Boolean CSP

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Maths CSP 2006

Outline

Outline

Boolean CSP are interesting

- non trivial
- many complete classifications
- Post's lattice

We can learn a lot from them :

- guess when the Galois connection applies
- a better description of Post relational lattice
- there are still some open questions !

S-formulas

- Let S be a set of Boolean relations.
- $C = R(x_1, \ldots, x_n)$ with $R \in S$ is an S-constraint
- $F = \bigwedge C_i$ where each C_i is an S-constraint is a CNF(S)-formula

SAT(S)

Input :	F an S-formula
Question :	Is F satisfiable ?

expressive power of S

$\langle S \rangle$ is the co-clone generated by S, that is the set of all relations R which can be implemented as :

$$R(x_1, \ldots, x_n) \equiv \exists x_{n+1} \ldots \exists x_m F(\vec{x}),$$

where F is an S-formula

Galois correspondence

- $\mathrm{Pol}(S)$ is the set of Boolean functions f that are *polymorphisms of* (which *preserve*) every relation in S. $\mathrm{Pol}(S)$ is a **clone**.
- $\mathrm{Inv}(B)$ is the set of relations that are preserved by every function in B, it is a co-clone

Pol and Inv form a Galois correspondence between closed sets of relations and closed sets of Boolean functions

$$\langle S \rangle = \mathrm{Inv}(\mathrm{Pol}(S)).$$

The expressive power of S depends on the (co-)clone generated by S

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Galois correspondence applies to complexity

$$\textsc{sat}(S) \equiv_m^{\log} \textsc{sat}(\langle S\rangle)$$

The complexity of $\textsc{sat}(S)$ depends on the clone $\mathrm{Pol}(S)$.

If S_1 and S_2 are two sets of relations such that S_1 is finite and $\mathrm{Pol}(S_2) \subseteq \mathrm{Pol}(S_1)$, then

$$\textsc{sat}(S_1) \leq \textsc{sat}(S_2).$$

$\Rightarrow$ In the Boolean case all clones are identified (Post 1941)

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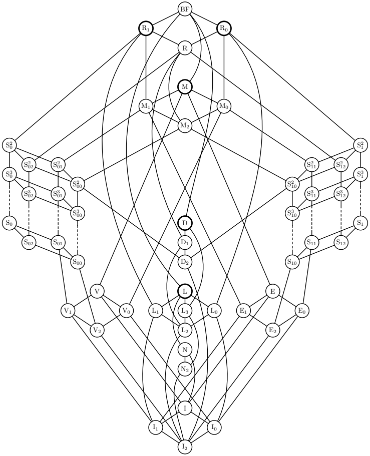
Post's lattice



Decision problem : SAT(S)

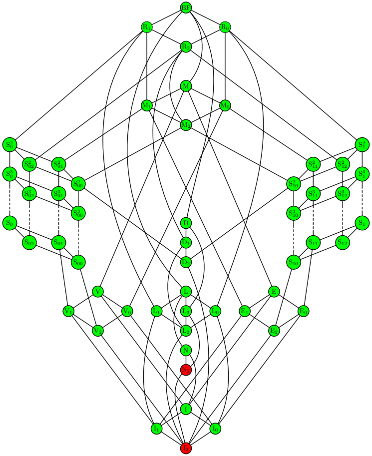
SAT(S)

Input :	F an S-formula
Question :	Is F satisfiable ?

Theorem (Schaefer, 1978)

- if S is 0-valid (1-valid), bijunctive, Horn(dual Horn) or affine, then SAT(S) is in P,
- otherwise SAT(S) is NP-complete.

Decision problem : SAT(S)



Counting problem : #SAT(S)

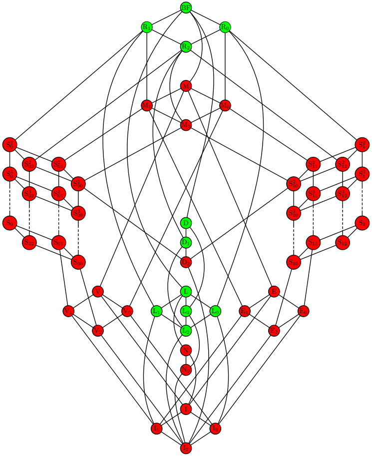
#SAT(S)

Input :	F an S-formula
Question :	How many satisfying assignments for F ?

Theorem

- if S is affine, then #SAT(S) is in FP,
- otherwise #SAT(S) is #P-complete.

Counting problem : #SAT(S)



Enumeration problem

Enumerate SAT(S)

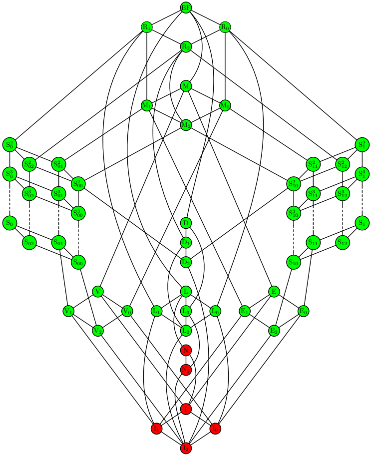
Input : F an S-formula

Question : Enumerate all the satisfying assignments.

Theorem

- if S is bijunctive, Horn (dual Horn) or affine, then one can enumerate all the solutions with polynomial delay
- otherwise such an algorithm does not exist unless P=NP.

Enumeration problem



Optimization problem : Max-SAT(S)

Max-SAT(S)

Input : F an S-formula

Question Find an assignment that satisfies a maximum number of S-clauses

Definition

A relation is 2-monotone if it can be expressed as a DNF-formula either of the form $(x_1 \wedge \ldots \wedge x_p)$ or $(\neg y_1 \wedge \ldots \wedge \neg y_q)$ or $(x_1 \wedge \ldots \wedge x_p) \vee (\neg y_1 \wedge \ldots \wedge \neg y_q)$.

Theorem

- if S is 0-valid (1-valid)or 2-monotone, then Max-SAT(S) is in PO,
- otherwise Max-SAT(S) is APX-complete.

Optimization problem : Max-SAT(S)

No picture !

2-monotone relations do not constitute a co-clone

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Summary

Pb	Class. in lattice	Does Galois apply ?
SAT(S)	yes	yes
#SAT(S)	yes	yes
Enumeration	yes	no
Max-SAT(S)	no	no

A. Bulatov, V. Dalmau
H. Schnoor, I. Schnoor

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Open Question 1

Can we identify the computational goals for which the Galois connection applies ?

Outline

Abduction

L denotes a finite set of Boolean relations

PQ-Abduction(L)

Input : An L-formula φ, a set of variables $A \subseteq Vars(\varphi)$ and a variable $q \in Vars(\varphi) \setminus A$

Question : Is there a set $E \subseteq Lits(A)$ such that $\varphi \wedge \bigwedge E$ is satisfiable but $\varphi \wedge \bigwedge E \wedge \neg q$ is not ?

If one exists, such a set E is called a *solution* of the abduction problem.

Known hard problems

- General problem (Eiter, Gottlob 1995)
  If $\mathcal{C}$ is the class of all propositional CNF formulas, then PQ-ABDUCTION($\mathcal{C}$) is $\Sigma_2\mathrm{P}$-complete.
- (dual) Horn abduction (Selman, Levesque 1990)
  If $\mathcal{C}$ is the class of all propositional Horn formulas, then PQ-ABDUCTION($\mathcal{C}$) is NP-complete. The same holds if $\mathcal{C}$ is the class of all propositional dual Horn formulas.

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Known tractable problems

- Affine (Zanuttini 2003)
  If L is an affine language, then the problem PQ-ABDUCTION(L) is in P.
- Bijunctive (Marquis 2000)
  If L is a bijunctive language, then the problem PQ-ABDUCTION(L) is in P.
- Definite Horn (Eiter, Gottlob 1995)
  If L is a definite Horn language, then the problem PQ-ABDUCTION(L) is in P.

Tools for easy cases

Definition

A clause $C = \bigvee_{i \in I} \ell_i$ is said to be a *prime implicate* of φ if $\varphi \wedge \bigwedge \{\neg \ell_i \mid i \in I\}$ is unsatisfiable but for all $i_0 \in I$, the formula $\varphi \wedge \bigwedge \{\neg \ell_i \mid i \in I, i \neq i_0\}$ is satisfiable.

Lemma

- An abduction problem (φ, A, q) has a solution if and only if there is a prime implicate of φ of the form $(\ell_1 \vee \dots \vee \ell_k \vee q)$ where for all i, ℓ_i is a literal built upon a variable in A.
- a solution is then $E = (\neg \ell_1 \wedge \dots \neg \ell_k)$

$\Rightarrow$ all the prime implicates of a given formula φ in CNF can be generated by repeatedly applying *resolution*.

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Implicative Hitting Set-Bounded IHS-B

Definition

A relation is

- *IHS-B*$-$ if it can be described by a formula in CNF whose clauses are all of one of the following types : (x_i), or $(\neg x_{i_1} \vee x_{i_2})$, or $(\neg x_{i_1} \vee \cdots \vee \neg x_{i_k})$
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- If L is an IHS-$B-$ language, then the problem PQ-ABDUCTION(L) is in P.
- If L is an IHS-$B+$ language, then the problem PQ-ABDUCTION(L) is in P.

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Tools for hardness results

Lemma

If L can be implemented by L', i.e., every relation in L can be expressed as a conjunctive query over L', then $\text{PQ-ABDUCTION}(L) \leq \text{PQ-ABDUCTION}(L')$

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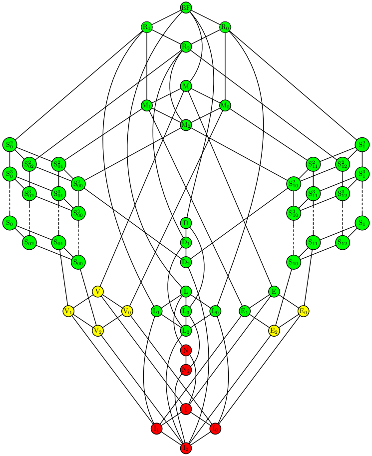
Main theorem

Let L be a finite constraint language. [Creignou, Zanuttini, 2005]

- if L is bijunctive, affine, definite Horn, IHS-$B+$ or IHS-$B-$, the problem PQ-ABDUCTION(L) is polynomial,
- otherwise, if L is Horn or dual Horn, the problem PQ-ABDUCTION(L) is NP-complete,
- in all other cases, the problem PQ-ABDUCTION(L) is $\Sigma_2\mathrm{P}$-complete.

Each of these conditions can be checked in polynomial time given a language written in extension.
Non trivial at this stage

Complexity of abduction



Global tractability versus tractability

Definition

A constraint language S is called *globally tractable* for a problem $\mathcal{A}$, if $\mathcal{A}(S)$ is tractable, and it is called *tractable* if for every finite $L \subseteq S$, $\mathcal{A}(L)$ is tractable

$\Rightarrow$ These two notions

- coincide for most of the computational problems
- do not coincide for the abduction problem (for IHS-$B+$ languages)

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Open Question 2

Can we identify/recognize the computational goals for which the notions of tractability and global tractability coincide ?

Open Question 3

(basic) Propositional circumscription

Input : φ an S-formula and c a clause

Question : Is c satisfied in every minimal model of φ ?

Conjecture (Kirousis, Kolaitis) :
a trichotomy P, coNP-complete or Π_2^{P}-complete holds

Outline

Bases and plain bases

Given a co-clone ICl, a set of relations $B \subseteq ICl$ is called

- a *basis* for ICl if for every $R \in ICl$

$$R(x_1, \ldots, x_n) \equiv \exists \{y_1, \ldots, y_m\} \mathcal{C}(\vec{x}, \vec{y})$$

  where $\mathcal{C}$ is some conjunction of constraints using only relations in B.
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Bases and plain bases differ

In the definition of plain basis :

- there is no existential variables
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Bases and plain bases

co-clone	(plain) basis	property
IS_{00}^n	$\{(\neg x), (\neg x \vee y)\} \cup \{(x_1 \vee \ldots \vee x_k) \mid k \leq n\}$	$\text{IHSB}+^n$
IS_{00}	$\{(\neg x), (\neg x \vee y)\} \cup \{(x_1 \vee \ldots \vee x_k) \mid k \in \mathbb{N}\}$	$\text{IHSB}+$
IS_{10}^n	$\{(x), (\neg x \vee y)\} \cup \{(\neg x_1 \vee \ldots \vee \neg x_k) \mid k \leq n\}$	$\text{IHSB}-^n$
IS_{10}	$\{(x), (\neg x \vee y)\} \cup \{(\neg x_1 \vee \ldots \vee \neg x_k) \mid k \in \mathbb{N}\}$	$\text{IHSB}-$

Boehler, Reith, Schnoor, Vollmer, 2005. Bases of minimal order
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When basis and plain basis differ

ICl	basis	plain basis
IE	$(\bar{x} \vee \bar{y} \vee z)$	$\{(\bar{x}_1 \vee \ldots \vee \bar{x}_k \vee y) \mid k \geq 1\}$
IE_0	$\{(\bar{x} \vee \bar{y} \vee z), (\bar{x})\}$	$\{N_k \mid k \geq 0\} \cup \{(\bar{x}_1 \vee \ldots \vee \bar{x}_k \vee y) \mid k \geq 1\}$
IE_1	$\{(\bar{x} \vee \bar{y} \vee z), (x)\}$	$\{(\bar{x}_1 \vee \ldots \vee \bar{x}_k \vee y) \mid k \geq 0\}$
IE_2	$\{(\bar{x} \vee \bar{y} \vee z), (x), (\bar{x})\}$	$\{N_k \mid k \geq 0\} \cup \{(\bar{x}_1 \vee \ldots \vee \bar{x}_k \vee y) \mid k \geq 0\}$

Advantages / Disadvantages

- infinite plain bases are necessary for co-clones $IL_{(c)}$, $IV_{(c)}$, $IE_{(c)}$, $IN_{(c)}$ and $II_{(c)}$, while they have finite (classical) bases
- for each co-clone there is a canonical plain basis

Minimal plain bases

Proposition

Every co-clone has a unique plain basis which is minimal with respect to inclusion.

one can identify a preferred representation of a relation in a co-clone with respect to this minimal plain basis

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Preferred representation

Definition (preferred representation)

Let *ICl* be a co-clone and let *R* be a relation in *ICl*. Then a conjunction of constraints φ is called a *preferred representation for R with respect to ICl* if φ represents *R* and every constraint in φ is built upon a relation out of the (unique) minimal plain basis of *ICl*.

Preferred representation and prime CNF

Proposition

Given a relation R of arity n and containing m vectors and a co-clone ICl to which R belongs, a preferred representation of R with respect to ICl can be found in time $O(m^2n^2)$.

Proof : Compute a prime CNF φ representing R. φ contains $O(mn)$ clauses and can be computed in time $O(m^2n^2)$ (Zanuttini, Hébrard 2002).

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Computing the minimal co-clone

Proposition

Given a relation R of arity n and containing m vectors, the minimal co-clone containing R can be found in time $O(m^2n^2)$.

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Expressibility

Input : Given a finite set of relations S and a relation R

Question : is R expressible by a conjunctive query over S (i.e., whether R is in the minimal co-clone containing every relation in S) ?

Proposition

The expressibility problem can be solved in polynomial time.

Proof : Compute C_S (C_R) the minimal co-clone containing all the relations in S (resp. R). the relation R is expressible by S if and only if $C_R \subseteq C_S$.

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Open question 4

INVERSE SATISFIABILITY

Input : Given a finite set of relations S and a relation R

Question : is R expressible by a CNF(S)-formula ? (with no existential variables, i.e., no projection)

A dichotomy theorem was obtained by Kavvadias and Sideri for the complexity of problem with constants. Does a dichotomy hold without the constants ? Are the Schaefer cases still tractable ?

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Input : Given a finite set of relations S and a relation R

Question : is R expressible by a CNF(S)-formula ? (with no existential variables, i.e., no projection)

A dichotomy theorem was obtained by Kavvadias and Sideri for the complexity of problem with constants. Does a dichotomy hold without the constants ? Are the Schaefer cases still tractable ?

Conclusion

- There is still something to learn from the Boolean case
  - from complexity to the lattice of (co)-clones
  - from the lattice to complexity
- There remains some open questions

Outline

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