{Inv}(\mathrm{L}_3) = \langle x \oplus y \oplus z \oplus w\rangle$, Reduction from circuit value problem for $\oplus$-circuits)

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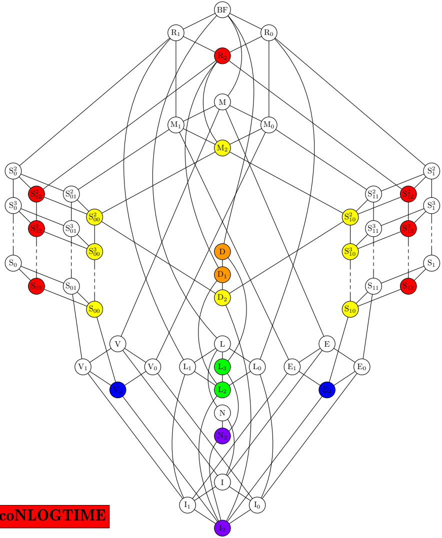
Thus:

- If $\mathrm{Inv}(\mathrm{L}_3) \subseteq \langle\Gamma\rangle \subseteq \mathrm{Inv}(\mathrm{L}_2)$, then $\mathrm{CSP}(\Gamma)$ is $\oplus\mathrm{L}$-complete.

Towards a Finer Classification

- If $I_0 \subseteq \mathrm{Pol}(\Gamma)$ or $I_1 \subseteq \mathrm{Pol}(\Gamma)$, then every constraint formula over Γ is satisfiable, and therefore $\mathrm{CSP}(\Gamma)$ is trivial.
- If $\mathrm{Pol}(\Gamma) \in \{I_2, N_2\}$, then $\mathrm{CSP}(\Gamma)$ is NP-complete.
- If $\mathrm{Pol}(\Gamma) \in \{V_2, E_2\}$, then $\mathrm{CSP}(\Gamma)$ is P-complete.
- If $\mathrm{Pol}(\Gamma) \in \{L_2, L_3\}$, then $\mathrm{CSP}(\Gamma)$ is $\oplus$L-complete.
- If $S_{00} \subseteq \mathrm{Pol}(\Gamma) \subseteq S_{00}^2$ or $S_{10} \subseteq \mathrm{Pol}(\Gamma) \subseteq S_{10}^2$ or $\mathrm{Pol}(\Gamma) \in \{D_2, M_2\}$, then $\mathrm{CSP}(\Gamma)$ is NL-complete.
- If $\mathrm{Pol}(\Gamma) \in \{D_1, D\}$ or $S_{02} \subseteq \mathrm{Pol}(\Gamma) \subseteq R_2$ or $S_{12} \subseteq \mathrm{Pol}(\Gamma) \subseteq R_2$, then $\mathrm{CSP}(\Gamma)$ is in L.

Classification of CSP-Satisfiability



NP complete
P complete
NL complete
$\oplus$L complete
L complete
L complete / coNLOGTIME

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Question: Stricter reductions?

Example 1: $\Gamma_1 = \{x, \overline{x}\}$:

A Γ_1-formula F is unsatisfiable iff it contains clauses x and $\overline{x}$ for some x, hence $\mathrm{CSP}(\Gamma_1) \in \mathrm{coNLOGTIME}$.

Within LOGSPACE

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Example 2: $\Gamma_2 = \{x, \overline{x}, =\}$:

Then $\mathrm{CSP}(\Gamma_2)$ can express undirected graph reachability as follows: Given G, s, t, construct F to consist of clauses $\overline{s}$, t, and $u = v$ for every edge $(u, v) \in G$.

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Thus: Provably different complexity: $\mathrm{CSP}(\Gamma_2) \not\leq_m^{\mathrm{AC}^0} \mathrm{CSP}(\Gamma_1)$,

but $\mathrm{Pol}(\Gamma_1) = \mathrm{Pol}(\Gamma_2) \quad (= \mathrm{R}_2)$.

- If $\Gamma' \subseteq \langle\Gamma\rangle$ then $\mathsf{CSP}(\Gamma') \leq_m^{\mathsf{AC}^0} \mathsf{CSP}\big(\Gamma \cup \{=\}\big) \leq_m^{\log} \mathsf{CSP}(\Gamma)$.

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The Equality Constraint

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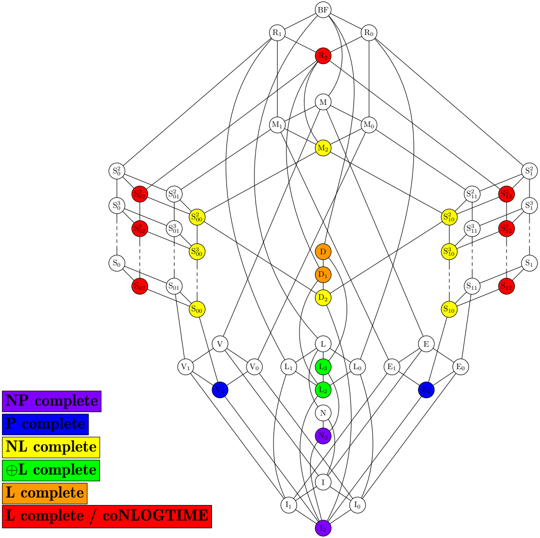
- If Γ can express equality then $\mathrm{CSP}(\Gamma)$ is hard for L, otherwise $\mathrm{CSP}(\Gamma) \in \mathrm{coNLOGTIME}$.

- If $\mathrm{Pol}(\Gamma) \in \{D_1, D\}$, then $\mathrm{CSP}(\Gamma)$ is L-complete.
- If $S_{02} \subseteq \mathrm{Pol}(\Gamma) \subseteq R_2$ or $S_{12} \subseteq \mathrm{Pol}(\Gamma) \subseteq R_2$, then either $\mathrm{CSP}(\Gamma)$ is in coNLOGTIME, or $\mathrm{CSP}(\Gamma)$ is L-complete. There is an algorithm deciding which case occurs.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Post's lattice: $L_2 \subseteq R_2$, hence $\mathrm{Inv}(R_2) \subseteq \mathrm{Inv}(L_2)$.

Hence:

- Undirected graph accessibility is in $\oplus$L, in other words: SL $\subseteq \oplus$L [Karchmer, Wigderson, 1993].

The Power of $\oplus$L

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(Today we even know SL $\subseteq$ L.)

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In the polynomial time setting:

Isomorphism conjecture/Berman-Hartmanis-Conjecture:

- If $A \leq_m^{\mathrm{p}} B$ and $B \leq_m^{\mathrm{p}} A$ then A and B are isomorphic via a polynomial-time computable bijection.

Implies $\mathrm{P} \neq \mathrm{NP}$.

Isomorphism Theorem holds for $\leq_m^{\mathrm{AC}^0}$-reducibility:

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- For every constraint language Γ, $CSP(\Gamma)$ is AC^0-isomorphic either to $0\Sigma^\star$ or to the standard complete set for one of the complexity classes NP, P, $\oplus$L, NL, or L.

There are only six different CSP-problems!

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- playground for the study of many issues related to counting classes
- CSP isomorphism problems yield good candidates for "intermediate problems"

References

The classification presented in this talk is given in

- E. Allender, M. Bauland, N. Immerman, H. Schnoor, H. Vollmer. The complexity of satisfiability problems: refining Schaefer's theorem; Proceedings 30th Mathematical Foundations of Computer Science, Springer Lecture Notes in Computer Science Vol. 3618, pp. 71–82, 2005.

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More on isomorphism for Boolean CSP:

- E. Böhler, E. Hemaspaandra, S. Reith, H. Vollmer. The complexity of Boolean constraint isomorphism; Proceedings 21st Symposium on Theoretical Aspects of Computer Science, Springer Lecture Notes in Computer Science Vol. 2996, pp. 164–175, 2004.

Open Questions for Boolean CSP

- Fine classification for Boolean counting problem
- Infinite constraint languages?
- Uniform Boolean CSP?