Constraint Programming Models for Graceful Graphs

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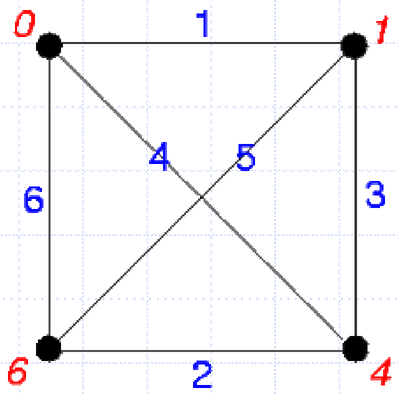
Modelling in Constraint Programming

- How a problem is formulated as a CSP can have a huge impact on how easy it is to solve
- We often try to modify an existing model to make it easier to solve
- Thinking of a completely different way of modelling a problem is much harder….

Graceful Labelling of a Graph

- A labelling f of the nodes of a graph with q edges is graceful if:
  - f assigns each node a unique label from $\{0,1,\ldots, q\}$
  - when each edge xy is labelled with $|f(x) - f(y)|$, the edge labels are all different



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Initial Model as a CSP

- A variable for each node, $x_1, x_2, \ldots, x_n$ each with domain $\{0, 1, \ldots, q\}$
- Auxiliary variables for each edge, $d_1, d_2, \ldots, d_q$ each with domain $\{1, 2, \ldots, q\}$
- $d_k = |x_i - x_j|$ if edge k joins nodes i and j
- $x_1, x_2, \ldots, x_n$ are all different
- $d_1, d_2, \ldots, d_q$ are all different

- Find values for each of $x_1, x_2, \ldots, x_n$ so that the constraints are satisfied

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Symmetry

- The CSP has symmetry from two sources:
  - the symmetry of the graph (if any)
  - a value symmetry: in any solution, we can replace every assignment $<x_i, j>$ by $<x_i, q - j>$
- We have used graceful graphs problems to investigate how to eliminate the effects of symmetry on search
  - adding constraints to the CSP
  - using dynamic symmetry breaking methods

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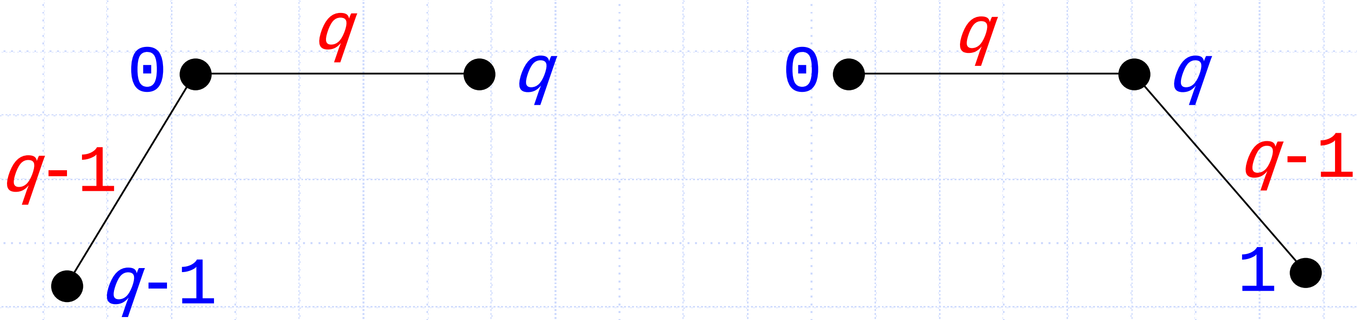
A non-CP Approach to Graceful Graphs

- Beutner & Harborth, *Graceful Labellings of Nearly Complete Graphs*, Results in Mathematics (41),34-39, 2002
- Focus on the edges:
- In any gracefully labelled graph with q edges:
  - the edge labelled q joins nodes labelled 0 and q
  - the edge labelled q-1 joins nodes labelled 0 and q-1 *or* 1 and q
  - the edge labelled q-2 joins nodes labelled 0 and q-2 *or* 1 and q-1 *or* 2 and q
  - etc.

Eliminating Value Symmetry

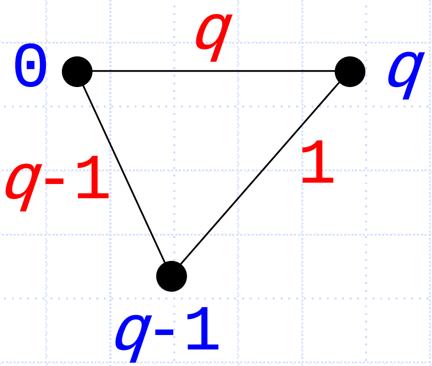
- The edge labelled q-1 joins nodes labelled 0 and q-1 *or* 1 and q



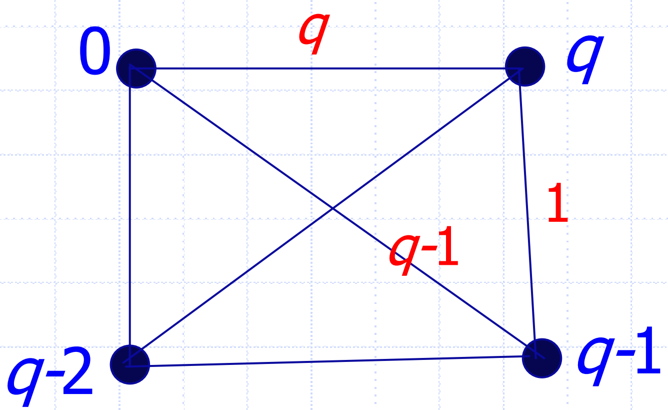
- In *any* gracefully labelled graph, we must have one of these two configurations
- They are symmetrically equivalent
- We can break the value symmetry by choosing one arbitrarily

K_n is not graceful for $n > 4$ (I)

- Breaking symmetry as above, we get the triangle below (since all nodes are connected)
- This is the graceful labelling of K_3 (with $q = 3$)

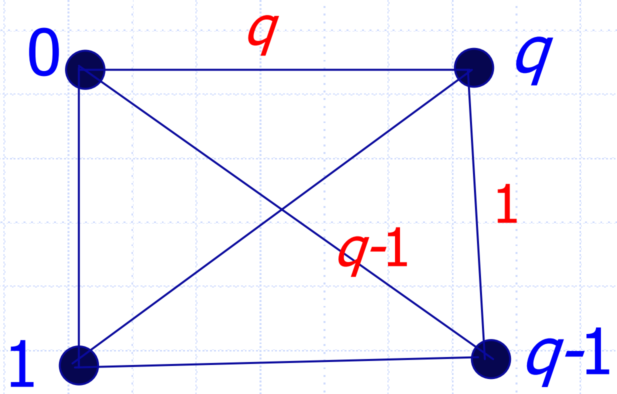


K_n is not graceful for $n > 4$ (II)



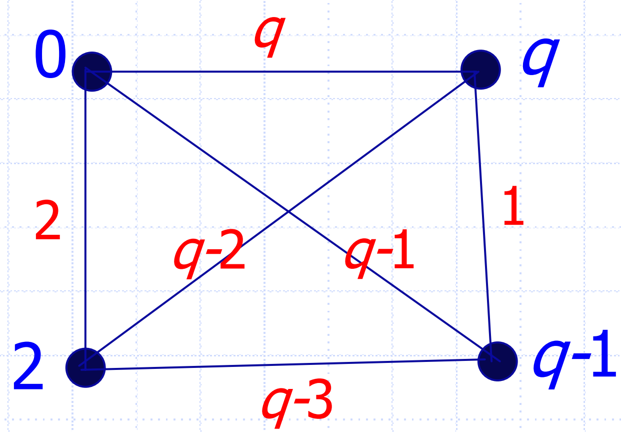
- For $n \geq 4$, the edge labelled $q-2$ requires an extra node labelled *either*:
  - $q-2$

K_n is not graceful for $n > 4$ (II)



- For $n \geq 4$, the edge labelled $q-2$ requires an extra node labelled *either*:
  - $q-2$
  - *or* 1

K_n is not graceful for $n > 4$ (II)



- For $n \geq 4$, the edge labelled $q-2$ requires an extra node labelled *either*:
  - $q-2$
  - *or* 1
  - *or* 2
- Adding a node labelled 2 gives the graceful labelling of K_4

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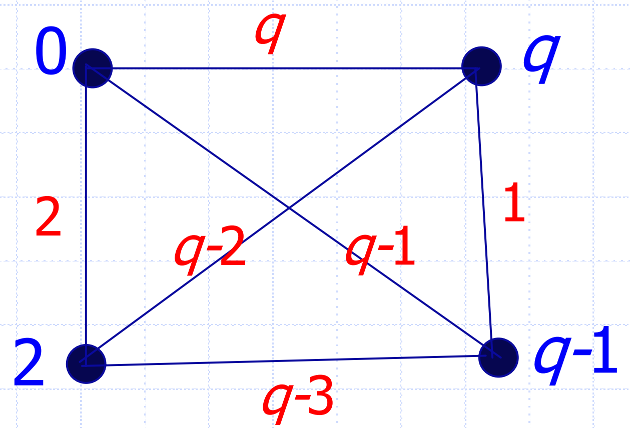
K_n is not graceful for $n > 4$ (III)



- To get an edge labelled q-4, we need a new vertex labelled q-4, q-2, 3 or 4; for K_5, only q-4 does not give a repeated edge label. But then we get edges labelled q-6 and 4: for K_5 these are the same
- For $n \geq 6$, we cannot get an edge labelled q-5 without repeating edge labels

A Different Viewpoint

- The proof focuses on the labels:
  - which node labels does this edge label join?
- The CSP model focuses on the nodes (and edges):
  - what is the label on this node (and on this edge)?
- Can we build a new CSP model using the new viewpoint?

Variables of new model

- $e_k = l$ if the edge labelled k joins nodes labelled l and $l+k$
- $n_l = i$ if the node label l is attached to node i (or a dummy value if node label l is not used)
- $e_k = l$ iff the values of n_l and n_{l+k} are adjacent nodes in the graph
- $n_1, n_2, \ldots, n_q$ are all different, unless they have the dummy value

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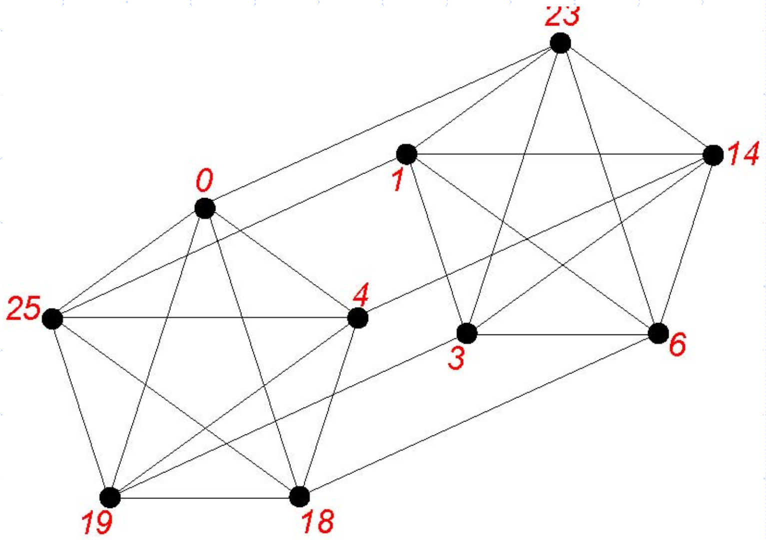
Graceful Labelling of K_n x P_2

- K_3 x P_2 and K_4 x P_2 have several graceful labellings (excluding symmetric equivalents)
- K_5 x P_2 has just one
- K_6 x P_2 and K_7 x P_2 have none
- I conjecture that K_n x P_2 is not graceful for $n > 5$



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Finding All Graceful Labellings of $K_n \times P_2$

	Labellings	Old model (backtracks)	New model (backtracks)
$K_3 \times P_2$	4	24	21
$K_4 \times P_2$	15	467	210
$K_5 \times P_2$	1	14051	1037
$K_6 \times P_2$	0	-	2614
$K_7 \times P_2$	0	-	4036

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Conclusions

- The original model seemed the ‘natural’ CSP model of the problem
- The model based on edge labels is less obvious, & is more difficult to write, but is much faster
- An example of a mathematical approach to the problem yielding a good CSP model
- But we can’t yet prove that $K_n \times P_2$ is not graceful for $n \geq 6$ ……

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