Cost function optimization & local consistency

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Overview

- **Introduction and definitions**
  - What is it ?
  - How do we solve it ?
- **Local consistency**
  - Fundamental operations on soft CN
  - How are fair valuation structures ?
  - Equivalence preserving operations
  - Local consistencies
  - Max flow as a local consistency ?
- **Conclusion**

Why soft constraints?

- CSP framework: for *decision* problems
- Many problems are *overconstrained* or *optimization* problems

- Economics (combinatorial auctions)
  - Given a set G of goods and a set B of bids…
    - Bid (B_i, V_i), B_i requested goods, V_i value
  - … find the best subset of compatible bids

  - Best = maximize revenue (sum)

Why soft constraints?

- Satellite scheduling
  - Spot 5 is an earth observation satellite
  - It has 3 on-board cameras
  - Given a set of requested pictures (of different importance)…
    - Resources, data-bus bandwidth, setup-times, orbiting
  - …select a subset of compatible pictures with max. importance (sum)

Why soft constraints?

- Probabilistic inference (bayesian nets)
  - Given a probability distribution defined by a DAG of conditional probability tables
  - And some evidence
  - ...find the *most probable* explanation for the evidence (product)

Haplotyping

Why soft constraints?

- Ressource allocation (frequency assignment)
  - Given a telecommunication network
  - …find the best frequency for each communication link avoiding interferences
  - Best can be:
    - Minimize the maximum frequency (max)
    - Minimize the global interference (sum)

w/o mentionning MaxCut/Clique, Vertex cover…

Soft constraint network (CN)

- (X,D,C)
  - $X=\{x_1,\ldots, x_n\}$ variables
  - $D=\{D_1,\ldots, D_n\}$ finite domains
  - $C=\{f,\ldots\}$ e cost functions
    - f_S, f_{ij}, f_i $f_\varnothing$ scope S,$\{x_i,x_j\}$,$\{x_i\}$, $\varnothing$
    - $f_S(t)$: $\rightarrow$ E (ordered by $\preccurlyeq$, $\bot \preccurlyeq \top$)

identity

annihilator

- commutative
- associative
- monotonic

- **Obj. Function**: $F(X) = \oplus f_S\ (X[S])$
- **Solution**: $F(t) \neq \top$
- **Soft CN**: find minimum cost solution

Specific frameworks

Instance	E	$\oplus$	$\bot \preccurlyeq \top$
Classic CN	$\{t,f\}$	and	$t \preccurlyeq f$
Possibilistic	[0,1]	max	$0 \preccurlyeq 1$
Fuzzy CN	[0,1]	usual min	$1 \preccurlyeq 0$
Weighted CN (bounded or not)	[0,k]	+	$0 \preccurlyeq k$
Bayes net	[0,1]	×	$1 \preccurlyeq 0$

Lexicographic CN, probabilistic CN…

Basic operations

- combination / join
- projection

Cost function implication

f_P implied by g_Q ($f_P \preccurlyeq g_Q$, g_Q tighter than f_P)

iff for all $t \in \ell(P \square Q)$, $f_P(t[P]) \preccurlyeq g_Q(t[Q])$

x_i	x_j	$g(x_i,x_j)$
b	b	2
b	g	3
b	r	2
g	b	4
g	g	5
g	r	4
r	b	T=6
r	g	T=6
r	r	T=6

implies

x_i	$f(x_i)$
b	1
g	2
r	3

or

x_i	$g[x_i]$
b	2
g	4
r	T=6

T=1 : classic CSP case

Combination (join with $\oplus$, + here)

x_i	x_j	$f(x_i,x_j)$
b	b	6
b	g	0
g	b	0
g	g	6

$\oplus$

x_j	x	$g(x_j,x_k)$
b	b	6
b	g	0
g	b	0
g	g	6

x_i	x_j	x_k	$h(x_i,x_j,x_k)$
b	b	b	12
b	b	g	6
b	g	b	0
b	g	g	6
g	b	b	6
g	b	g	0
g	g	b	6
g	g	g	12

$= 0 \oplus 6$

Projection (elimination)

x_i	x_j	$f(x_i,x_j)$
b	b	4
b	g	6
b	r	0
g	b	2
g	g	6
g	r	3
r	b	1
r	g	0
r	r	6

Min

$f[\mathbf{x_i}]$

x_i	$g(x_i)$
b	0
g	2
r	0

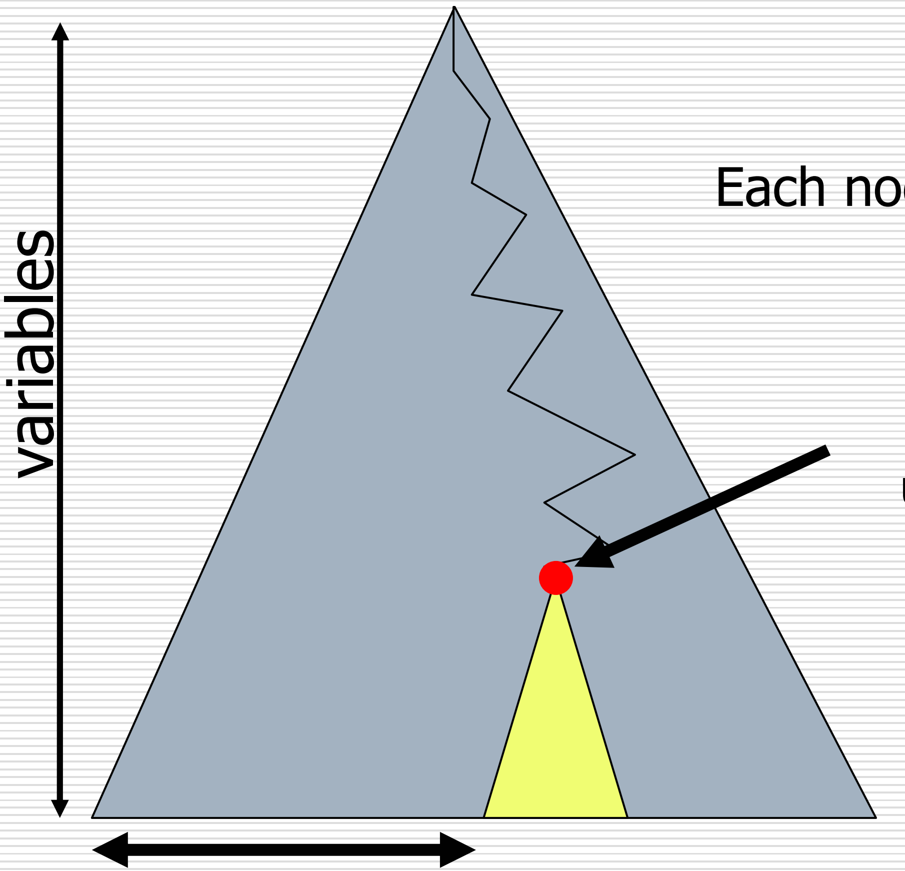
$g[\varnothing]$

$h_\varnothing$
0

Systematic search

Branch and bound(s)

Systematic search



Each node is a soft constraint subproblem

(LB) Lower Bound = $f_\varnothing$
under estimation of the best solution in the sub-tree

If $f_\varnothing$ LB ≥ UB then prune

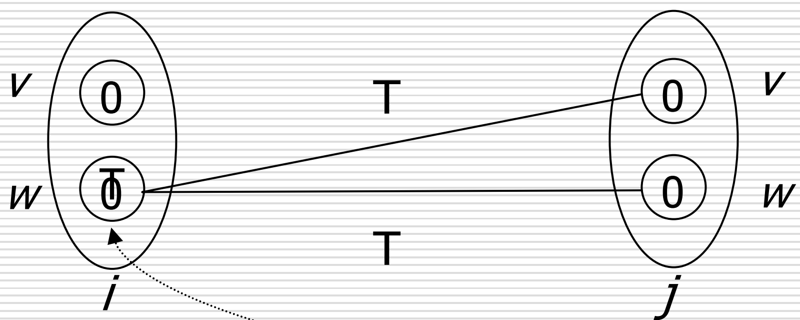
(UB) Upper Bound = best solution so far = **T**

Local consistency

- Very useful in classical CSP
  - Node consistency (1-consistency)
  - Arc consistency…(2-consistency in binary CSP)
  - Related to tractable classes (AC: tree, 2SAT)
- Produces an equivalent more explicit problem:
  - Preserves solutions (costs), may detect unfeasibility (gives a better $f_\emptyset$)

Classical arc consistency (binary CSP)

- for any x_i and c_{ij}
  - $c=(c_{ij} \bowtie c_j)[x_i]$ brings no new information on x_i



$c_{ij} \bowtie c_j$

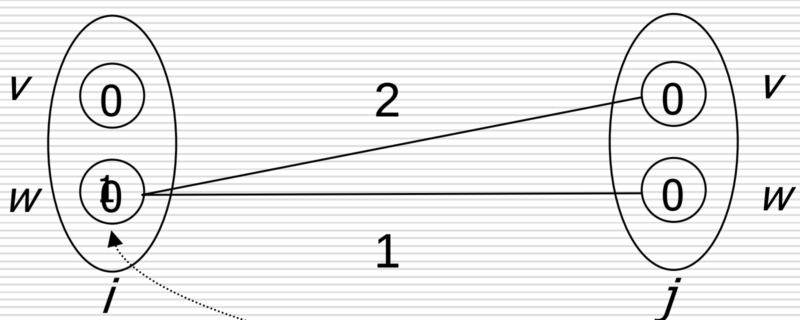
x_i	x_j	c_{ij}
v	v	0
v	w	0
w	v	T
w	w	T

$(c_{ij} \bowtie c_j)[-x_j]$

x_i	$c(x_i)$
v	0
w	T

Arc consistency and soft constraints

- for any x_i and f_{ij}
  - $f=(f_{ij} \oplus f_j)[x_i]$ brings no new information on x_i



$f_{ij} \oplus f_j$

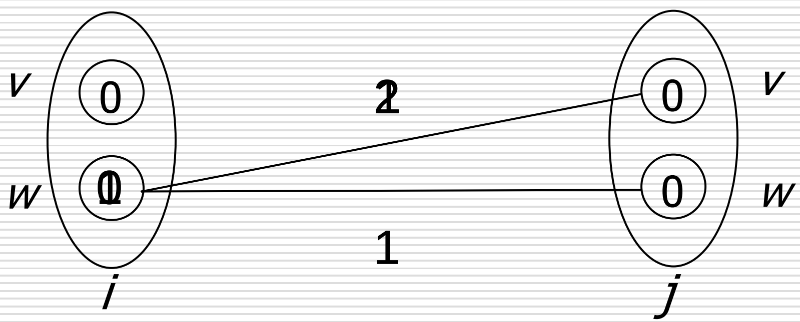
x_i	x_j	f_{ij}
v	v	0
v	w	0
w	v	2
w	w	1

$(f_{ij} \oplus f_j)[x_i]$

X_i	$f(x_i)$
v	0
w	1

Always equivalent iff $\oplus$ idempotent [Rosenfeld 76, Bistarelli et al. 95]

Extraction of implied constraints



$f_{ij} \oplus f_j$

x_i	x_j	f_{ij}
v	v	0
v	w	0
w	v	2
w	w	1

$\ominus$

$f_{ij} \oplus f_j [x_i]$

x_i	$f(x_i)$
v	0
w	1

x_i	x_j	f_{ij}
v	v	0
v	w	0
w	v	1
w	w	0

- Combination+Extraction: equivalence preserving transformation [S,CS]
- Requires the existence of a pseudo difference $(a \ominus b) \oplus b = a$: fair valued CN.

Fair valuation structures

- Totally characterized [CS,C]
  - Slices that interact together idempotently
  - Each slice (discrete case) is isomorphic to
    - $S_\infty = (N \cup \{\infty\}, +)$ infinite top
    - $S_k = ([0,k], +_k)$ finite top

This means only **3 types** of structures to consider.

Tractability only known for finite costs+infinite T

(idempotent case should be…easy)

(K,Y) **equivalence preserving inference**

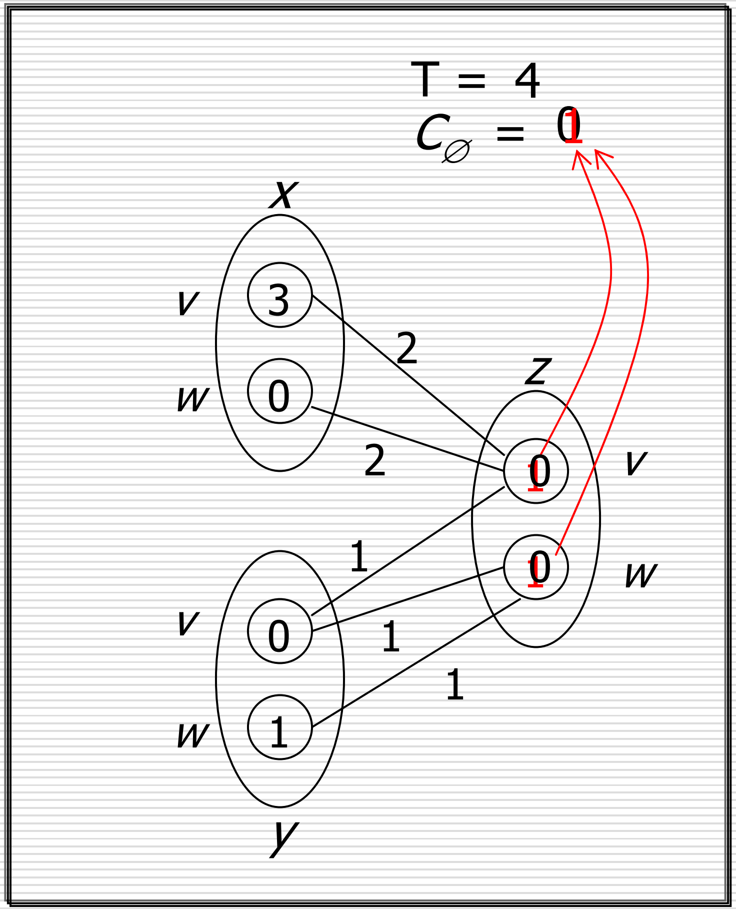
- For a set K of constraints and a scope Y
  - Replace K by (⊕K)
  - Add (⊕K)[Y] to the CN (implied by K)
  - Extract (⊕K)[Y] from (⊕K)

- Yields an equivalent network
- All implicit information on **Y** in **K** is explicit

- Repeat for a class of (K,Y) until fixpoint

Node Consistency (NC*): $(f_i, \varnothing)$ EPI

- For any variable X_i
  - $\forall a,\ f_\varnothing + f_i(a) < T$
  - $\exists\ a,\ f_i(a) = 0$



- Complexity:

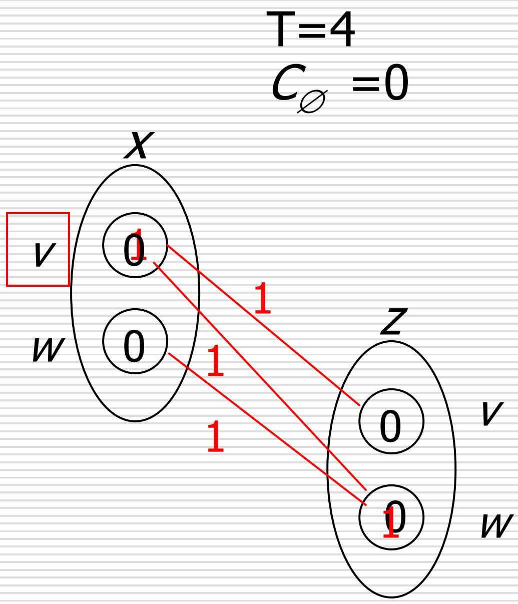
  O(nd)

Full AC (FAC*): $(\{f_{ij}, f_j\}, x_i)$ EPI

- NC*
- For all f_{ij}
  - $\forall a \, \exists \, b$

    $f_{ij}(a,b) + f_j(b) = 0$

    (full support)



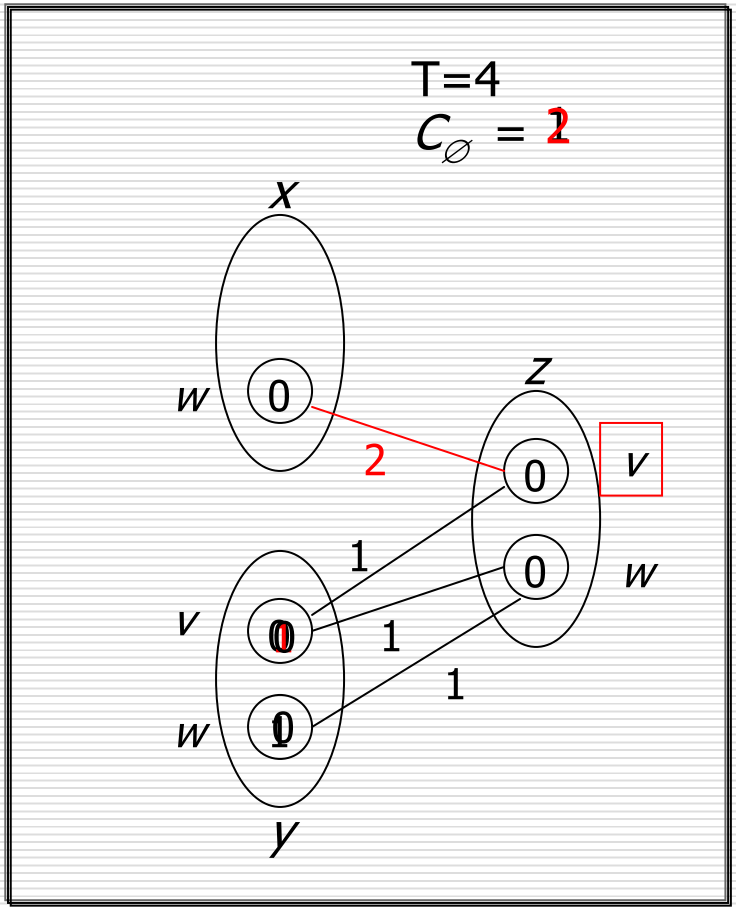
That's our starting point!
No termination !!!

Arc Consistency (AC*): ({f_{ij}},x_i) EPI

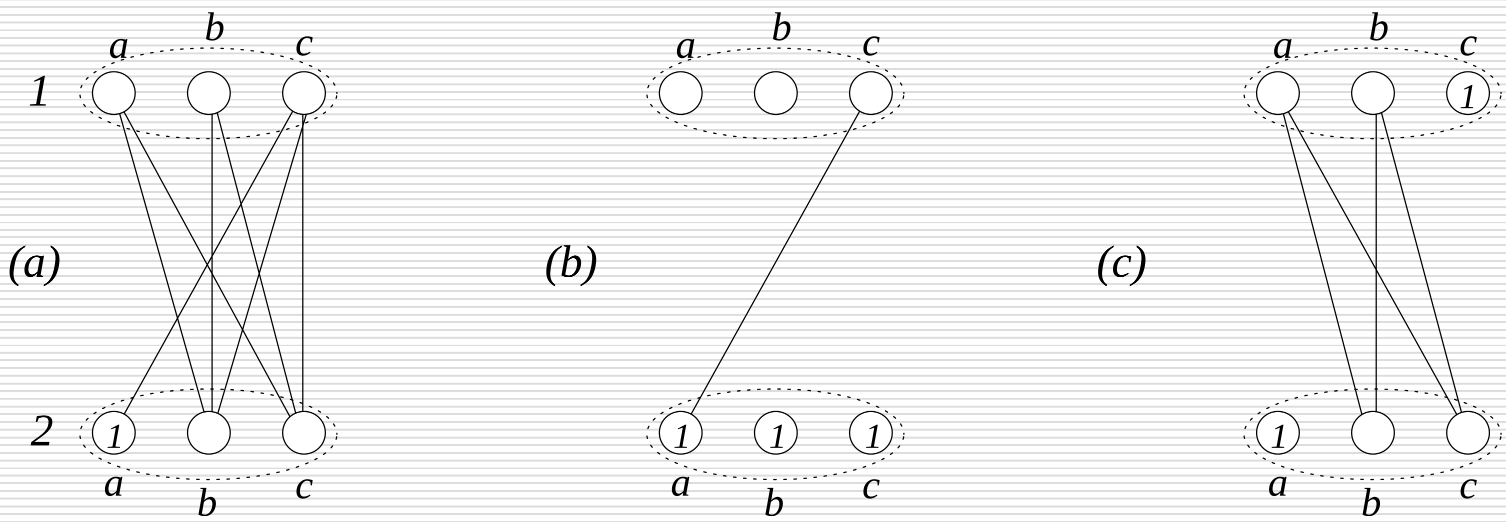
- NC*
- For all f_{ij}
  - $\forall a \exists b$

    $f_{ij}(a,b) = 0$
- b is a *support*
- complexity:

  $\mathbf{O}(n^2 d^3)$



Confluence is lost



Finding an AC closure that maximizes the lb is an NP-hard problem [CS]

Directional AC (DAC*): $(\{f_{ij},f_j\},x_i)_{i<j}$ EPI

- NC*
- For all f_{ij} $(i<j)$ $\forall a \exists b$ $\quad f_{ij}(a,b) + f_j(b) = 0$

- b is a *full-support*
- complexity: $\mathbf{O}(ed^2)$ and it solves trees.

Simultaneous AC operations ap.

- Use rationals for costs. Infinite T.
- Value a, variable i, cost function f_{ij}.
  - w_{iaj} = cost sent from $f_i(a)$ to f_{ij} - from f_{ij} to $f_i(a)$.
  - w_i = amount sent from f_i to $f_\varnothing$.

  - Max $\sum w_i$
    - For all (a,i) $f_i(a) - \sum_j w_{iaj} - w_i \geq 0$
    - For all f_{ij},a,b $f_{ij}(a,b) + w_{iaj} + w_{jbi} \geq 0$

Solved in pol. time by LP for bounded arities

Extra properties

- More powerful than any seq. of AC preserving transformation (infinite T)
- No better equivalent problem with the same scopes (finite costs, infinite T)
  - Quite expensive in practice.

Pb	nvar / ncost	EDAC	cpu"	LPb	cpu"	ub
CELAR06	100/1332	0	0.02	3.5	621	3389*
CELAR07	200/2865	10000	0.04	31453	3529	343592

Max-Flow as local consistency ?

If d=2 (Max-2SAT), the LP is a max-flow problem

- What if d > 2 ?
- What if finite T ?

Current only tractable language of MaxCSP

- 2 Monotone (maxSAT)
- Submodular (maxCSP): solved by a max-flow algorithm

Any connection ?

References

- Tutorial slides on cost function optimization (soft constraints) are available here: http://www.math.unipd.it/~frossi/cp-school/Ecole2005.ppt.
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