Near-Unanimity Functions and CSP

A short survey

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Abstract

A near unanimity function, or nuf, is an n-variable operation that satisfies the identities

$$f(x,\ldots,x,y) = f(x,\ldots,x,y,x) = \cdots f(y,x,\ldots,x) = x.$$

Relational structures invariant under such operations satisfy remarkable properties. We present a survey of various results in connection with the complexity of Constraint Satisfaction Problems.

Outline

- Preliminaries
- Tractability results: bounded width
- Complexity
- Decidability issues
- Restrictions on structures
- 1-tolerant nuf's: FO-definability
- Generalisations of nuf's
- Problems

Preliminaries

- CSP’s
- Algebras
- A conjecture
- Nuf’s
- Two fundamental properties of nuf’s

CSP's

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Let $H = \langle A; \theta_i (i \in I) \rangle$ be a fixed relational structure:

for each $i \in I$ (the *signature of* H),
θ_i is a relation on A of arity r_i (a subset of A^{r_i})

$\Gamma = \{\theta_i : i \in I\}$ is the set of *constraint relations*.

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a *homomorphism* is a map $f : X \to A$ such that $f(\rho_i) \subseteq \theta_i$ for all $i \in I$.

We write $G \to H$ ($G \not\to H$) when G admits (does not admit) a homomorphism to H.

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$$\underbrace{\begin{bmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \cdots & \vdots \\ a_{r,1} & \cdots & a_{r,n} \end{bmatrix}}_{\text{columns in } \theta} \overset{f}{\longmapsto} \underbrace{\begin{bmatrix} b_1 \\ \vdots \\ b_r \end{bmatrix}}_{\theta}$$

Applying f to the rows of the matrix whose columns are in θ yields a tuple of θ.

Algebras, cont'd

f preserves $\theta \equiv \theta$ is *invariant under* f

A structure H is invariant under an operation f or *admits* f if all its basic relations are invariant under f.

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A structure H is invariant under an operation f or *admits* f if all its basic relations are invariant under f.

Example: the structure $\langle\{0,1\};\theta\rangle$ with $\theta = \{(0,1),(1,0),(1,1)\}$ is

- invariant under $f(x,y) = x \vee y$
- not invariant under $g(x,y) = x \wedge y$

Algebras, cont'd

Given a set Γ of relations on A, construct:

$F =$ all operations on A preserving all relations in Γ.

The algebra associated to $CSP(\Gamma)$:

$$\mathbb{A} = \langle A; F \rangle$$

Algebras, cont'd

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The algebra associated to $CSP(\Gamma)$:

$$\mathbb{A} = \langle A; F \rangle$$

Constraint relations $\subseteq$ subalgebras of powers of $\mathbb{A}$.

The operations one can build from F (and projections) via composition are called the *terms* of $\mathbb{A}$.

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(FV, BJK)

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i.e. target structure admits “nice” terms

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$\mathbb{A}$ admits terms obeying “nice” identities

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more formally, in what follows, we’ll refer to such algebras as *nice enough*.

Nuf's

Let $n \geq 3$.

A near unanimity function, or nuf, is an n-variable operation that satisfies the identities

$$f(x,\ldots,x,y) = f(x,\ldots,x,y,x) = \cdots f(y,x,\ldots,x) = x.$$

Notice: an nuf is *idempotent*, i.e. satisfies

$$f(x,x,\ldots,x) = x.$$

Nuf's, cont'd

Concept first appears:

Huhn, 1972; Baker & Pixley 1975.

Case $n = 3$ widely studied: majority operations

- 50's and 60's: majority decision: voting paradoxes, etc.
- Wille 1970: majority terms of lattices, congruences
- Bandelt et al.: absolute retracts
- Pouzet et al.: Metric point of view
- Etc.

Elementary facts about nuf's

- Prototypical example: the *median*

$$m(x,y,z) = \max(\min(x,y),\min(x,z),\min(y,z))$$
$$= (x \wedge y) \vee (x \wedge z) \vee (y \wedge z)$$

- If a structure H admits an nuf with n variables, it admits one with $n+1$ variables (just add a fictitious variable)

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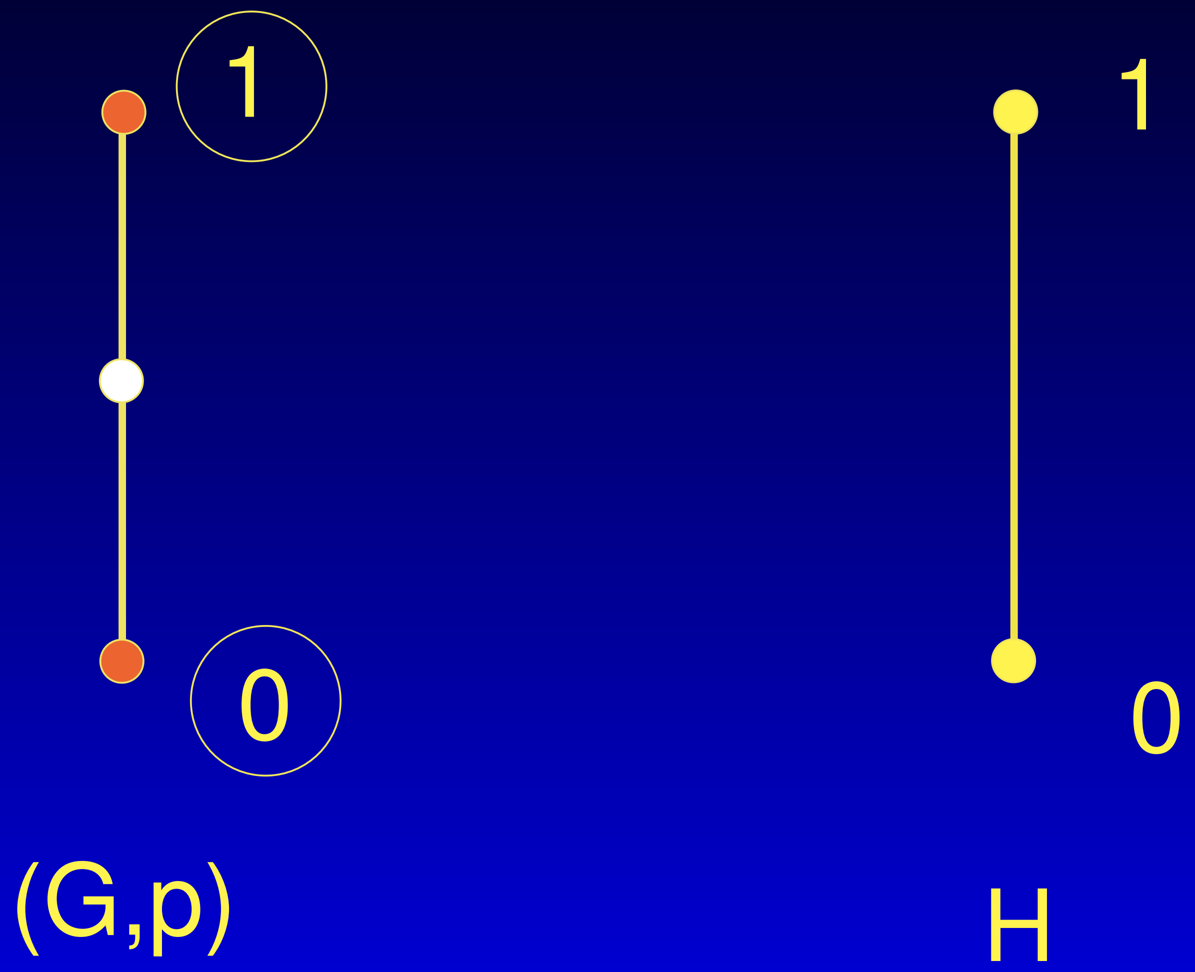
- G is a structure of the same type as H;
- p is a partial map from G to H;
- p does not extend to a full homomorphism from G to H and
- the pair (G, p) is minimal with the above properties.

A zig-zag



Nuf's and zig-zags

Theorem [Zádori 1997; G. Tardos 1986]

Let H be a structure.
Then the following are equivalent:

1. *H admits an nuf of arity $n+1$;*
2. *every zig-zag (G, p) for H satisfies $|dom\, p| \leq n$.*

Nuf's and zig-zags, cont'd

Sketch of proof.
$(1) \Rightarrow (2)$: let f be an nuf of arity $n+1$.

- let (G, p) be a zig-zag; suppose for a contradiction $dom\, p = \{x_1, \ldots, x_{n+1}\}$;

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- let (G, p) be a zig-zag; suppose for a contradiction $dom\, p = \{x_1, \ldots, x_{n+1}\}$;
- let p_i be the restriction of p to $dom\, p \setminus \{x_i\}$: it extends to some $\widehat{p_i}$;

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- let p_i be the restriction of p to $dom\, p \setminus \{x_i\}$: it extends to some $\widehat{p_i}$;
- then $f(\widehat{p_1}, \cdots, \widehat{p_{n+1}})$ is an extension of p:

$$
\begin{aligned}
& f(\widehat{p_1}, \cdots, \widehat{p_{n+1}})(x_i) \\
=& f(p(x_i), \ldots, p(x_i), *, p(x_i), \ldots, p(x_i)) \\
=& p(x_i).
\end{aligned}
$$

Nuf's and zig-zags, cont'd

Sketch of proof, cont'd

$(2) \Rightarrow (1)$:

- let $G = A^{n+1}$ and define the partial map

$$p(x, \ldots, x, y, x \ldots, x) = x$$

for all tuples of this form;

Nuf's and zig-zags, cont'd

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  for all tuples of this form;
- the restriction of p to any set with at most n tuples is extendible (appropriate projection)
- (G, p) non-extendible $\Rightarrow$ contains a zig-zag;
- hence p is extendible; H admits $n+1$-ary nuf.

n-decomposability

Theorem [Baker & Pixley 1975]

Let $\mathbb{A}$ be a finite algebra.
Then the following are equivalent:

1. *$\mathbb{A}$ has an nuf term of arity $n+1$;*
2. *every subalgebra of a power of $\mathbb{A}$ is determined by its projections onto n indices.*

i.e. if Γ admits an nuf of arity $n+1$, then constraints in Γ are determined by their projections on n factors.

Tractability

Theorem [Feder & Vardi 1993, 1998]
If Γ *is invariant under an nuf of arity* $n+1$ *then* $CSP(\Gamma)$ *has width* n*. In particular,* $CSP(\Gamma)$ *is tractable.*

In fact, $CSP(\Gamma)$ has *bounded strict width*:
i.e. once local consistency has been achieved, partial solutions may be extended greedily to global solutions.

(strong $n+1$-consistency $\Rightarrow$ global consistency, see Jeavons, Cohen & Cooper 1998)

Tractability, cont'd

Furthermore:

Theorem [FV 1993, 1998; JCC 1998]

Let Γ be a set of constraint relations.
Then the following are equivalent:

1. *Γ is invariant under an nuf of arity $n+1$;*
2. *$CSP(\Gamma)$ has strict width n.*

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- both problems have strict width 2.
- $HORNSAT$ does not have strict width n $(\forall n)$:

$$(1,1,\ldots,1) \notin \{\overline{X_1} \vee \cdots \vee \overline{X}_{n+1}\}$$

but contains $(1,1,\ldots,1,0,1,\ldots,1)$ for any placement of the 0.

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Can we do better ?

Example: if $|A| = 2$, and Γ is invariant under a majority operation (i.e. the median), then $CSP(\Gamma)$ is a subproblem of $2 - SAT$, hence

$$CSP(\Gamma) \in \mathbf{NL}.$$

Majority implies NL

Theorem [Dalmau & Krokhin, 2006]

Let H be a structure admitting a majority operation. Then $CSP(H)$ is in **NL**.

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Let H *be a structure admitting a majority operation.*
Then $CSP(H)$ *is in* **NL**.

In fact, they show more:
H has bounded path width duality.

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H is said to have *finite (tree)* duality if it has a complete set of obstructions that is finite (consists of trees).

H has *bounded path width duality* if it has a complete set of obstructions that consists of structures with ... bounded path width.

Related results, cont'd

Dalmau has shown that on 2 elements, problems invariant under an nuf (of any arity) have bounded path duality, and hence are in **NL**.

Related results, cont'd

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This prompts:

Question. If H admits an nuf, is $CSP(H)$ in **NL** ?

(We'll see later that a slight strengthening of the notion of nuf guarantees this)

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For algebras, however:

Theorem [Maróti, 2005] *Let* $\mathbb{A} = \langle A; F \rangle$ *be an algebra with* A *and* F *finite. Then it is decidable to determine if* $\mathbb{A}$ *has an nuf term operation.*

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(Makes use of Lovász's proof of Kneser's conjecture !!)

Partial results

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Partial results

- For *fixed* arity: poly-time algorithm (Feder & Vardi 1993)
- H a poset: poly-time decidable (Kun & Szabó 2001)
- H a symmetric, reflexive graph: poly-time decidable (L., Loten & Zádori 2005)

The last two algorithms consist of a “dismantling” of the structure H^2 to its diagonal.

Restrictions on structures

- list-homomorphism problems on graphs
- retraction problems on posets and graphs

List-homomorphism problems

Let $H = \langle A; \Gamma \rangle$ be a fixed structure. Consider the following decision problem:

- Input: A structure G and $\forall g$ a list $L_g \subseteq A$;
- Question: Is there a homomorphism $f : G \longrightarrow H$ such that $f(g) \in L_g \, \forall g$?

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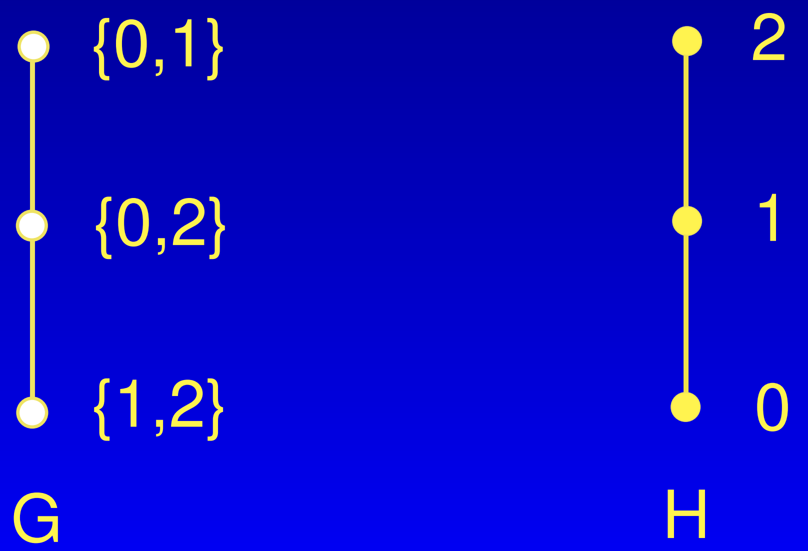


List-homomorphisms, cont'd

List-hom problems translate into CSP as follows:

add to constraint relations of H all unary constraint relations, i.e. create

$$H^* = \langle A; \Gamma \cup \{L : L \subseteq A\} \rangle.$$

The list-homomorphism problem for H is $CSP(H^*)$.

CSP's of this form are also known as *conservative*: an operation f preserves every subset of A iff it is conservative, i.e.

$$f(x_1, \ldots, x_n) \in \{x_1, \ldots, x_n\}.$$

List-homomorphisms, cont'd

Theorem [Bulatov 2003] *If the algebra associated to a conservative CSP is "nice enough" then the CSP is tractable; otherwise it is* **NP**-*complete.*

The dividing line in this dichotomy result has been investigated in detail for list-homomorphism problems on various types of graphs, e.g. symmetric graphs:

List-hom for symmetric graphs

Theorem [Feder, Hell & Huang 2003] *Let H be a graph. Then the following are equivalent:*

1. *H is a bi-arc graph;*
2. *H admits a conservative majority operation.*

If this holds then the list homomorphism problem for H is tractable, otherwise it is **NP***-complete.*

Conservative nuf's on graphs

Theorem [Brewster, F+H+H & MacGillivray 2003]
Let H be a graph. Then the following are equivalent:

1. *H is a bi-arc graph;*
2. *H admits a conservative nuf.*

FHH $+\epsilon$:

Fact [Kun, L. & Memartoluie 2004]
Let H be a graph. Then the following are equivalent:

1. *H is a bi-arc graph;*
2. *the algebra associated to H^* is "nice enough".*

(predicted by Bulatov's result under $\mathbf{P} \neq \mathbf{NP}$)

Retraction problems

If a structure H is reflexive (i.e. has loops), then the problem $CSP(H)$ is trivial: there is always a homomorphism, namely, a constant map.

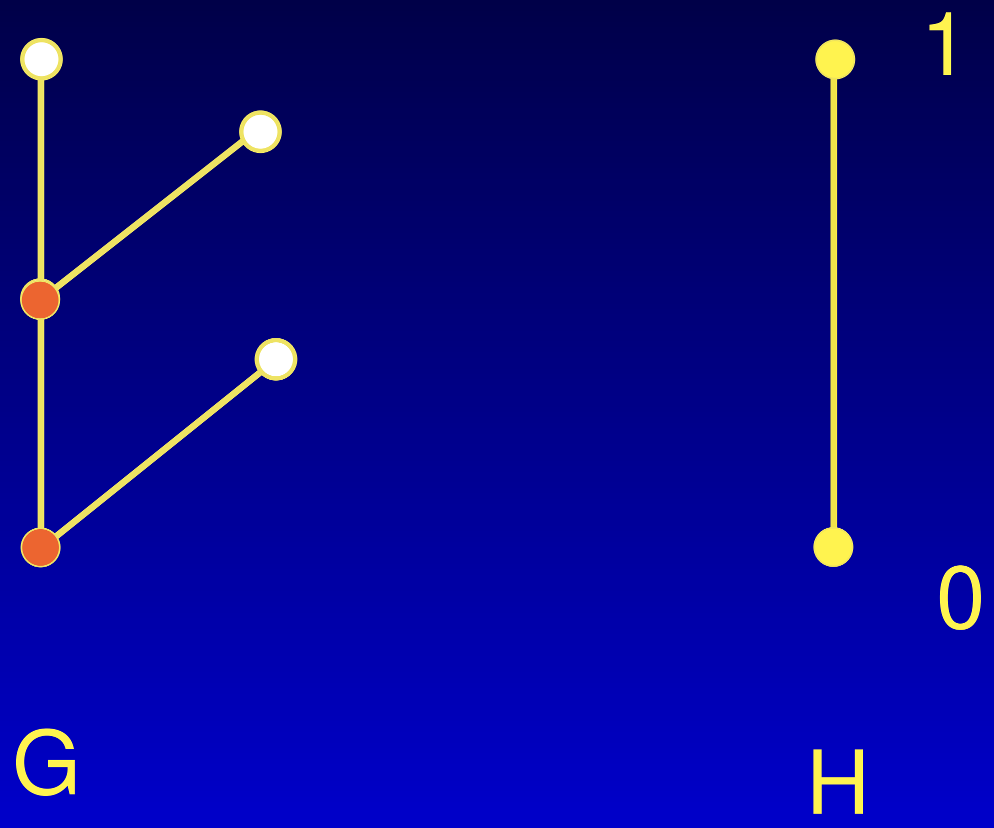
A natural decision problem for reflexive structures is the *retraction problem*:

- Input: A structure G containing a copy of H;
- Question: Is there a retraction $f : G \longrightarrow H$?

i.e. a homomorphism $f : G \longrightarrow H$ such that $f(x) = x$ for all $x \in H$.

Retraction problems, cont'd

Example:



Retraction problems, cont'd

Retraction problems translate into CSP as follows:

add to constraint relations of H all unary constraint relations *of size 1*, i.e. create

$$H' = \langle A; \Gamma \cup \{\{a\} : a \in A\}\rangle.$$

The retraction problem for H is $CSP(H')$.

Remark We're cheating a bit: this is better called the *one or all list homomorphism problem.*

FO-definability

Let $\mathcal{K}$ be a class of structures of the same type as H.

$\mathcal{K}$ is *first order definable (FO definable)* if there exists a first order sentence Φ in the language of H such that $G \in \mathcal{K}$ iff G satisfies Φ.

We say that $CSP(H)$ is FO-definable if

$$\mathcal{K} = \{G : G \text{ admits a homomorphism to } H\}$$

is FO-definable.

Retraction problems on graphs

Theorem [Dalmau, Krokhin & L. 2004]
Let H be a symmetric, reflexive graph. Then the following are equivalent:

1. *the retraction problem for H is FO-definable;*
2. *H is connected and admits an nuf.*

Retraction problems on graphs

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1. *the retraction problem for H is FO-definable;*
2. *H is connected and admits an nuf.*

There is an analog for posets but *inputs must be restricted to posets*:

i.e. existence of an nuf does not guarantee FO-definability (e.g. 2-COL) ...

1-tolerant nuf's

FO-definability of general CSP's can be captured by *1-tolerant nuf's*.

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Let $H = \langle A; \Gamma \rangle$ be a structure. A map $f : A^n \to A$ is a *1-tolerant homomorphism* if it satisfies the following condition: for every $\theta \in \Gamma$,

1-tolerant nuf's

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Let $H = \langle A; \Gamma \rangle$ be a structure. A map $f : A^n \to A$ is a *1-tolerant homomorphism* if it satisfies the following condition: for every $\theta \in \Gamma$,

$$\underbrace{\begin{bmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ldots & \vdots \\ a_{r,1} & \cdots & a_{r,n} \end{bmatrix}}_{n-1 \text{ columns in } \theta} \overset{f}{\longmapsto} \underbrace{\begin{bmatrix} b_1 \\ \vdots \\ b_r \end{bmatrix}}_{\theta}$$

Applying f to the rows of a matrix with *at least* $n-1$ columns in θ yields a tuple of θ.

1-tolerant powers

For convenience, consider the *n-th 1-tolerant power* of H:

- ${}^1H^n$ is a structure similar to H;
- universe is A^n;
- n-tuples are θ-related as in the diagram:

$$\underbrace{\begin{bmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \cdots & \vdots \\ a_{r,1} & \cdots & a_{r,n} \end{bmatrix}}_{n-1 \text{ columns in } \theta}$$

Critical obstructions

An obstruction G for H is *critical* if it is minimal with the property $G \not\to H$, i.e. for every proper substructure $G' \subseteq G$ we have $G' \to H$.

For convenience: the tuples in relations of structures we call *hyperedges*.

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Lemma

Let H be a structure. Tfae:

1. *The critical obstructions of H have at most n hyperedges;*
2. *H admits an $n+1$-ary 1-tolerant operation.*

Proof of the lemma

$(1) \Rightarrow (2)$:

- H admits no $n+1$-ary 1-tolerant operation $\Leftrightarrow$ there is no homomorphism from the 1-tolerant power ${}^{1}H^{n+1}$ to H;

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- hence there is a critical obstruction C such that $c : C \longrightarrow {}^1H^{n+1}$;

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- $\forall\, 1 \leq i \leq n+1\ \exists$ hyperedge e_k of C which is not respected by $\pi_k \circ c$;
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- 1-tolerance implies e_k is respected by $\pi_j \circ c$ for $j \neq k$;
- C has at least $n+1$ hyperedges.

Proof of the lemma, cont'd

$(2) \Rightarrow (1)$:

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- hence ${}^1H^{n+1} \not\to H$.

Core structures

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A structure is *rigid* if its only automorphism is the identity.

Lemma *If H is a core with tree duality then it is rigid.*

Lemma [Nešetřil & Tardif 2000]
If H has finite duality then it has tree duality.

FO-definable cores

Theorem [L., Loten & Tardif, 2006]
Let H be a core structure.
The following are equivalent:

1. $CSP(H)$ *is FO-definable;*
2. H *admits a 1-tolerant nuf.*

Furthermore, there is a polynomial-time algorithm to recognise these structures.

Sketch of Proof

1. $CSP(H)$ is FO-definable;
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- By Atserias (or Rossman) 2005, we have that *H has finite duality iff $CSP(H)$ is FO-definable.*
- Hence by the lemma it suffices to prove that, if H is a core with finite duality then *every 1-tolerant homomorphism is actually an nuf;*
- follows because H is a core and has tree duality, and hence is rigid.

(use $\phi_{i,y} : H \longrightarrow {}^1H^n$, $x \mapsto (x, \ldots, x, y, x \ldots, x)$.)

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We briefly describe (without proof) the algorithm to recognise core structures H with FO-definable CSP.

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Example. Suppose θ is 4-ary and that b dominates a. If $(a, b, c, a) \in \theta$ then

$$(b, b, c, a),\ (a, b, c, b),\ (b, b, c, b) \in \theta.$$

An algorithm, cont'd

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Dismantling algorithm

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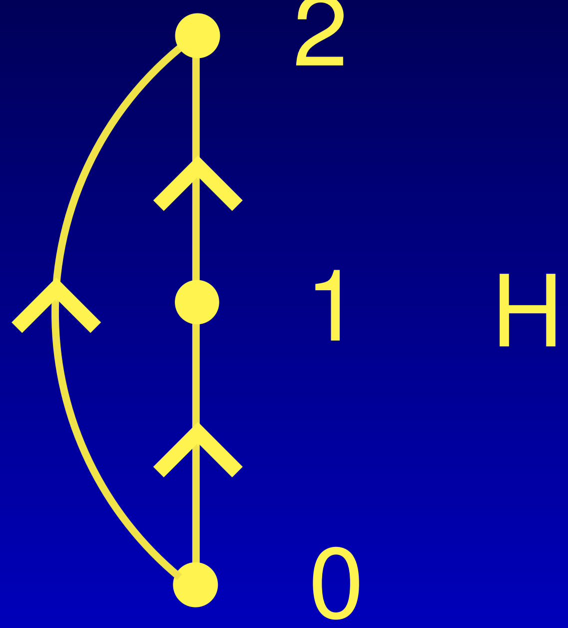
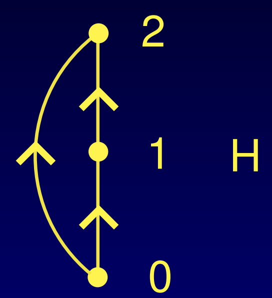
If the resulting structure is Δ, then $CSP(H)$ is FO-definable; otherwise it is not.

Example 1.

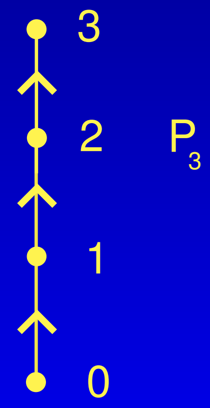
Let $H = \langle \{0,1,2\}, \{(0,1),(1,2),(0,2)\} \rangle$, the irreflexive transitive tournament on 3 vertices.



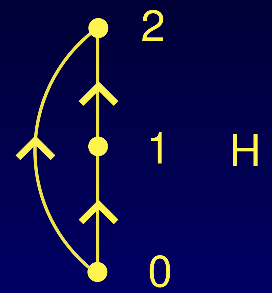
Example 1, cont'd



$CSP(H)$ is FO-definable: a complete set of obstructions is given by $\{P_3\}$ where P_3 is the path of length 3:



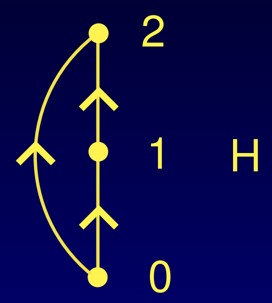
Example 1, cont'd



Thus a structure $G \not\to H$ iff it satisfies

$$\exists\, a \exists\, b \exists\, c \exists\, d\, [(a,b) \in \theta \wedge (b,c) \in \theta \wedge (c,d) \in \theta].$$

Example 1, cont'd



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H admits a 4-ary 1-tolerant nuf, but no 3-ary;

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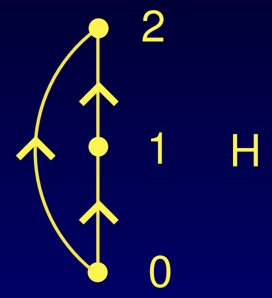

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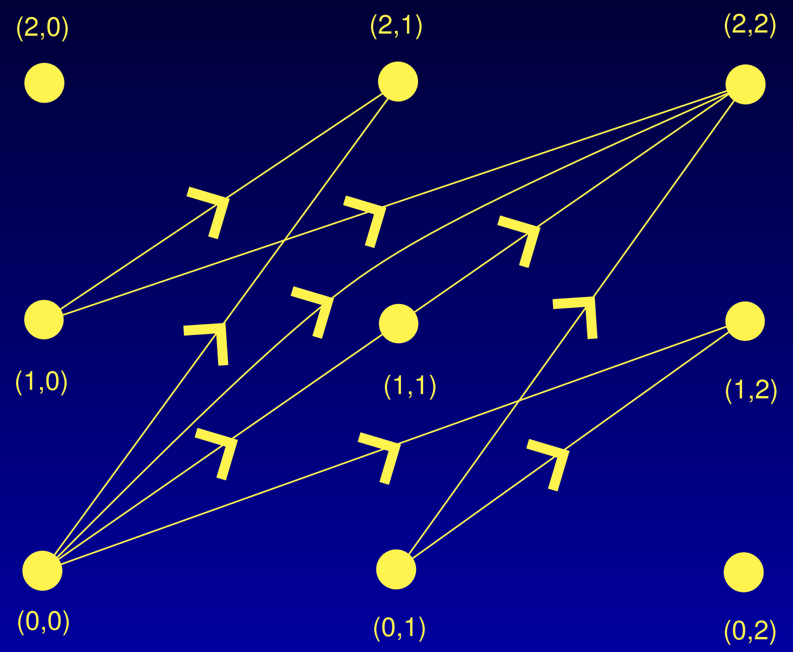
H admits a 4-ary 1-tolerant nuf, but no 3-ary;

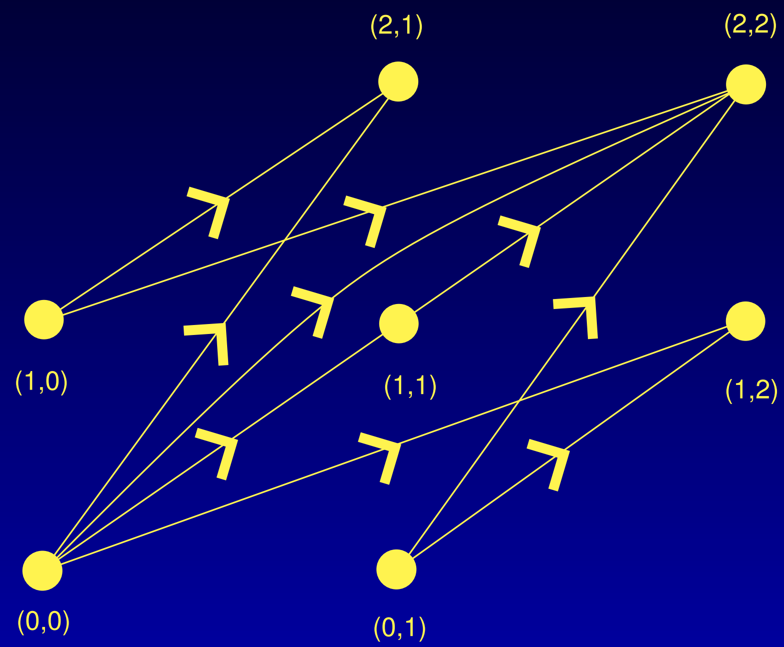
But it *does* admit a majority operation.
($m(x,y,z) = 1$ if $\{x,y,z\} = \{0,1,2\}$ and maj. else.)

Example 1, cont'd



Dismantling H^2 to the diagonal.

Example 1, cont'd



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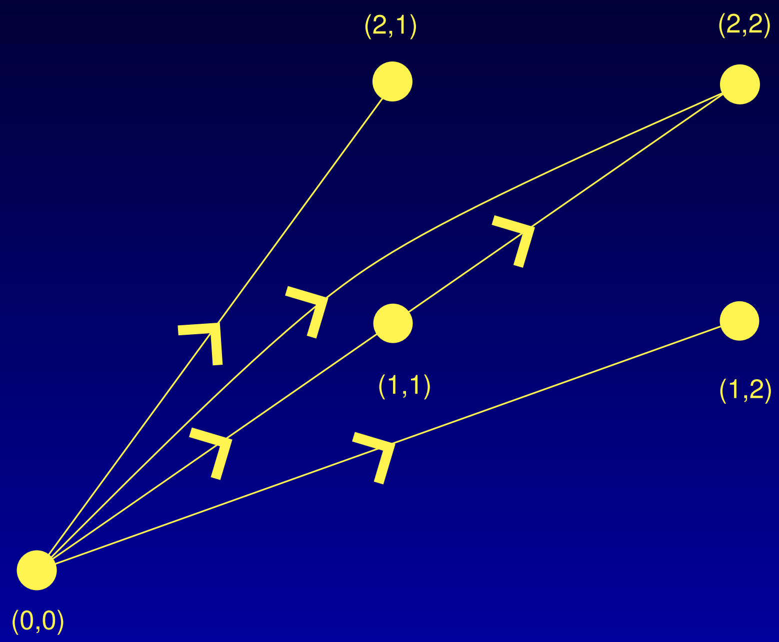

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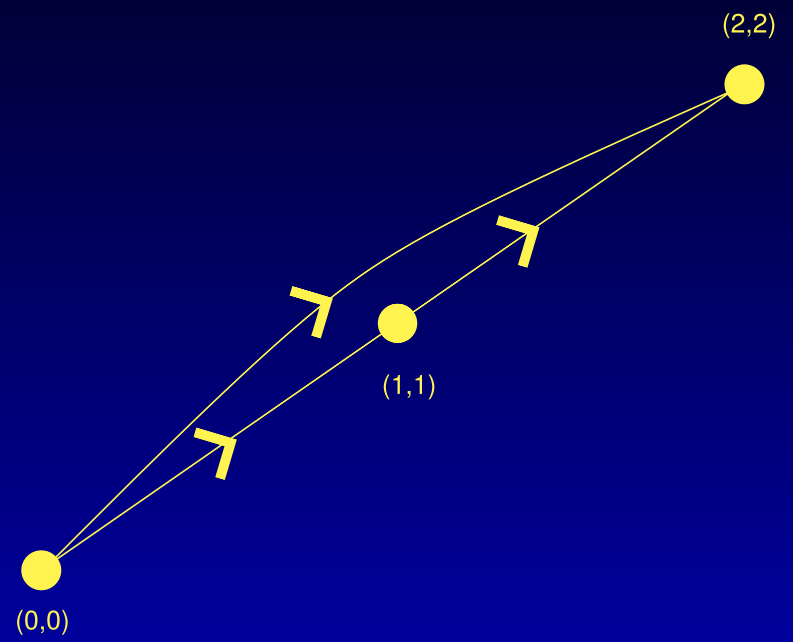

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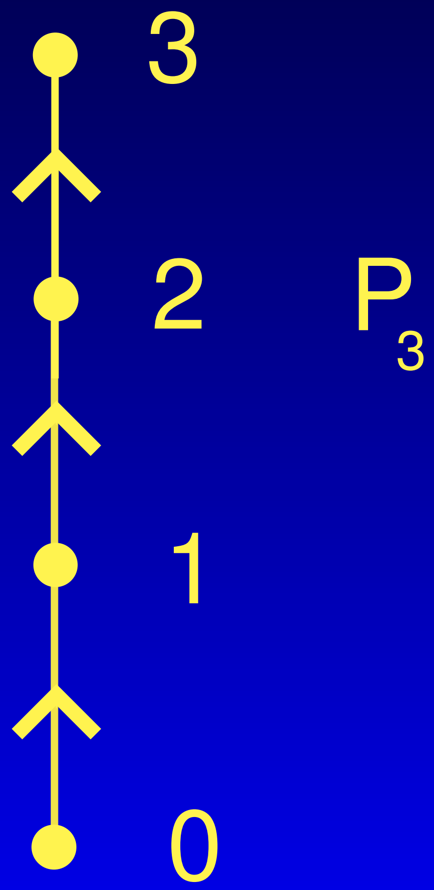
Example 2.

Let $H = \langle \{0, 1, 2, 3\}, \{(0, 1), (1, 2), (2, 3)\} \rangle$, the irreflexive path of length 3.



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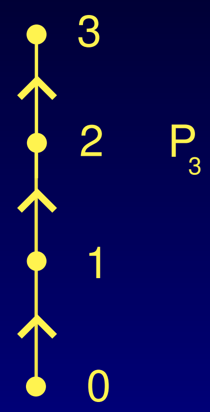

H is a path: admits a majority operation (Feder 2001);

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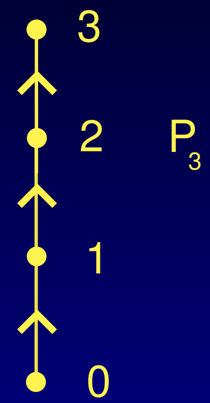

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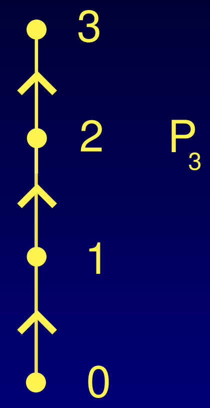

H is a path: admits a majority operation (Feder 2001);

$CSP(H)$ is *not* FO-definable:

H is a core, and H^2 dismantles only down to $H^2 \setminus \{(0,3),(3,0)\}$.

Generalisations of nuf's

- Congruence-distributivity
- Generalised majority-minority terms
- Weak nuf's

Congruence-distributivity

Algebras in congruence-distributive varieties are characterised by the existence of a sequence of 3-ary *Jónsson terms* $d_0, \ldots, d_m$ for some $m \geq 2$. The following is known:

$\mathbb{A}$ has an nuf term
$\Rightarrow$ $\mathbb{A}$ has Jónsson terms
$\Rightarrow$ $\mathbb{A}$ is "nice enough".

Hence conjecturally they are tractable, and have enough structure to provide a good testing ground for the conjecture.

C-D, cont'd

Let $CD(m)$ denote the class of finite algebras that admit a sequence of m Jónsson terms.

It is immediate (from the Jónsson condition) that algebras in $CD(2)$ have a majority term, and hence yield tractable CSP's. The next case, $m = 3$, is taken care of by the following result:

C-D, cont'd

Theorem [Kiss & Valeriote 2006]
Let $\mathbb{A}$ *be a finite algebra in* $CD(3)$*; let* Γ *denote the set of all subalgebras of finite powers of* $\mathbb{A}$*.*

Then $CSP(\Gamma)$ *has relational width* $|A^2|$*, and hence is globally tractable.*

C-D, cont'd

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This prompts:

Question. If $\mathbb{A}$ admits Jónsson terms, does it yield a tractable CSP ? Does the CSP have bounded width ?

GMM operations

An operation f is a *generalised majority-minority (GMM)* operation if it satisfies the following: for every pair $\{a, b\}$ either

$$f(x, \ldots, x, y) = f(y, x, \ldots, x) = y$$

or

$$f(x, \ldots, x, y) = \cdots = f(y, x, \ldots, x) = x$$

holds for all $x, y \in \{a, b\}$.

Notice that nuf's are a special case of GMM.

GMM operations, cont'd

Theorem [Dalmau 2005]
If Γ *is invariant under a GMM operation then* $CSP(\Gamma)$ *is tractable.*

GMM operations, cont'd

Theorem [Dalmau 2005]
If Γ *is invariant under a GMM operation then* $CSP(\Gamma)$ *is tractable.*

NOTE: the principal author of the above paper is *not* Ho Weng Kin.

http://www.lfcs.inf.ed.ac.uk/events/lics/2005/ Kin Dalmau-GeneralizedMajority.html

Weak nuf's

We'll say an idempotent operation f is a weak nuf if it satisfies the identities

$$f(x,\ldots,x,y) = f(x,\ldots,x,y,x) = \cdots = f(y,x,\ldots,x).$$

[Note: what's missing to get an nuf is that these are all equal to x]

Once again, conjecturally, CSP's invariant under such a term should be tractable.

Weak nuf's, cont'd

Theorem [Kiss & Valeriote 2006]
If $CSP(\Gamma)$ has relational width k, then Γ is invariant under a weak nuf of arity k.

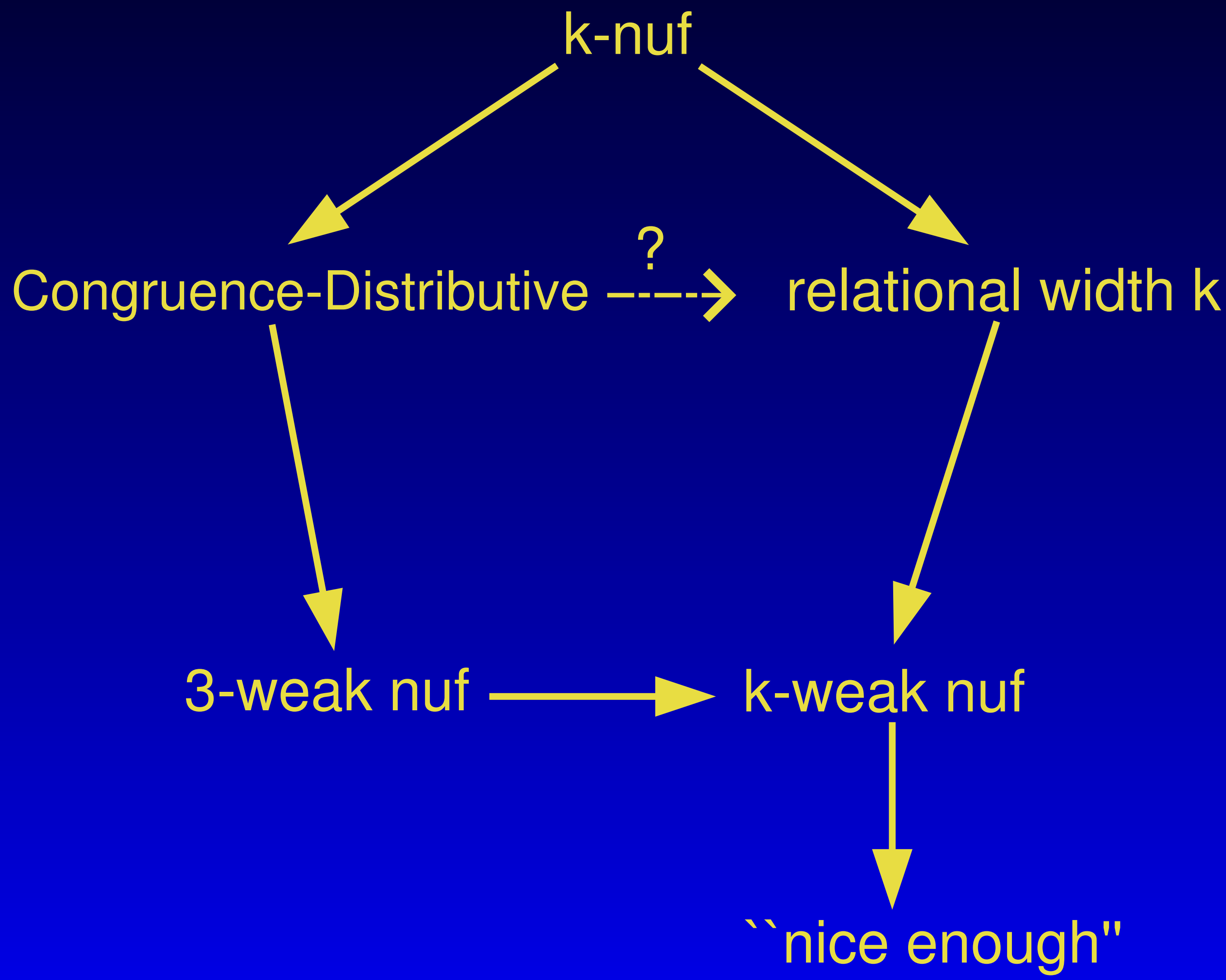
Weak nuf's, cont'd

Theorem [Kiss & Valeriote 2006]
If $CSP(\Gamma)$ has relational width k, then Γ is invariant under a weak nuf of arity k.

Theorem [McKenzie, 2006]
Let $\mathbb{A}$ be an algebra in a congruence-distributive variety. Then $\mathbb{A}$ has a weak majority term.

(More as we speak ? ....)

A recap


k-nuf
Congruence-Distributive
?
relational width k
3-weak nuf
k-weak nuf
``nice enough''

Some Problems

Problem 1. If H admits an nuf, is $CSP(H)$ in **NL** ? What about bounded path width duality ?

Problem 2. Is the problem "does H admit an nuf ?" decidable ? What about for FO-definable H ?

Problem 3. Investigate the notion of weak majority operation and weak nuf: imply tractability ? ($\not\Rightarrow$ bounded width (M. Valeriote))

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