# Classification of Bipartite Boolean Constraint Satisfaction through Delta Matroid Intersection

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# Schaefer's Classification Problems

**Domain: {0,1}**
**Constants: 0,1**

## Polynomial Cases

- **Horn clauses:** $\bar{x} \vee \bar{y} \vee z, \quad \bar{x} \vee \bar{y} \vee \bar{z}$
  **closure function** $f(0,0) = f(0,1) = f(1,0) = 0 \qquad f(1,1) = 1$

- **Anti-Horn clauses:** $x \vee y \vee \bar{z}, \quad x \vee y \vee z$
  **closure function** $f(0,0) = 0 \qquad f(0,1) = f(1,0) = f(1,1) = 1$

- **2-satisfiability:** $x \vee y, \quad x \vee \bar{y}, \quad \bar{x} \vee \bar{y}$
  **closure function** $g(x, y, z) =$ **majority** $(x, y, z)$

- **Linear equations modulo 2:** $x + y + z = 0 \pmod 2, \quad x + y + z = 1 \pmod 2$
  **closure function** $h(x, y, z) = x + y + z \pmod 2$

## All other problems are NP-complete

- **3-satisfiability:** $x \vee y \vee z, \quad \bar{x} \vee y \vee \bar{z}, \quad x \vee y \vee \bar{z}$

- **One-in-3-satisfiability:** $\{(0, 0, 1), (0, 1, 0), (1, 0, 0)\}$

- **Not-all-equal satisfiability:** $\{(x, y, z) \neg (0, 0, 0), (1, 1, 1)\}$

# What happens to NP-complete problems when restricted to two occurrences per variable?

- **One-in-3-satisfiability**

  0 0 1
  0 1 0 graph matching
  1 0 0

- **Not-all-equal satisfiability**

  1, 2 graph matching

- **3-satisfiability**

  polynomial delta-matroid parity

**All three become polynomial delta-matroid parity problems !!!**

# Towards a classification with two occurrences per variable

1. *If* not in Schaefer's polynomial cases *then* can simulate all clauses

$$x \vee \bar{y} \vee \bar{z}, \quad x \vee y \vee \bar{z} \vee \bar{t}$$

2. *If* not delta-matroid *then* can simulate

$$R = (x \leq \cong y, z): \quad (0, 0, 0), (1, 1, 1) \in R \qquad (x, y, z) \in R \Rightarrow x \leq y, z$$

1. and 2. simulate satisfiability with three occurrences per variable

NP-complete !!!

# Bipartite case classification with two occurrences per variable

One-in-three satisfiability → graph matching → bipartite graph matching

**One left constraint and one right constraint**

Delta-matroid parity → delta-matroid intersection

**Delta-matroids:**

$[n] = \{1, 2, \ldots, n\}$

$B = \{A \subseteq [n]\}$ bases

$A_1 \in B,\ A_2 \in B,\ i \in (A_1 \,\Delta\, A_2) \Rightarrow \exists\, j \in (A_1 \,\Delta\, A_2) : (A_1 \,\Delta\, \{i, j\}) \in B$

**Bases:**

$i \in A$: $\quad x_i = 1$

$i \notin A$: $\quad x_i = 0$

# Delta-matroid intersection more general than delta-matroid parity

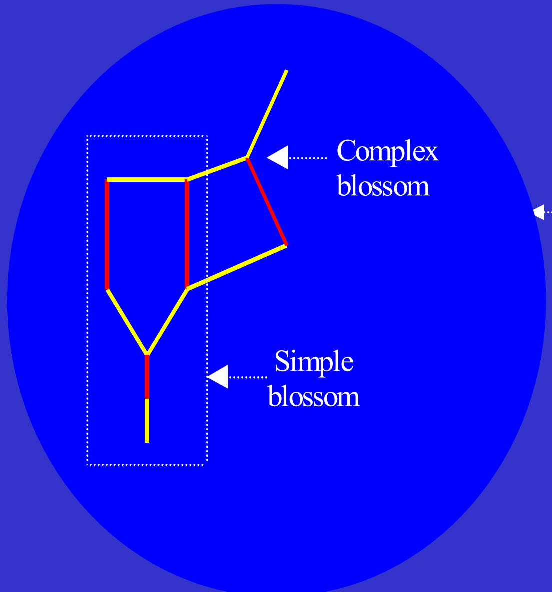
if = delta-matroid [*x* = *y*] = {(0, 0), (1, 1)} not allowed or

not = delta-matroid [*x* not = *y*] = {(0, 1), (1, 0)} not allowed

if = disallowed: simple blossom augmentation gives polynomial algorithm

if not = disallowed: simple blossom augmentation gives polynomial algorithm

if =, not = disallowed in one delta-matroid, other arbitrary: simple blossom augmentation gives polynomial algorithm




simple blossom collapses and does augmentation in these cases

# Bipartite classification with oracles

1. NP-complete cases

2. Schaefer-derived cases:
   - Horn clauses, anti-Horn clauses, 2-satisfiability, linear equations modulo 2
   - One side has only monadic constraints
   - Upward closed 2-sat in one side, other side 2-sat downward closure (or vice versa)

3. 2-sat upward closed and delta-matroid downward closed in one side, reverse in the other side

4. Delta-matroid derived cases:
   - Delta-matroid intersection without equality
   - Delta-matroid intersection without equality, inequality in one side
   - Upward delta-matroid in one side, downward closure of other side is delta-matroid
   - Delta matroid parity with equality

   **Open!!!**
   - Local odd and even delta-matroids
   - Local-zebra and linear-zebra delta-matroids (not with oracle)
   - Delta-matroid without inequality
   - etc.

# Open problems

*** Zebra cases do not work with oracles, but polynomial.  Other such problems?**

**Linear delta-matroids seem to work with oracles and not just linear representation.**

*** *k*-partite for $k \geq 3$: solved classification, polynomial with oracles or NP-complete**

*** Multi-domain case:   $\{0_A, 1_A\}, \{0_B, 1_B\}\ldots$**

**With 2 (or more) occurrences per variable, solved  when relations satisfy symmetry: exchanging first and second occurrences of variables gives another valid relation.**

**What about without symmetry?**

**List constraints have been classified when subsets of lists of size at most 3 are also lists.**

**What about if only subsets of size at most 2 are required lists. Also are NP-complete cases still provable with 3 occurrences, and what about 2 occurrences?**