CSP & HOMOMORPHISMS

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PRAGUE

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GRAPH

DUALITY, DENSITY, MAC's

(NOT ONLY FOR GRAPHS)

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PRAGUE

I. DUALITIES

II. HOMOMORPHISM ORDER (DENSITY, UNIVERSALITY, MAC's)

III. RESTRICTED DUALITIES

(HELL, N.: GRAPHS AND HOMOMORPHISMS
OXFORD UNIV. PRESS
2004)

CSP(H)

GIVEN: A STRUCTURE G

QUESTION: DOES THERE EXISTS A HOMOMORPHISM $G \longrightarrow H$?

STRUCTURE : GRAPHS
RELATIONS
RELATIONAL SYSTEMS
$(X, (R_i ; i \in I)) = \mathbf{A}$
OPERATIONS, ...

HOMOMORPHISM : MAPS PRESERVING ALL RELATIONS

$$(x_1, \dots, x_t) \in R(\mathbf{A}) \Rightarrow (f(x_1), \dots, f(x_t)) \in R(\mathbf{B})$$

$\boxed{\text{RECALL}}$: H SMALL

CSP(H) HUGE PROBLEM

$H = \Delta$ $\quad$ $CSP(\Delta) \equiv$ 3 COLORING

H =  $\quad$ HOM(G, ) = # INDEPENDENT SETS IN G

7 TERNARY EXPRESS ON $\{0,1\}$ $\quad$ 3-SAT

$$\{A \,;\; A \longrightarrow H\} = CSP(H)$$

IS HEREDITARY CLASS

$$\Downarrow$$

DUAL DESCRIPTION BY
FORBIDDEN SUBSTRUCTURES

$$FORB(\mathcal{F}) = \{A \,;\; F \in \mathcal{F} \Rightarrow F \nrightarrow A\}$$

$$\boxed{FORB(\mathcal{F}) = CSP(H)}$$

$\forall H \; \exists \mathcal{F} \; \ldots$

FOR EXAMPLE $\mathcal{F} = \{F \,;\; F \nrightarrow H\}$.

$\exists ?$ "SIMPLE" $\mathcal{F}$?

EXAMPES OF "SIMPLE" $\mathcal{F}$:

- $\mathcal{F}$ PATHS — HELL, ZHU
- $\mathcal{F}$ TREES — HELL, N., ZHU
- $\mathcal{F}$ k-TREES — HELL, N., ZHU; FEDER, VARDI

PATH DUALITY

TREE DUALITY

BOUNDED TREE WIDTH DUALITY

PATH WIDTH DUALITY

SYNTACTIC DESCRIPTIONS

ATSERIAS, DALMAU, KROKHIN, ...

EXTREMAL CASE

$\mathcal{F}$ A FINITE SET

$\boxed{\text{FINITARY DUALITY}}$

$$\mathrm{FORB}(\mathcal{F}) = \mathrm{CSP}(H)$$

$$\mathrm{FORB}(\mathcal{F}) = \mathrm{CSP}(\mathcal{D})$$

$$\Updownarrow$$

FOR EVERY STRUCTURE A HOLDS:

$\neg\mathrm{Forb}(\mathcal{F})$ EITHER THERE EXISTS $F \in \mathcal{F}$ SUCH THAT $F \longrightarrow A$

OR THERE EXISTS $D \in \mathcal{D}$ SUCH THAT $A \longrightarrow D$.

$\mathcal{D}$ $\boxed{\text{DUAL OF}}$ $\mathcal{F}$, $(\mathcal{F}, \mathcal{D})$ $\boxed{\text{DUAL PAIR}}$

FINITARY DUALITY

$\Downarrow$

CSP($\mathcal{D}$)

— NP ∩ co NP

— P

⟺ FIRST ORDER DEFINABLE (ATSERIAS; ROSSMAN)

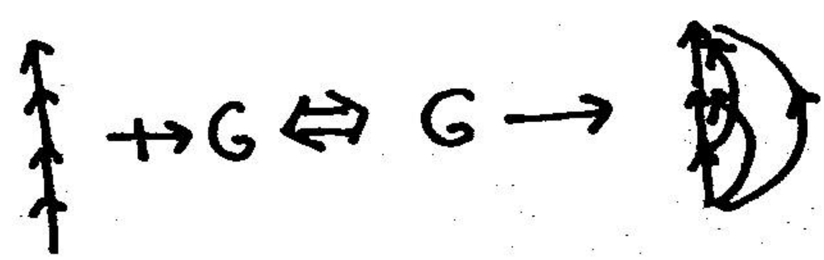
?TOO RESTRICTIVE?.

EXPRESSING POWER
(OF DUALITIES)

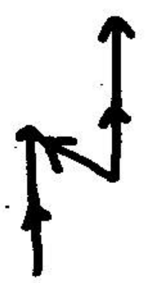
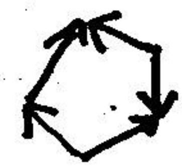
- LP, FARKAS LEMMA (HOCHSTATTLER, N.)
- MENGER THM.
- $\boxed{\text{NOT}}$ UNDIRECTED GRAPHS
- $\boxed{\text{NOT}}$ FINITE ALGEBRAS (KUN, N.)

FOR RELATIONS IMPORTANT EXAMPLES
FROM VERY START

$$\mathrm{FORB}(\vec{P}_n) = \mathrm{CSP}(\vec{T}_n)$$



(GALLAI, ROY; HASSE; VITKAVER)

 $\not\to G \Leftrightarrow G \longrightarrow$ 

(KOMÁREK)

CHARACTERIZATION

THM (N., TARDIF)

1. FOR EVERY TREE T THERE EXISTS (UNIQUE UP TO HOM-EQUIVALENCE) DUAL D_T :

$$T \not\to G \Leftrightarrow G \longrightarrow D_T$$

2. (UP TO HOM-EQUIVALENCE) THERE ARE NO OTHER (SINGLETON) DUAL PAIRS.

FOR ORIENTED GRAPHS

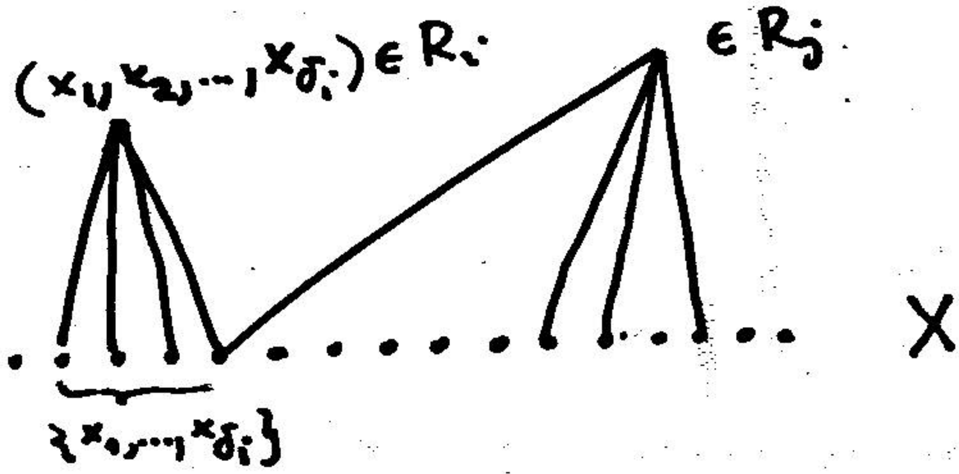
FOR RELATIONAL STRUCTURES

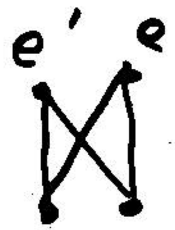
$\boxed{\text{DEFINITION OF TREE}}$

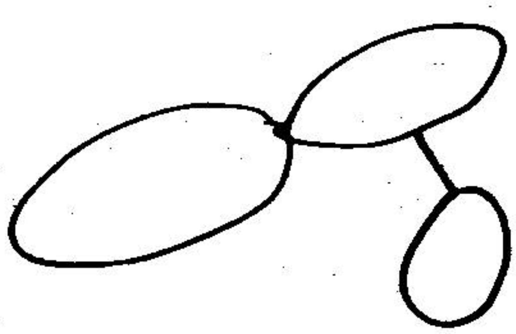
$A = (X, R_1, \ldots, R_t)$ IS A TREE

IFF

ITS INCIDENCE GRAPH IS A TREE







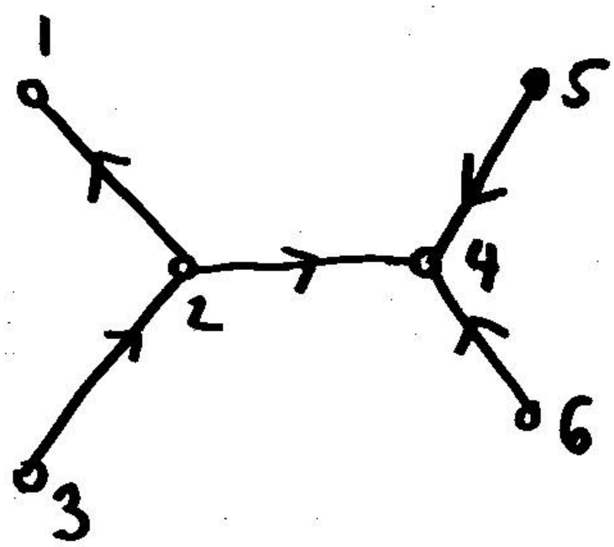
$x \times y$

FASCINATING DUALS

- EFFICIENT CONSTRUCTION
  EXPLICITE
- $|D_T| \leq 2^{|T| \cdot \log |T|}$
- $\exists T \quad |D_T| \geqslant 2^{|T|/\log |T|}$

  CORE
- DIAMETER $\leq |T|+3$
  (ŠVEJDAROVÁ, N.)
- GIVEN H
  IS H DUAL ?

  DECIDABLE
  NPC PROBLEM
  (LAROSE, LOTEN, TARDIF)



$T = (V, E)$

D_T

VERTICES :

$f : V \longrightarrow V \qquad f(v) \sim v$

FOR EXAMPLE: $\begin{array}{c|c|c|c|c|c} 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 2 & 4 & 2 & 6 & 4 & 4 \end{array}$ f

$(f, g) \in E(D_T)$

$(u, v) \in E \Rightarrow \big(f(u), g(v)\big) \neq (v, u)$

$(f, g) \in E(D_T)$

$\begin{array}{c|c|c|c|c|c} 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 2 & 1 & 2 & 6 & 4 & 4 \end{array}$ g

$\boxed{\text{COROLLARY}}$ (CURIOUS)

4 COLOUR THM

$\Updownarrow$

EVERY PLANAR GRAPH G HAS ORIENTATION $\vec{G}$ SUCH THAT $\vec{G}$ DOES NOT CONTAIN

T_1

&

T_2

$\chi(D_{T_2}) \leq 4$

(NOT 5 !!)

$\exists$ H SUCH THAT ALL CRITICAL GRAPHS G, $G \not\to H$ HAVE AT MOST $\sim \log |H|$ VERTICES

CHARACTERIZATION OF ALL FINITARY DUALITIES

$$(\mathcal{F}, D) \qquad \mathcal{F} = \{F_1, \dots, F_t\}$$

$$\Updownarrow$$

$$D \sim D_{F_1} \times \dots \times D_{F_t}$$

$\boxed{\text{THM}}$ (FONIOK, N., TARDIF)

$(\mathcal{F}, \mathcal{D})$ IS A DUALITY

$$\Updownarrow$$

$\mathcal{F}$ IS A FAMILY OF FORESTS

$\mathcal{D}$ IS A SET OF TRANSVERSALS OF $\mathcal{F}$

EXAMPLE: T_1, T_2, T_3, T_4 TREES

$\boxed{\mathrm{I}}$ $T_i \not\to T_j$ FOR $i \neq j$

$$\mathcal{F} = \{T_1 + T_2,\ T_1 + T_3,\ T_4\}$$

$$\mathcal{D} = \{D_1 \times D_4,\ D_2 \times D_3 \times D_4\}$$

$\boxed{\mathrm{II}}$ $T_1 \to T_3$, OTHERWISE $T_i \not\to T_j$

$$\mathcal{F} = \{T_1 + T_2,\ T_3 + T_4\}$$

$$\mathcal{D} = \{D_1,\ D_2 \times D_3,\ D_2 \times D_4\}.$$

$\boxed{\text{PART II}}$

DUALITY & HOMOMORPHISM ORDER

$$A \leq B \equiv A \longrightarrow B$$

$\mathcal{C}$ .... ALL CORE STRUCTURES
+
EXISTENCE OF HOMOMORPHISM

$\mathcal{C}$ LATTICE
+
HEYTING

$+, \times$

$$A \times B \to C \Leftrightarrow A \to C^{B}$$

SPECTACULAR PROPERTIES OF $\mathcal{C}$

– $\boxed{\text{UNIVERSAL}}$

(EVERY COUNTABLE POSET INCLUDED

$\Updownarrow$

EVERY FINITE POSET REPRESENTABLE ON LINE)

HUBIČKA, N.

ALL FINITE ORIENTED PATHS ARE (COUNTABLY) UNIVERSAL

– $\boxed{\text{DENSE}}$ (MOSTLY)

$$A < B \Rightarrow \exists C \quad A < C < B$$

$\boxed{\text{THM}}$ (WELZL)

$\mathcal{C}$ FOR UNDIRECTED GRAPHS IS DENSE WITH TRIVIAL EXCEPTION

SPARSE INCOMPARABILITY LEMMA

VERSION 1

$\forall \ell\ \forall K\ \forall B$ CORE $\exists B'$ GIRTH$(B') \geqslant \ell$

$$\begin{tikzcd} B \arrow[r] & C \\ B' \arrow[u] \arrow[ur, "\exists"'] & \end{tikzcd} \qquad \forall |C| \leq K$$

VERSION 2

$\forall \ell\ \forall K\ \forall B$ projective $\exists B'$ GIRTH$(B') \geqslant \ell$

$$\begin{tikzcd} B \arrow[r] & C \\ B' \arrow[u] \arrow[ur, "1\text{-}1"'] & \end{tikzcd} \qquad \forall |C| \leq K$$

SIL VERSION 1 $\Rightarrow$ DENSITY

$A < B$ $\qquad B'$ SIL

$C = A + B'$

PROOF OF SIL

(ERDÖS) BOLLOBAS, SAUER
N. RÖDL (RIGID ~ PROJECTIVE)

N.
MÜLLER ; FEDER, VARDI

N. ZHU (VERSION 2)
MATOUŠEK, N. (GRAPHS)
KUN (SET SYSTEMS)

(NOT TRUE FOR INFINITE !)

A COMBINATORIAL CLASSIQUE

$\boxed{\text{GAP}}$ (A,B) A < B

AND

THERE IS NO C WITH

A < C < B.

$\boxed{\text{GAP THM}}$ (N.; TARDIF)

FOR ANY SIGNATURE Δ

GAPS OF $\mathcal{C}_\Delta$ CHARACTERIZED.

$\boxed{\text{THM}}$ $(N_\cdot, T_\cdot)$

THERE IS ONE-TO-ONE CORRESPONDENCE BETWEEN CONNECTED GAPS AND SINGLETON DUALITIES.

PROOF: $A < B$ CONNECTED GAP IF B CONNECTED

$$\Downarrow$$

(B, A^B) DUALITY

(F, D) DUALITY

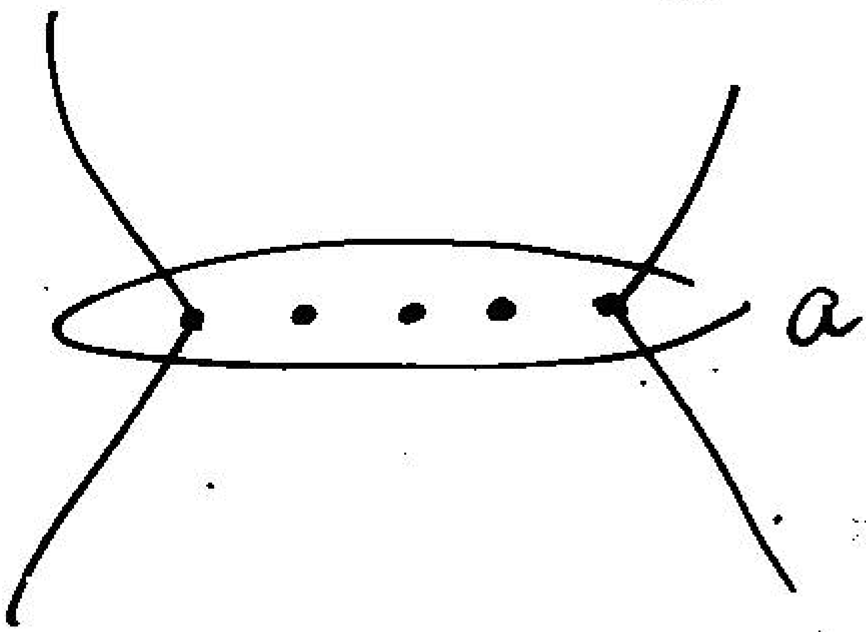
$$\Downarrow$$

$(F \times D, F)$ GAP

CONNECTIVITY OF $\mathcal{C}$

A FINITE CUT $\equiv$ MAXIMAL ANTICHAIN a

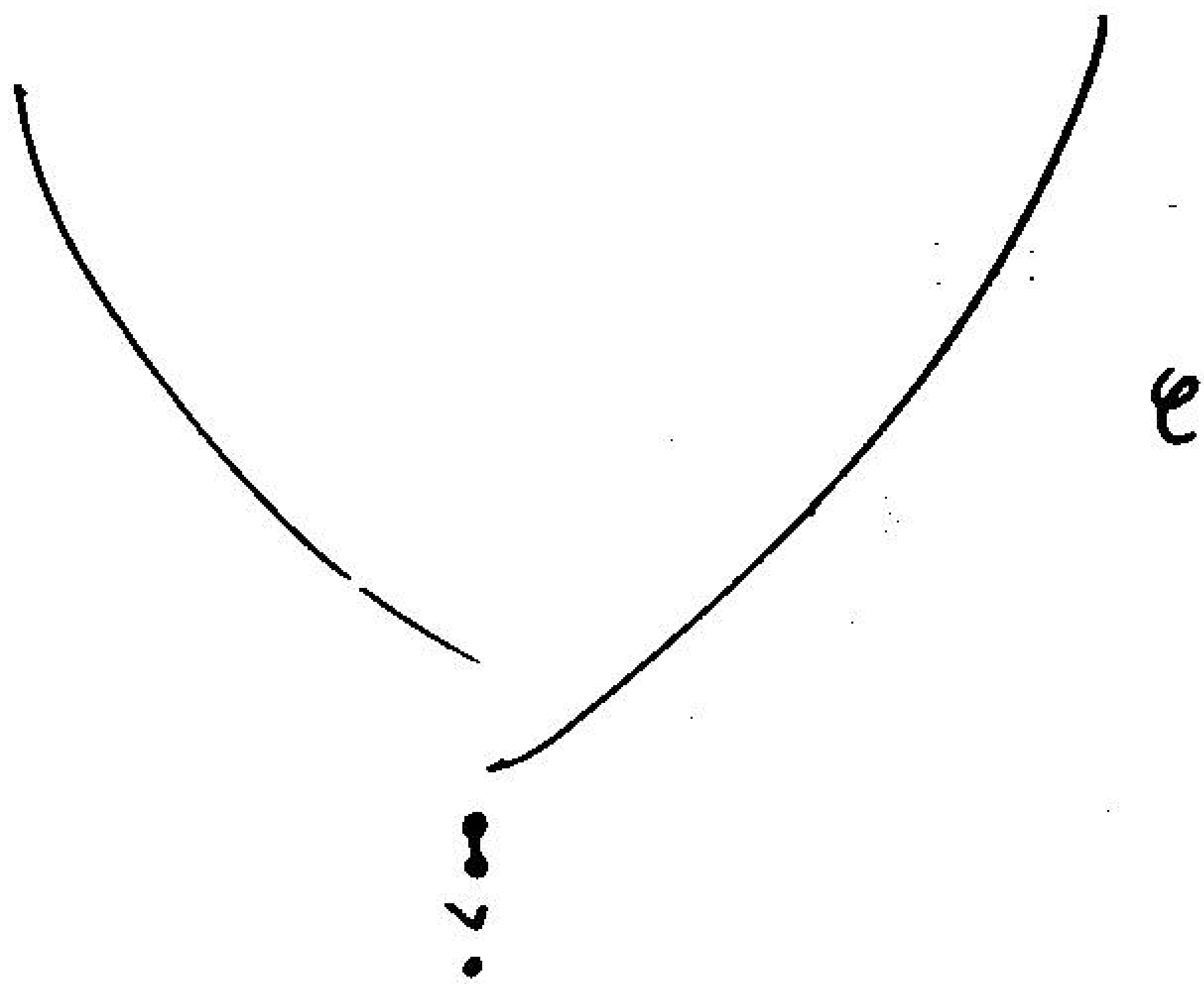
(EVERY $c \in \mathcal{C}$ EITHER ABOVE OR UNDER a)



UNDIRECTED GRAPHS

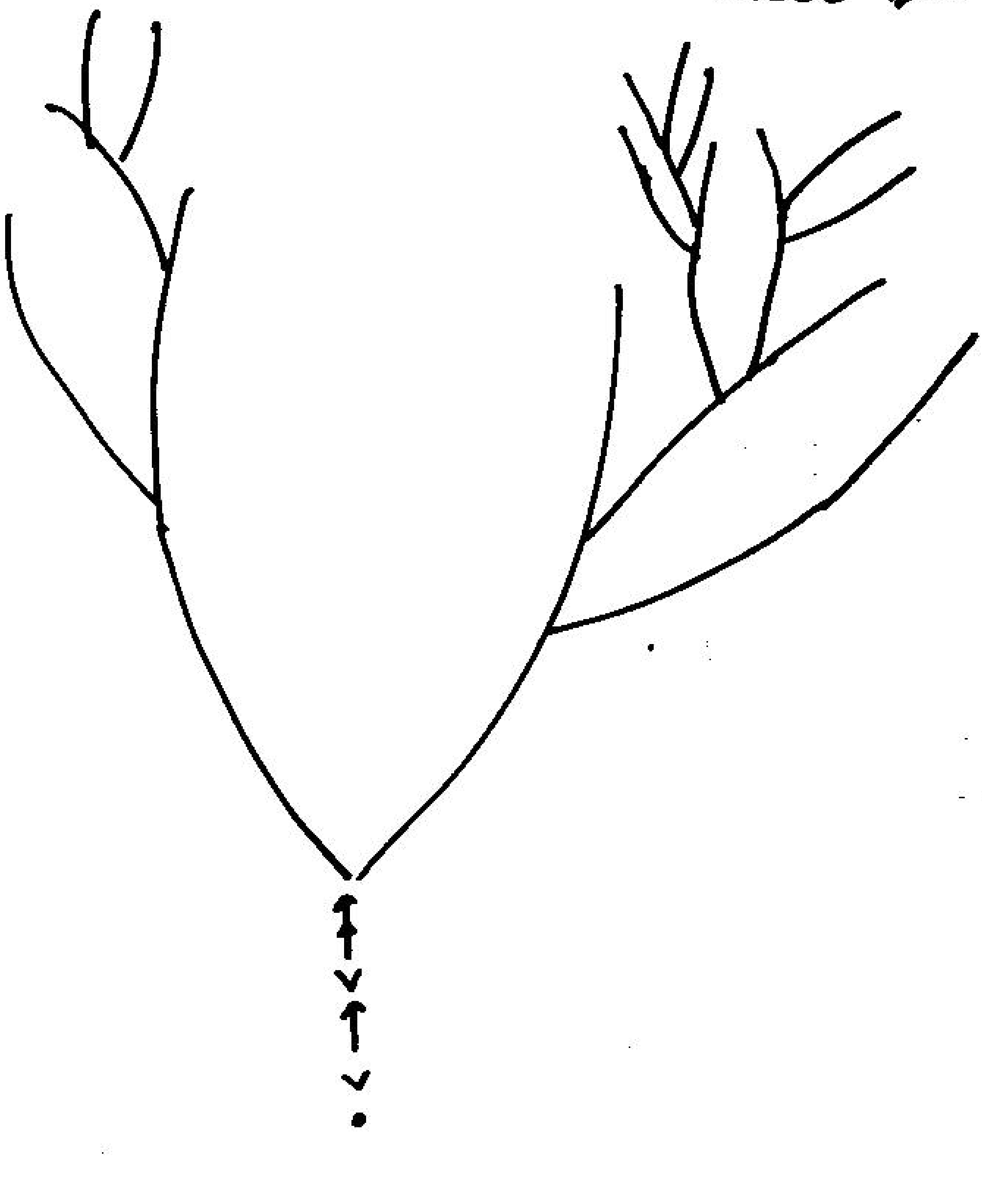
NO FINITE MAC's

(∞ CONNECTIVITY)



FOR RELATIONS (ALREADY ORIENTED GRAPHS)

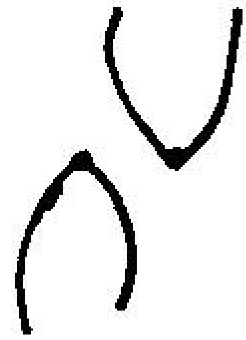
INFINITELY MANY MAC's OF ALL SIZES ≥ 2



$\boxed{\text{EXAMPLE}}$

(T, D) DUALITY $\Rightarrow$ {T, D} MAC

"SPLITTED MAC"
(ERDÖS, SOUKUP, ...)



MAC's FOR COUNTABLE GRAPHS ?

$\boxed{\text{CONJECTURE}}$ (N., SHELAH)

EVERY FINITE MAC OF COUNTABLE GRAPHS CONTAINS A FINITE GRAPH.

(NO MAC OF SIZE 1!)

$\boxed{\text{THM}}$ (N., T., FONIOK)

ALL MAC's IN $\mathcal{C}$
ARE FINITARY DUALITIES

ONLY FOR SIGNATURE $\Delta = (R)$
(ONE k-NARY RELATION)

DUALITY $\leftrightsquigarrow$ DENSITY

MAX. ANTICHAINS

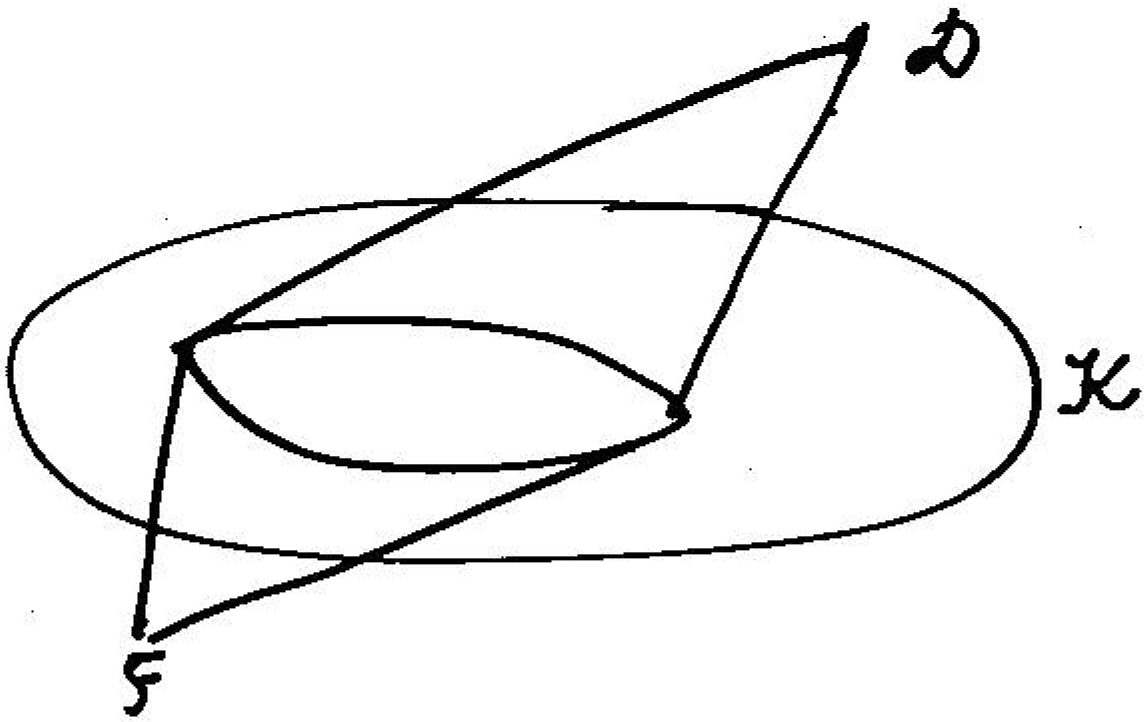
III. RESTRICTED DUALITIES

$\mathcal{K}$ A CLASS OF STRUCTURES

$(\mathcal{F}, \mathcal{D})$ DUALITY FOR $\mathcal{K}$
RESTRICTED DUALITY

FOR EVERY $A \in \mathcal{K}$
EITHER $F \longrightarrow A$ FOR SOME $F \in \mathcal{F}$
OR $A \longrightarrow D$ FOR SOME $D \in \mathcal{D}$

$$\mathrm{FORB}(\mathcal{F}) \cap \mathcal{K} = \mathrm{CSP}(\mathcal{D}) \cap \mathcal{K}$$



$\boxed{\text{EXAMPLE}}$

$\mathcal{K}$ ~~AK~~ PLANAR GRAPHS

$\chi(G) \leq 3$ G PLANAR (GRÖTZSCH)
$\Delta \nsubseteq G$

$FORB(\Delta) \cap \mathcal{K} \subseteq CSP(\Delta)$

$FORB(\Delta) \cap \mathcal{K} = CSP(H_{16}) \cap \mathcal{K}$

H_{16} CLEBSCH GRAPH

(GUENIN, NASSERASR)

$\boxed{\text{EXAMPLE}}$

$\mathcal{K} = \{ G ;\ \text{MAX DEGREE} \leq d \}$

$\boxed{\text{THM}}$ (HÄGGKVIST, HELL)

$F_1, \dots, F_t$ CONNECTED GRAPHS

THEN THERE EXISTS $D_{F_1,\dots,F_t} = D$

SUCH THAT

$(\{F_1, \dots, F_t\}, D)$ IS RESTRICTED DUALITY FOR $\mathcal{K}$.

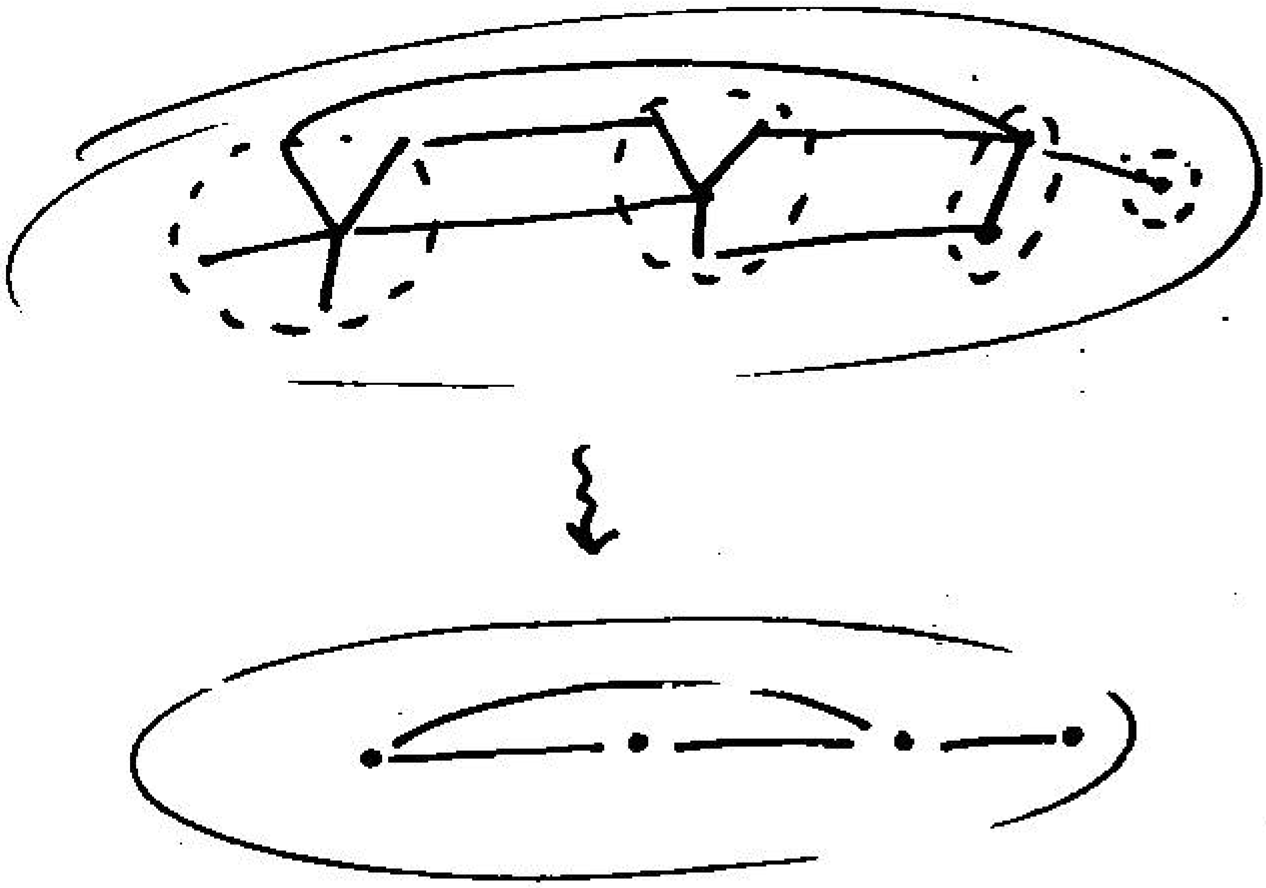
NOT TRUE FOR $\underbrace{\text{DEGENERATED}}$ CLASSES OF GRAPHS

$$\mathrm{MAD}(G) = \max \left\{ \frac{|E'|}{|V'|} ;\ (V', E') \subseteq G \right\}$$

$$\nabla_0(G) = \mathrm{MAD}(G)$$

$$\nabla_1(G) = \max\left\{ \frac{|E'|}{|V'|} \; ; \; (V', E') \right\}$$

STAR FOREST CONTRACTION



$$\nabla_k(G) = \max\left\{ \frac{|E'|}{|V'|} \;;\; \underset{\substack{\text{CONTRACTION} \\ \text{OF BALLS} \\ \text{OF RADIUS} \leq k}}{(V', E')} \right\}$$

$$\nabla_0(G) = \mathrm{MAD}(G) \leq \nabla_1 \leq \nabla_2 \leq \cdots$$

$\mathcal{K}$ HAS $\boxed{\text{BOUNDED EXPANSION}}$

IF $\forall k \ \max\limits_{G \in \mathcal{K}} \nabla_k(G) < \infty$

EXAMPLE

- BOUNDED DEGREE d — $\nabla_k \leq d^k$
- MINOR CLOSED (NOT ALL GRAPHS) — $\nabla_k \leq$ CONST.
- MESHES GRAPHS d-DIMENSIONAL — $\nabla_k \leq k^d$
- TOPOLOGICAL SUBGRAPHS — $\nabla_k \leq 2^{2^k}$

⋮

$\boxed{\text{THM}}$ (P. OSSONA DE MENDEZ, N.)

EVERY CLASS $\mathcal{K}$ WITH BOUNDED EXPANSION HAS $\boxed{\text{ALL}}$ FINITARY DUALITIES.

EXPLICITELY:

FOR EVERY CHOICE OF CONNECTED $F_1, \dots, F_t$ THERE EXISTS D SUCH THAT FOR EVERY $G \in \mathcal{K}$ HOLDS:

$$\exists i \; F_i \longrightarrow G \quad \text{IFF} \quad G \not\longrightarrow D.$$

RICH SUPPLY OF FIRST ORDER DEFINABLE RESTRICTED CSP.