An Investigation of the Multimorphisms of Tractable and Intractable Classes of Valued Constraints

Mathematics of Constraint Satisfaction:

Algebra, Logic and Graph Theory

20 - 24 March 2006

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Agenda

- The valued constraint formalism (8 mins)
- Reasoning about tractability (5 mins)
- Expressibility/Multimorphisms (10 mins)
- Open Questions (4 minutes)

Constraint Satisfaction

- We have a set of variables
- We have a set of domain values for each variable
- We have an oracle that determines validity
- We have to find any feasible assignment

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Constraint Optimization

- We have a set of variables
- We have a set of domain values for each variable
- We have an oracle that determines cost
- We have to find any optimal assignment

Assignment Costs: Axioms

Local Computation

$\perp$ is the best value.

$\top$ is the worst value.

$\otimes$ models projection and is commutative, associative and idempotent.

$\oplus$ models aggregation and is commutative and associative;

$$\forall\, a : (a \otimes \top = a) \wedge (a \oplus \perp) = a;$$

$$\forall\, a : (a \otimes \perp = \perp) \wedge (a \oplus \top) = \top;$$

$\oplus$ distributes over $\otimes$:

$$\forall\, a,b,c : (a \oplus (b \otimes c) = (a \oplus b) \otimes (a \oplus c)).$$

We then define:

$$(a \leq b) \Leftrightarrow (a \otimes b = a).$$

With respect to $\leq$ we can show $\otimes$ and $\oplus$ are monotonic.

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VCSP framework

- Here we insist $\leq$ is a total order.
- Then the costs are a valuation structure.
- We write:
  - 0 to mean $\perp$ (the best value);
  - ∞ to mean $\top$ (the worst value);
  - Projection ($\otimes$) becomes minimum;
- If $\oplus$ is strictly monotonic (except on ∞) we can also subtract costs (we get $\ominus$).

Constraint Satisfaction

- consists of (CSP);
  - A set of problem variables;
  - A domain of values;
  - A set of constraints.

- Each constraint has a:
  - Scope: list of concerned variables;
  - Relation: validity of each assignment.

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Valued Constraint Satisfaction

- consists of (VCSP):
  - A set of problem variables;
  - A domain of values;
  - A set of constraints;
  - A set of costs (valuation structure).

A totally ordered set with a strictly monotonic aggregation operator

- Each constraint has a:
  - Scope: list of concerned variables;
  - Cost Function: cost of each assignment.
  - **Multiplicity**: How many times we apply it.

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An Optimiz… m

Unary Valued Constraints

Classical CSP

…t Homomorphisms to Proper Interval Graphs and Bigraphs
…gory Gutin, Pavol Hell, Arash Rafiey, Anders Yeo.

For graphs G and H, a mapping $f: V(G) \to V(H)$ is a homomorphism of G to H if

$$uv \in E(G) \text{ implies } f(u)f(v) \in E(H).$$

If, moreover, each vertex $u \in V(G)$ is associated with costs $c_i(u); i \in V(H)$, then the cost of the homomorphism f is

$$\sum\{u \in V(G)\}\, c_{f(u)}(u).$$

For each fixed graph H, we have the *minimum cost homomorphism problem*, written as MinHOM(H).

The problem is to decide, for an input graph G with costs

$$c_i(u);\ u \in V(G);\ i \in V(H),$$

whether there exists a homomorphism of G to H and, if one exists, to find one of minimum cost.

Aggregation

Optimisation

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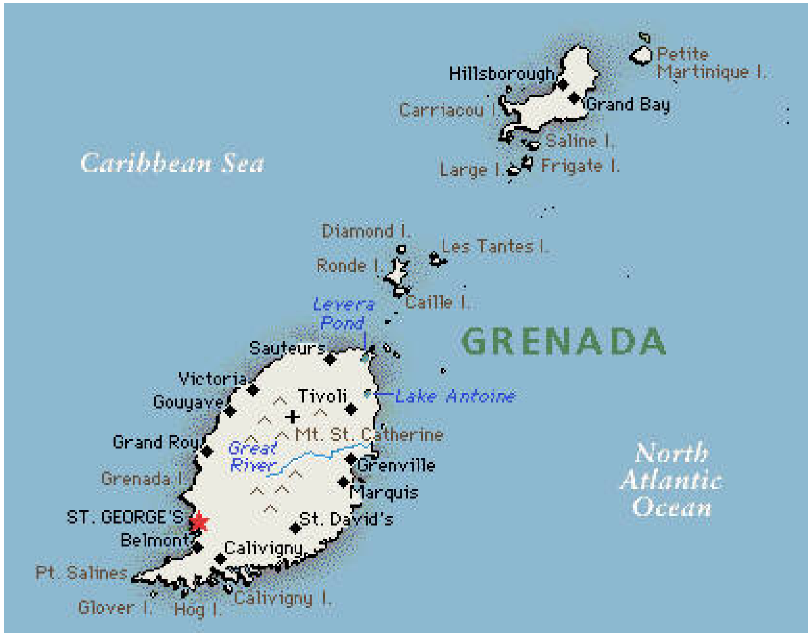
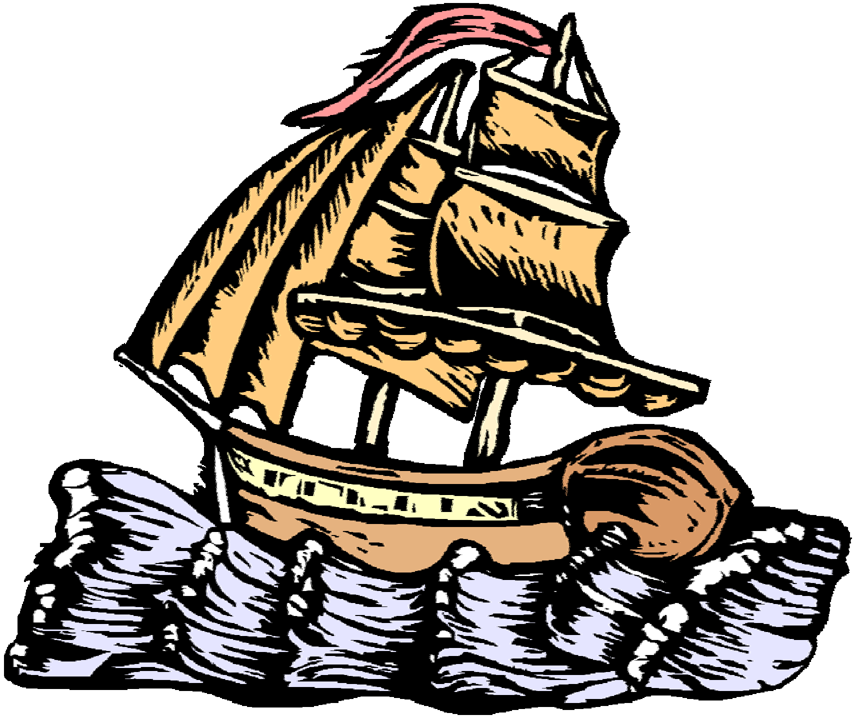
A voyage of Discovery

- In general the VCSP is NP-hard.
- It generalizes CSP.





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Reasoning about Tractability
+ 8 minutes

Valued Constraint Languages

- For any domain D, and valuation structure χ, a k-ary cost function is a mapping μ from D^k to χ.
- A valued constraint language (for D and χ) is any set Γ of cost functions.

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Tractability?

- We generalised the notion of a polymorphism to a multimorphism.
- All tractable Boolean languages (and many others) are characterised by single multimorphisms.
- Intractable Boolean languages have no multimorphisms (to speak of).

Multimorphisms and Expressibility

?

?

?

Do Multimorphisms capture Expressibility in the same way that Polymorphism did for crisp contraints?

?

?

?

No

Well – at least I don't think so

Fractional Polymorphism

- Here we generalise the concept of a multimorphism.
- A k-ary **fractional polymorphism** of ϕ is a (+ve) weighted set of k-ary polymorphisms of Feas(ϕ) with total weight equal to k.

Fractional polymorphisms

- An n-ary cost function assigns costs to n-tuples
- A k-ary fractional polymorphism maps k-tuples to weighted sets of k-tuples
- We take k (not necessarily different) n-tuples and add up their costs.
- We apply the fractional polymorphism to them component-wise and add up the weighted costs of the obtained weighted n-tuples
- The weighted sum of the costs cannot be worse than original sum of the costs

Fractional polymorphism: Example

0: Start with any two 3-tuples

1	2	3
5	1	2

Cost: $\phi(x,y,z) := x + 2y + 3z$

Fractional Polymorphism: {(min,1),(max,1)}

Fractional polymorphism : Example

1: Apply the cost function

$$\begin{bmatrix}\phi \\ \phi\end{bmatrix} \Rightarrow \begin{bmatrix}1 & 2 & 3 \\ 5 & 1 & 2\end{bmatrix} = \begin{bmatrix}14 \\ 13\end{bmatrix}$$

Cost: $\phi(x,y,z) := x + 2y + 3z$

Fractional Polymorphism: $\{(\min,1),(\max,1)\}$

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Fractional polymorphism : Example

2: Add the results

ϕ
ϕ

$\Rightarrow$

1	2	3
5	1	2

=

14	27
13	

Cost: $\phi(x,y,z) := x + 2y + 3z$

Fractional Polymorphism: {(min,1),(max,1)}

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Fractional polymorphism : Example

3: Apply the fractional polymorphism

F	F	F

$\Downarrow$

ϕ		1	2	3		14	27
ϕ	$\Rightarrow$	5	1	2	=	13	
ϕ		1	1	2		1*9	27
ϕ		5	2	3		1*18	

Cost: $\phi(x,y,z) := x + 2y + 3z$

Fractional Polymorphism: {(min,1),(max,1)}

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Fractional polymorphism : Example

4: Check the transformed result is no bigger

F	F	F

$$\Downarrow$$

ϕ		1	2	3		14	27
ϕ	$\Rightarrow$	5	1	2	=	13	
ϕ		1	1	2		9	27
ϕ		5	2	3		18	

Cost: $\phi(x,y,z) := x + 2y + 3z$

Fractional Polymorphism: $\{(\min,1),(\max,1)\}$

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Multimorphisms and Polymorphisms

- A **k-ary polymorphism** is a **singleton fractional polymorphism**.
- A **multimorphism** is a **fractional polymorphism with integral weights**.

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Expressibility and Fractional Polymorphisms
+ 13 minutes

Expressibility

- We say that ϕ is **expressible** over Γ if there is a VCSP, P, and a list of variables, σ, for which the projection onto σ of Sol(P) has cost function ϕ +/- some constant
- For classical CSPs this notion captured the capability of join and project. Here it captures the equivalent notions

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Expressibility

- Every cost function expressed by Γ has the fractional polymorphisms of Γ.
- If, furthermore, any cost functions with the fractional polymorphisms of Γ can be expressed by Γ then the fractional polymorphisms capture complexity.
- The tractability of a valued constraint language would then be determined by its fractional polymorphisms.

The Fractional Polymorphisms Conjecture

Conjecture

For any language Γ, any finite set Γ^* of cost functions improved by the fractional polymorphisms of Γ is polynomial time reducible to Γ.

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Fractional Polymorphisms of Feas(Γ)

- Feas(Γ) is the set of cost functions where finite values are replaced by zero.
- It is worth observing that any fractional polymorphism of Γ is also a fractional polymorphism of Feas(Γ).
- …because a fractional polymorphism is a weighted set of polymorphisms of Feas(Γ).

Does Γ express Feas(Γ)?

- If Γ **does not** express Feas(Γ) then we cannot have the full conjecture holding. Feas(Γ) provides a counterexample.
- The conjecture fails.

k-th order Indicator Problem

- Variables: $V = D^k$
- Scope $\sigma \in V^*$ matches $\rho \in \Gamma$ if each of the k lists of components of σ is in ρ
- The constraint $<\sigma,\rho>$ is applied whenever σ matches ρ

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The valued Indicator Problem

- Consider the $|\mathrm{Feas}(\phi)|$ indicator problem over the language $\mathrm{Feas}(\Gamma)$
- Make this into a family of VCSPs by assigning a Multiplicity Variable, $x(\sigma,\gamma)$, to each matched scope σ of the (valued) constraint γ in Γ (*and add in a constant*)
- Is there an assignment to the Multiplicity Variables so that the VCSP obtained expresses ϕ on σ_ϕ where $V[\sigma_\phi] = \mathrm{Feas}(\phi)$?

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Expressibility and Fractional Polymorphisms

New Theorem

- Either the valued indicator problem expresses a cost function that finitely matches ϕ (matches ϕ wherever it is finite)
- Or, by a variant of Farkas' Lemma we get that there is a fractional polymorphism of Γ that is not a fractional polymorphism of ϕ.

Does Γ express Feas(Γ)?

- Suppose that Γ expresses Feas(Γ) then, if Pol(Feas(Γ)) $\subseteq$ Pol(Feas(ϕ)) we can express Feas(ϕ) over Γ
- Feas(ϕ) + a finite match of ϕ is equal to ϕ
- In this case we get that the fractional polymorphisms of Γ and polymorphisms of Feas(Γ) exactly capture expressibility.
- The conjecture holds.

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Corollaries

- Any language Γ without any fractional polymorphisms (to speak of) is intractable. (Such a language expresses XOR).
- If Γ expresses (e.g. includes) Feas(Γ) then the expressibility of Γ is known.
- Finite expressiveness is captured by fractional polymorphisms.
- XOR has no fractional polymorphisms to speak of and so expresses any finite cost function.

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Feas(Γ)

- Feas(Γ) is not in general expressible.
- So Γ cannot express all cost functions with the fractional polymorphisms of Γ.

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Conclusions and Open Questions
+23 minutes

Conclusions - Expressibility

- Fractional Polymorphisms characterise finite expressibility (and so reducibility)
- Fractional Polymorphisms, and classical Polymorphisms, characterise expressibility for languages with feasibility
- Feasibility is not (in general) expressible

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Conclusions - Complexity

- No fractional polymorphisms means intractable
- Adding all finite cost functions closed under the same fractional polymorphisms does not change the complexity
- Adding all cost functions closed under the same fractional polymorphisms (and polymorphisms) to a language with feasibility does not change the complexity

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Open Questions

- What exactly characterises expressibility …. and so reducibility?
  - Is there something a bit stronger than fractional polymorphism?
- For which languages Γ is Feas(Γ) expressible?
  - What characterises those non-expressible feasibility cost functions?

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The Valued Indicator Problem expresses ϕ

If the following equations are satisfied:

- $\forall \mathcal{F} \in \mathrm{Pol}_{|\mathrm{Feas}(\phi)|}(\Gamma)$,

  $$\sum_{\gamma\in\Gamma}\sum_{\sigma \text{ matches } \mathrm{Feas}(\gamma)} x(\sigma,\gamma)\, \gamma(\mathcal{F}(\sigma)) \geq \phi(\mathcal{F}(\sigma_\phi))$$

  with equality when F is a projection,

then with the multiplicities $x(\sigma,\gamma)$ the valued indicator problem expresses a function that finitely matches ϕ

The Valued Indicator Problem **does not** express ϕ

By Farkas' Lemma the following equations:

- $\forall\, \gamma \in \Gamma, \forall\, \sigma$ matches γ,
  $\sum_{\mathcal{F} \in \mathrm{Pol}(\Gamma)} y(\mathcal{F})\ \gamma(\mathcal{F}(\sigma)) = 0$

  $\forall\, \gamma \in \Gamma$,
  $\sum_{\mathcal{F} \in \mathrm{Pol}(\Gamma)} y(\mathcal{F})\ \phi(\mathcal{F}(\sigma_\phi)) < 0$

  where $y(\mathcal{F}) \geq 0$ whenever $\mathcal{F}$ is not a projection.

This solution precisely defines a fractional polymorphism of Γ that is not a fractional polymorphism of ϕ

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