On CSP vs. MMSNP

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Over 10 years ago Feder and Vardi introduced the logic **MMSNP** as

- a logical 'manifestation' of

$$\textbf{CSP} = \{CSP(\mathcal{A}) : \mathcal{A} \text{ a finite structure}\}$$

so as to hopefully

- use logical tools and techniques from finite model theory to obtain a dichotomy result.

This talk is about better understanding the difference between problems in **MMSNP** and problems in **CSP**.

(Joint work with Florent Madelaine, Durham University.)

Defining MMSNP

What does it mean for a homomorphism to exist from a graph $\mathcal{G}$ to some fixed graph $\mathcal{H}$?

"There exists a map f from $|\mathcal{G}|$ to $|\mathcal{H}|$ such that ...

- For each vertex h of $|\mathcal{H}|$, define the set M_h of vertices of $\mathcal{G}$ mapped to h by f.

... for every pair (u, v) of vertices of $\mathcal{G}$ ...

- Allow only universal quantification over vertices.

... if (u, v) is an edge of $\mathcal{G}$ then $(f(u), f(v))$ is an edge of $\mathcal{H}$."

- We are only interested in 'positive' information (about edges, not non-edges) so concentrate on edges (and not non-edges) in any formula.
- Don't allow the equality relation as homomorphisms don't 'distinguish' between different elements.

Monotone monadic SNP without inequality (MMSNP) is the fragment of ESO consisting of formulae of the form

$$\exists \mathbf{M} \forall \mathbf{t} \bigwedge_i \neg \big(\alpha_i(\sigma, \mathbf{t}) \wedge \beta_i(\mathbf{M}, \mathbf{t}) \big),$$

where **M** is a tuple of monadic relation symbols (not in σ), **t** is a tuple of (first-order) variables and for every negated conjunct $\neg(\alpha_i \wedge \beta_i)$:

- α_i consists of a conjunction of positive atoms involving relation symbols from σ and variables from **t**
- β_i consists of a conjunction of atoms or negated atoms involving relation symbols from **M** and variables from **t**.

(Dropping any one of ‘monotone’, ‘monadic’ and ‘without inequalities’ results in a class of problems that does not have a dichotomy.)

An example: consider the signature $\sigma_2 = \langle E(-,-)\rangle$.

Define Φ_0 as

$$\begin{aligned}\exists R\, \exists W\, \exists B\, \forall x\, \forall y\, \big(&\neg(\neg R(x) \wedge \neg W(x) \wedge \neg B(x)) \\ &\wedge \neg(R(x) \wedge B(x)) \wedge \neg(R(x) \wedge W(x)) \wedge \neg(B(x) \wedge W(x)) \\ &\wedge \neg(E(x,y) \wedge R(x) \wedge R(y)) \wedge \neg(E(x,y) \wedge W(x) \wedge W(y)) \\ &\wedge \neg(E(x,y) \wedge B(x) \wedge B(y))\big).\end{aligned}$$

The problem defined by Φ_0 is 3-COLOURABILITY, which is in **CSP**.

In fact, **CSP** $\subseteq$ **MMSNP** (simply write out the definition of a homomorphism as an **MMSNP** formula, hinted at above).

An example: define Φ_1 over σ_2 as

$$\exists C\ \forall x\ \forall y\ \forall z\ \big(\neg\big(E(x,y) \wedge E(y,z) \wedge E(z,x) \wedge C(x) \wedge C(y) \wedge C(z)\big) \\ \wedge \neg\big(E(x,y) \wedge E(y,z) \wedge E(z,x) \wedge \neg C(x) \wedge \neg C(y) \wedge \neg C(z)\big)\big).$$

The problem defined by Φ_1 is NON-MONOCHROMATIC TRIANGLE, which is *not* in **CSP**:

- Feder and Vardi showed this using a probabilistic argument
- Madelaine and S. did so constructively.

So, **CSP** $\subset$ **MMSNP**, ... but Feder and Vardi proved that for every $\Omega \in$ **MMSNP**, there exists $\Omega' \in$ **CSP** such that:

- Ω reduces to Ω' via a polynomial-time Karp reduction
- Ω' reduces to Ω via a randomized polynomial-time Turing reduction.

$* * *$ *Gabor Kun has recently removed the word 'randomized'*.

Another realisation of **MMSNP**

A coloured structure $(\mathcal{A}, a^{\mathcal{T}})$ is a σ-structure $\mathcal{A}$ together with a homomorphism $a^{\mathcal{T}}$ to a σ-structure $\mathcal{T}$.

A $\mathcal{T}$-pattern is a structure $(\mathcal{A}, a^{\mathcal{T}})$ so that $\mathcal{A}$ has no isolated elements.

A representation is a pair $(\mathscr{F}, \mathcal{T})$, where $\mathscr{F}$ is a set of forbidden patterns and $\mathcal{T}$ is the target.

A structure $(\mathcal{A}, a^{\mathcal{T}})$ is valid w.r.t. $(\mathscr{F}, \mathcal{T})$ if no forbidden pattern of $\mathscr{F}$ maps homomorphically into $(\mathcal{A}, a^{\mathcal{T}})$.

A structure $\mathcal{A}$ is valid w.r.t. $(\mathscr{F}, \mathcal{T})$ if there exists a homomorphism $a^{\mathcal{T}} : \mathcal{A} \to \mathcal{T}$ such that no forbidden pattern of $\mathscr{F}$ maps homomorphically into $(\mathcal{A}, a^{\mathcal{T}})$.

Observation: a graph $\mathcal{G}$ is in the problem NON-MONOCHROMATIC TRIANGLE iff its vertices can be coloured black or white so that

- there is no homomorphism of the 'all-white' triangle into $\mathcal{G}$, and
- there is no homomorphism of the 'all-black' triangle into $\mathcal{G}$.

Moreover, the 'forbidden patterns' can be 'read from' the defining formula Φ_1:

$$\exists C\ \forall x\ \forall y\ \forall z\ \Big(\neg\big(E(x,y)\wedge E(y,z)\wedge E(z,x)\wedge C(x)\wedge C(y)\wedge C(z)\big) \\ \wedge\neg\big(E(x,y)\wedge E(y,z)\wedge E(z,x)\wedge\neg C(x)\wedge\neg C(y)\wedge\neg C(z)\big)\Big).$$

The forbidden patterns problem, **FPP**$(\mathscr{F},\mathcal{T})$, given by the representation $(\mathscr{F},\mathcal{T})$ is the problem whose yes-instances are exactly those structures that are valid w.r.t. $(\mathscr{F},\mathcal{T})$.

Another example: define Φ_2 as

$$\exists C\ \forall x\ \forall y\ \left(\neg(E(x,y) \wedge C(x)) \wedge \neg(E(x,x) \wedge C(x) \wedge C(y))\right).$$

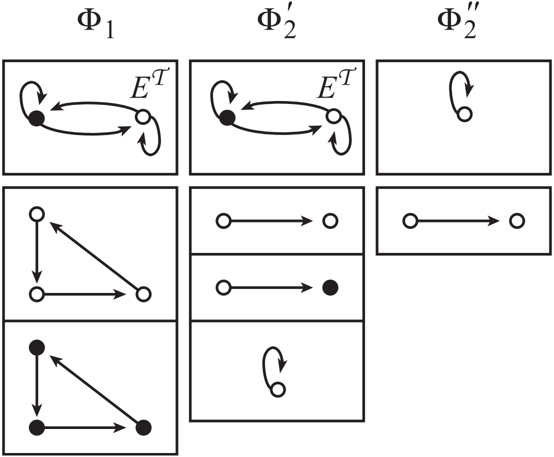
It is not so easy to see how the problem defined by Φ_2 can be realised as a forbidden patterns problem; but Φ_2 is equivalent to the disjunction of Φ_2' and Φ_2'' defined as

$$\begin{aligned} \Phi_2' \quad = \quad & \exists C\ \forall x\ \forall y\ \left(\neg(E(x,y) \wedge C(x) \wedge C(y))\right. \\ & \left.\wedge \neg(E(x,y) \wedge C(x) \wedge \neg C(y)) \wedge \neg(E(x,x) \wedge C(x))\right) \end{aligned}$$

and

$$\begin{aligned} \Phi_2'' \quad = \quad & \exists C\ \forall x\ \forall y\ \left(\neg(E(x,y) \wedge C(x) \wedge C(y))\right. \\ & \left.\wedge \neg(E(x,y) \wedge C(x) \wedge \neg C(y)) \wedge \neg(\neg C(y))\right). \end{aligned}$$

We can see how to realise the problem defined by Φ_2 as the union of the forbidden patterns problems corresponding to Φ_2' and to Φ_2'':



In fact, any sentence of **MMSNP** can be written as a finite disjunction of primitive sentences, where a primitive sentence is such that for every negated conjunct $\neg(\alpha \wedge \beta)$:

- for every first-order variable x that occurs in $\neg(\alpha \wedge \beta)$ and for every monadic symbol C in **M**, exactly one of $C(x)$ and $\neg C(x)$ occurs in β
- unless x is the only first-order variable that occurs in $\neg(\alpha \wedge \beta)$, an atom of the form $R(\mathbf{t})$, where x occurs in $\mathbf{t}$ and R is a relation symbol from σ, must occur in α.

So, primitive sentences are the ones we can 'read' (to obtain forbidden patterns) and we restrict ourselves to primitive sentences of **MMSNP**.

Theorem

The class of problems captured by the primitive fragment of the logic **MMSNP** *is exactly the class* **FPP** *of forbidden patterns problems.*

Proof: Let $\Phi = \exists \mathbf{M} \forall \mathbf{t} \varphi$ be a primitive sentence of **MMSNP**.

A conjunction $\chi(\mathbf{M}, x)$ of atoms and negated atoms involving only relation symbols from **M** and the sole first-order variable x, where for each relation symbol C in **M**, exactly one of $C(x)$ or $\neg C(x)$ occurs, is referred to as an **M-colour**.

So, associated with every negated conjunct $\neg(\alpha \wedge \beta)$ in Φ (more precisely, with β) and every variable occurring in this negated conjunct, is a unique **M**-colour; in fact, β can be written as the conjunction of these **M**-colours.

Construct the structure $\mathcal{T}$ from Φ as follows:

- its domain T consists of all **M**-colours $\chi(\mathbf{M}, x)$ that are not explicitly forbidden in Φ, *i.e.*, some $\neg(\alpha \wedge \beta)$ is s.t. α is empty and β is the **M**-colour χ; and
- for every relation symbol R of arity m in σ, set $R^{\mathcal{T}} := T^m$.

Start with $\mathscr{F} := \emptyset$, and for every negated conjunct $\neg(\alpha \wedge \beta)$ in φ, add to $\mathscr{F}$ the structure $(\mathcal{A}_\alpha, a_\beta^{\mathcal{T}})$, where:

- the domain of $\mathcal{A}_\alpha$ consists of all first-order variables that occur in the negated conjunct $\neg(\alpha \wedge \beta)$
- for every relation symbol R in σ, there is a tuple $R^{\mathcal{A}_\alpha}(\mathbf{t})$ iff the atom $R(\mathbf{t})$ appears in α
- for every $x \in |\mathcal{A}_\alpha|$, set $a_\beta^{\mathcal{T}}(x) := \chi$, where χ is the **M**-colour of x in β.

Then the problem defined by Φ is $\mathbf{FPP}(\mathscr{F}, \mathcal{T})$. $\square$

Normal forms for representations

We have essentially reduced the question of whether a problem in the primitive fragment of **MMSNP** is in **CSP** to whether a problem defined as **FPP**$(\mathscr{F}, \mathcal{T})$ is in **CSP**.

We now develop a normal form for a representation $(\mathscr{F}, \mathcal{T})$ so as to enforce six properties.

Assume that all forbidden patterns are connected.

($\mathfrak{p}1$) Any structure is valid if, and only if, it is weakly valid.

A structure $\mathcal{A}$ is weakly valid for $(\mathscr{F}, \mathcal{T})$ if there exists a coloring $a : \mathcal{A} \to \mathcal{T}$ such that no forbidden pattern embeds into $(\mathcal{A}, a)$.

A homomorphic image of $(\mathcal{B}, b) \in \mathscr{F}$ is a structure $(\mathcal{C}, c)$ such that there is an epimorphism $\mathcal{B} \to \mathcal{C}$ with the properties that:

- for each symbol $R \in \sigma$ and for each tuple $R^{\mathcal{C}}(\mathbf{t}')$, there exists a tuple $R^{\mathcal{B}}(\mathbf{t})$ such that $h(\mathbf{t}) = \mathbf{t}'$;
- the following diagram commutes:

$$
\begin{tikzcd}
\mathcal{B} \arrow[r, two heads, "h"] \arrow[dr, "b^{\mathcal{T}}"'] & \mathcal{C} \arrow[d, "c^{\mathcal{T}}"] \\
& \mathcal{T}
\end{tikzcd}
$$

Denote all homomorphic images of patterns of $\mathscr{F}$ by $\mathbf{H}\mathscr{F}$.

Lemma

The representation $(\mathbf{H}\mathscr{F}, \mathcal{T})$ *is equivalent to* $(\mathscr{F}, \mathcal{T})$*, and* $(\mathbf{H}\mathscr{F}, \mathcal{T})$ *satisfies* $\mathfrak{p1}$*.*

$\mathfrak{p}2$ Every pattern of $\mathscr{F}$ is automorphic.

A retract of a structure $(\mathcal{B}, b)$ is a structure $(\mathcal{A}, a)$ for which there is a monomorphism $\iota : \mathcal{A} \to \mathcal{B}$ and an epimorphism $s : \mathcal{B} \to \mathcal{A}$ such that $s \circ \iota = \mathrm{id}_{\mathcal{A}}$, $b \circ \iota = a$ and $s \circ a = b$.

A retract $(\mathcal{A}, a)$ of $(\mathcal{B}, b)$ is a proper retract if $\mathcal{A} \not\approx \mathcal{B}$.

A coloured structure is automorphic if it has no proper retracts, and a core of a coloured structure is an automorphic retract.

Lemma

Every $\mathcal{T}$-coloured structure has a $\mathcal{T}$-coloured core that is unique up to $\mathcal{T}$-coloured isomorphism.

Let $(\mathscr{F}', \mathfrak{T})$ be obtained from $(\mathscr{F}, \mathfrak{T})$ by replacing a pattern of $\mathscr{F}$ with its core. We call this a core-reduction on $(\mathscr{F}, \mathfrak{T})$.

Lemma

If $(\mathscr{F}', \mathfrak{T})$ is a core-reduction of $(\mathscr{F}, \mathfrak{T})$ then

- *$(\mathscr{F}', \mathfrak{T})$ is equivalent to $(\mathscr{F}, \mathfrak{T})$*
- *if $(\mathscr{F}, \mathfrak{T})$ satisfies $\mathfrak{p}1$ then so does $(\mathscr{F}', \mathfrak{T})$.*

$\mathfrak{p}3$ It is not the case that there is a monomorphism $(\mathcal{B}_1, b_1) \to (\mathcal{B}_2, b_2)$, for any distinct patterns $(\mathcal{B}_1, b_1)$ and $(\mathcal{B}_2, b_2)$ of $\mathscr{F}$.

Let $(\mathscr{F}, \mathcal{T})$ be a representation and let $(\mathcal{B}_1, b_1)$ and $(\mathcal{B}_2, b_2)$ be distinct patterns of $\mathscr{F}$ such that there is a monomorphism $(\mathcal{B}_1, b_1) \to (\mathcal{B}_2, b_2)$.

Let $(\mathscr{F}', \mathcal{T})$ be obtained by removing $(\mathcal{B}_2, b_2)$ from $\mathscr{F}$.

Then $(\mathscr{F}', \mathcal{T})$ is an embed-reduction of $(\mathscr{F}, \mathcal{T})$.

Lemma

If $(\mathscr{F}', \mathcal{T})$ is an embed-reduction of $(\mathscr{F}, \mathcal{T})$ then

- $(\mathscr{F}', \mathcal{T})$ *is equivalent to* $(\mathscr{F}, \mathcal{T})$
- *if* $(\mathscr{F}, \mathcal{T})$ *satisfies* $\mathfrak{p}1$ *then so does* $(\mathscr{F}', \mathcal{T})$.

$\mathfrak{p}4$ No pattern of $\mathscr{F}$ is conform.

A pattern $(\mathcal{A}, a)$ is conform iff $\mathcal{A}$ consists solely of an antireflexive tuple $R^{\mathcal{A}}(\mathbf{t})$; that is, there exists a relation symbol R in σ such that $R^{\mathcal{A}} = \{\mathbf{t}\}$, where every element of $|\mathcal{A}|$ occurs in $\mathbf{t}$ exactly once, and for every other relation symbol R' in σ, we have $R'^{\mathcal{A}} = \emptyset$.

Let $(\mathscr{F}, \mathfrak{T})$ be a representation and let $(R(\mathbf{t}), \pi^{\mathfrak{T}})$ be a conform pattern of $\mathscr{F}$.

Let $\mathfrak{T}'$ be $\mathfrak{T}$ with $R(\pi^{\mathfrak{T}}(\mathbf{t}))$ removed.

Let $\mathscr{F}'$ be the patterns of $\mathscr{F}$ that are also $\mathfrak{T}'$-patterns; that is, the patterns $(\mathcal{B}, b) \in \mathscr{F}$ for which $b(\mathbf{u}) \neq \pi(\mathbf{t})$, for any tuple $R^{\mathcal{B}}(\mathbf{u})$.

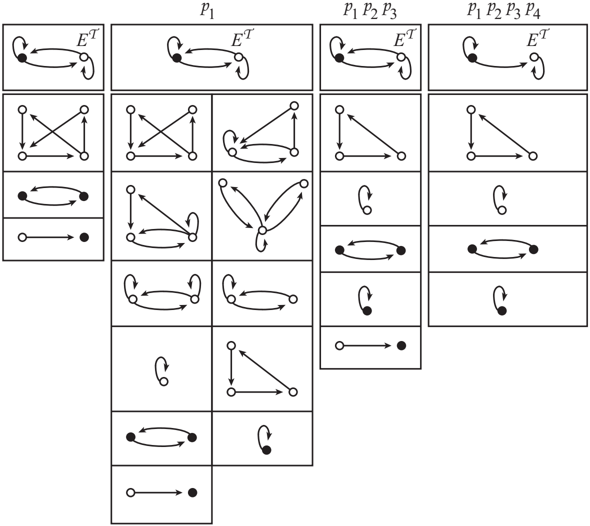
We say that $(\mathscr{F}', \mathfrak{T}')$ is a conform-reduction of $(\mathscr{F}, \mathfrak{T})$.

Lemma

If $(\mathscr{F}', \mathfrak{T})$ is a conform-reduction of $(\mathscr{F}, \mathfrak{T})$ then

- $(\mathscr{F}', \mathfrak{T})$ *is equivalent to* $(\mathscr{F}, \mathfrak{T})$
- *if* $(\mathscr{F}, \mathfrak{T})$ *satisfies* $\mathtt{p1}$ *then so does* $(\mathscr{F}', \mathfrak{T})$.

$(\mathcal{F}, \mathcal{T})$ $(\mathbf{H}\mathcal{F}, \mathcal{T})$



$\mathfrak{p}5$ Every forbidden pattern is biconnected.

Let the forbidden pattern $(\mathcal{A}, a^{\mathcal{T}})$ of $(\mathscr{F}, \mathcal{T})$ be such that $\mathcal{A}$ has an articulation point x such that $(\mathcal{A}, a^{\mathcal{T}}) = (\mathcal{B}, b^{\mathcal{T}})|_x(\mathcal{C}, c^{\mathcal{T}})$.

Let $\mathcal{T}'$ be built from $\mathcal{T}$ so that

- $a^{\mathcal{T}}(x) = b^{\mathcal{T}}(x) = c^{\mathcal{T}}(x)$ is replaced in $\mathcal{T}'$ by k_0 and k_1
- set $R^{\mathcal{T}'}(\mathbf{t})$ whenever $R^{\mathcal{T}}(\tilde{\mathbf{t}})$ holds, with $\mathbf{t}$ obtained from $\tilde{\mathbf{t}}$ by replacing every occurrence of $a^{\mathcal{T}}(x)$ with either k_0 or k_1.

Let $\mathscr{F}'$ be built from $\mathscr{F}$ so that

- $(\mathcal{A}, a^{\mathcal{T}}) = (\mathcal{B}, b^{\mathcal{T}})|_x(\mathcal{C}, c^{\mathcal{T}})$ is replaced by $(\mathcal{B}, b^{\mathcal{T}'})$ and $(\mathcal{C}, c^{\mathcal{T}'})$ where $b^{\mathcal{T}'}(x) = k_0$ and $c^{\mathcal{T}'}(x) = k_1$.
- Every occurrence of the colour $a^{\mathcal{T}}(x)$ in any forbidden pattern (including the two above) is replaced by both k_0 and k_1 (so lots of new forbidden patterns corresponding to one old one).

$(\mathscr{F}', \mathcal{T}')$ is a Feder-Vardi reduction of $(\mathscr{F}, \mathcal{T})$.

Lemma

If $(\mathscr{F}', \mathfrak{T})$ *is a Feder-Vardi-reduction of* $(\mathscr{F}, \mathfrak{T})$ *then*

- $(\mathscr{F}', \mathfrak{T})$ *is equivalent to* $(\mathscr{F}, \mathfrak{T})$
- *if* $(\mathscr{F}, \mathfrak{T})$ *satisfies* $\mathfrak{p}\mathfrak{1}$ *then so does* $(\mathscr{F}', \mathfrak{T})$*.*

However, 'splitting' forbidden pattern of $\mathscr{F}$ results in more forbidden patterns in $\mathscr{F}'$, albeit 'more connected' ones and ones very similar in structure.

However, by working with compact representations, where forbidden patterns are actually families of forbidden patterns, the members of which only differ in their colourings, and a consequent notion of a Feder-Vardi-reduction relating to compact representations, we can show that any sequence of Feder-Vardi-reductions can be assumed to be finite.

Hence, we are now in a position to enforce properties $\mathfrak{p}1 - \mathfrak{p}5$.

Essentially, we continually 'hit' ($\mathbf{H}\mathscr{F}, \mathfrak{T}$) with core-, embed-, conform- and Feder-Vardi-reductions so as to enforce properties $\mathfrak{p}1 - \mathfrak{p}5$.

It can be shown that this process eventually halts so that we may assume that $(\mathscr{F}, \mathfrak{T})$ satisfies the properties.

Our final property is

$\mathfrak{p}6$ The representation is automorphic.

Any representation $(\mathscr{F}, \mathfrak{T})$ satisfying properties $\mathfrak{p}1 - \mathfrak{p}6$ is normal.

Every representation can be (effectively) reduced to an equivalent normal representation.

Corollary

Let $(\mathscr{F}, \mathfrak{T})$ *be a normal representation. If* $\mathscr{F} \neq \emptyset$ *then the target* $\mathfrak{T}$ *is not valid w.r.t.* $(\mathscr{F}, \mathfrak{T})$.

Recap

Any problem in **MMSNP** can be described by a disjunction of primitive sentences.

The primitive sentences of **MMSNP** equate to **FPP**.

Assume all representations are connected.

Any problem in **FPP** can be defined by a normal representation.

If a problem Ω in **FPP** can be defined by a (normal) representation $(\mathscr{F}, \mathfrak{T})$ with $\mathscr{F} = \emptyset$ then $\Omega =$ **CSP**$(\mathfrak{T})$.

Theorem

If a problem Ω in **FPP** *can be defined by a normal representation $(\mathscr{F}, \mathfrak{T})$ with $\mathscr{F} \neq \emptyset$ then $\Omega \notin$* **CSP**.

Henceforth, the representation $(\mathscr{F}, \mathfrak{T})$ is normal, connected and $\mathscr{F} \neq \emptyset$.

Constructing counter-examples

A family of structures $\mathscr{W}$ is said to be a witness family for $(\mathscr{F}, \mathcal{T})$ iff $\mathscr{W} \subseteq$ **FPP**$(\mathscr{F}, \mathcal{T})$ and for any structure $\mathcal{B}$, there exists $\mathcal{W} \in \mathscr{W}$ such that either

- $\mathcal{W} \not\to \mathcal{B}$, or
- for some $h : \mathcal{W} \to \mathcal{B}$, the homomorphic image $h(\mathcal{W})$ does not belong to **FPP**$(\mathscr{F}, \mathcal{T})$.

Suppose that $\mathscr{W}$ is a witness family for $(\mathscr{F}, \mathcal{T})$ and **FPP**$(\mathscr{F}, \mathcal{T}) =$ **CSP**$(\mathcal{B})$.

There exists $\mathcal{W} \in \mathscr{W}$ such that either

- $\mathcal{W} \not\to \mathcal{B}$ - contradiction

or

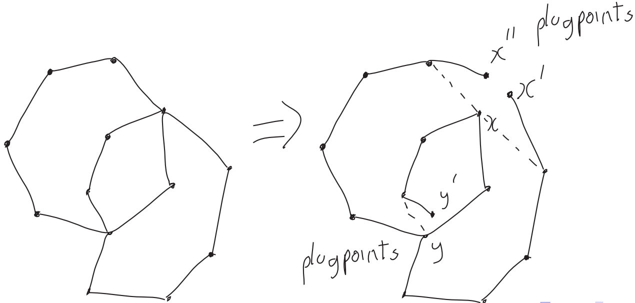
- $h : \mathcal{W} \to \mathcal{B}$ and $h(\mathcal{W}) \notin$ **FPP**$(\mathscr{F}, \mathcal{T})$ - contradiction, as $h(\mathcal{W}) \to \mathcal{B}$.

As $(\mathscr{F}, \mathcal{T})$ is normal, $\mathcal{T}$ is not valid w.r.t. $(\mathscr{F}, \mathcal{T})$.

So, some forbidden pattern $(\mathcal{A}, a^{\mathcal{T}})$ embeds into $(\mathcal{T}, \iota^{\mathcal{T}})$ via some monomorphism f.

As $(\mathcal{A}, a^{\mathcal{T}})$ is biconnected and non-conform, it contains a cycle; let C be the image of this cycle under f (and so C is a cycle).

'Break' C at some articulation point x, into x and x', so that x' inherits x's colour. Continue until no forbidden pattern embeds, and call the resulting gadget $(\mathcal{G}, g^{\mathcal{T}})$ (which is valid for $(\mathscr{F}, \mathcal{T})$).



Suppose that $\mathcal{G}$ has p_i plugpoints of type i, for $i = 1, 2, \ldots, k$.

Let $\sigma' = \langle R \rangle$, with the arity of R equal to $p_1 + p_2 + \ldots + p_k$.

From Feder and Vardi:

Theorem

Fix r and s. For every structure $\mathcal{B}$ of size n, there exists a structure $\mathcal{B}'$ of size n^a (where a depends solely on r and s) such that:

- *the girth of $\mathcal{B}'$ is greater than r;*
- *$\mathcal{B}' \to \mathcal{B}$; and*
- *for every structure $\mathcal{C}$ of size at most s, $\mathcal{B} \to \mathcal{C}$ if, and only if, $\mathcal{B}' \to \mathcal{C}$.*

Furthermore, $\mathcal{B}'$ can be constructed from $\mathcal{B}$ in randomized polynomial time.

Let γ be the length of a longest cycle in any forbidden pattern of $\mathscr{F}$.

Suppose that $\mathbf{FPP}(\mathscr{F}, \mathcal{T}) = \mathbf{CSP}(\mathcal{B})$ where $|\mathcal{B}| = b$.

Using the above theorem, there is a σ'-structure $\mathcal{Q}$ of girth greater than γ with the property that for every map $h : |\mathcal{Q}| \rightarrow \{0, 1, \ldots, b-1\}$, there must exist at least one tuple $R^{\mathcal{Q}}(\mathbf{u}^1, \mathbf{u}^2, \ldots, \mathbf{u}^k)$ such that for every i,

$$h(u^i_1) = h(u^i_2) = \ldots = h(u^i_{p_i}).$$

Define $\mathcal{W}_{\mathcal{B}}$ as follows.

- Initialize the domain of $\mathcal{W}_{\mathcal{B}}$ to be that of $\mathcal{Q}$.
- For every tuple $R^{\mathcal{Q}}(\mathbf{u}^1, \mathbf{u}^2, \ldots, \mathbf{u}^k)$, plug a copy of the gadget $\mathcal{G}$ by identifying the p_i type-i plug-points of $\mathcal{G}$ with the p_i 'socket-points' $\mathbf{u}^i$ of $|\mathcal{Q}|$, for each $i = 1, 2, \ldots, k$.

We want a $\mathcal{T}$*-colouring of* $W_{\mathcal{B}}$:

$w : W_{\mathcal{B}} \to \mathcal{T}$ (inherit the colouring $\mathcal{G} \to \mathcal{T}$).

We want $(\mathcal{W}_{\mathcal{B}}, w) \in$ **FPP**$(\mathscr{F}, \mathcal{T})$:

Suppose that $(\mathcal{W}_{\mathcal{B}}, w)$ is not valid w.r.t. $(\mathscr{F}, \mathcal{T})$; so, some forbidden pattern $(\mathcal{A}, a)$ embeds into $(\mathcal{W}_{\mathcal{B}}, w)$.

As $(\mathcal{A}, a)$ is biconnected and non-conform, there exists a cycle in $(\mathcal{W}_{\mathcal{B}}, w)$ of length at most γ.

This cycle yields a cycle of length at least 2 and at most γ in $\mathcal{Q}$ - contradiction; so, $(\mathcal{W}_{\mathcal{B}}, w)$ is valid w.r.t. $(\mathscr{F}, \mathcal{T})$.

We want the witness property:

If $\mathcal{W}_{\mathcal{B}} \not\to \mathcal{B}$, we are done; so, suppose that $h : \mathcal{W}_{\mathcal{B}} \to \mathcal{B}$.

h induces a map $\hat{h} : |\mathcal{Q}| \to \{0, 1, \ldots, b-1\}$, and so there exists a tuple $R^{\mathcal{Q}}(\mathbf{u}^1, \mathbf{u}^2, \ldots, \mathbf{u}^k)$ for which for every i,

$$\hat{h}(u_1^i) = \hat{h}(u_2^i) = \ldots = \hat{h}(u_{p_i}^i).$$

Thus, $h(\mathcal{W}_{\mathcal{B}})$ contains a homomorphic image of $\mathcal{G}$, where all same-type plug-points map to the same element of $h(\mathcal{W}_{\mathcal{B}})$; so, $h(\mathcal{W}_{\mathcal{B}})$ contains a homomorphic image of $\mathcal{T}$ via $\tilde{h}$.

Let f be any homomorphism $f : h(\mathcal{W}_{\mathcal{B}}) \to \mathcal{T}$; so, $f \circ \tilde{h} : \mathcal{T} \to \mathcal{T}$.

By the normal form corollary, there is a forbidden pattern $(\mathcal{A}, a)$ such that $\tilde{f} : (\mathcal{A}, a) \to (\mathcal{T}, f \circ \tilde{h})$.

Consequently, $\tilde{h} \circ \tilde{f} : (\mathcal{A}, a) \to (h(\mathcal{W}_{\mathcal{B}}), f)$ and $h(\mathcal{W}_{\mathcal{B}}) \notin \mathbf{FPP}(\mathscr{F}, \mathcal{T})$.

Tidy-up

Theorem

Let Φ be any sentence of **MMSNP***. Corresponding to Φ is an effectively derivable set of normal representations. The following are equivalent.*

- *There are no forbidden patterns in any of the normal representations.*
- *The problem defined by Φ is in* **CSP***.*

If $|\mathscr{F}| = 1$ and $\mathfrak{T}$ is trivial then the question of whether $(\mathscr{F}, \mathfrak{T})$ is in **CSP** corresponds to the duality problems from yesterday.

In particular, given a tree X, we can build (the same) D_X (as Nešetřil) using Feder-Vardi-reductions.

Our method gives an incremental construction of D_X.

Open problems

Bodirsky has shown that every connected problem in **FPP** is a constraint satisfaction problem with a (nice) infinite template.

So, we can decide whether such infinite constraint satisfaction problems are actually in (finite) **CSP**.

There are problems not in **FPP** that are ‘nice’ infinite constraint satisfaction problems (the template is countable and ω-categorical) so:

Open Problem: Find a good logic to capture ‘nice’ infinite **CSP**.

Omitted from this talk is the notion of recolouring (a kind of morphism between representations).

Open Problem: Does the recolouring problem for representations correspond to the containment problem?

(This open problem is related to whether an infinite countable structure is a core.)