Fingerprint Schemes

- **Def.** A constraint language Γ has a *fingerprint scheme* if ...

- **Thm.** If Γ has a fingerprint scheme, then $QCSP^{q}(\Gamma)$ is in coNP under bounded alternation.

- **Thm.** If Γ has a polymorphism of one of the following types, then Γ has a fingerprint scheme.
    - Mal'tsev polymorphism
    - Near-unanimity polymorphism
    - Set function polymorphism

Fingerprint Schemes

- CSP φ on variables $\{v_1, \dots, v_n\}$, domain D
- Arc consistency: poly-time algorithm

$$\varphi \nearrow (S_1, \dots, S_n) \quad S_i \subseteq D \ \forall i$$
$$\searrow \text{FAIL}$$

- If Γ has set function polymorphism $f: \{S \subseteq D\} \to D$ then $(f(S_1), \dots, f(S_n))$ satisfies φ.

- Γ has fingerprint scheme if:
  - have inference algorithm mapping CSP φ to fingerprint
  - example: tuples $(S_1, \dots, S_n)$ are fingerprints
  - each fingerprint specifies a constraint
    $(S_1, \dots, S_n) \rightsquigarrow (v_1 \in S_1) \wedge \dots \wedge (v_n \in S_n)$
  - preorder $\sqsubseteq$ on fingerprints
    $(S_1, \dots, S_n) \sqsubseteq (S_1', \dots, S_n') \Leftrightarrow S_i \subseteq S_i' \ \forall i$
  - inference algorithm also maps
    CSP φ + fingerprint F $\to$ fingerprint F'
    s.t. $F' \sqsubseteq F$
  - $\sqsubseteq$-chains have polynomial length (wrt arity n)
  - ...

A Conjecture

Conj. [BJK '05]

Γ has no G-set $\Longrightarrow$ CSP(Γ) tractable

Conj.

$$\begin{tikzcd}
 & \Gamma \text{ has fingerprint scheme} \arrow[d, dashed, "\text{(known)}"] \\
\Gamma \text{ has no G-set} \arrow[ur, Rightarrow, bend left] & \text{CSP}(\Gamma) \text{ tractable}
\end{tikzcd}$$

(via inference algorithm

- Conjecture confirmed in 2-element case

Capturing coNP

- Work on $\exists$-restricted QCSP has application in descriptive complexity
- Grädel ['92] studied
  ~~Second-order Horn logic (SO-HORN)~~ -
  SO sentences of form

$$\boxed{Q_1 R_1 \ldots Q_m R_m \ \forall y_1 \ldots \forall y_s \ \bigwedge_{i=1}^{t} C_i}$$

where
  - $Q_i \in \{\forall, \exists\}$
  - R_i relation symbols
  - C_i <u>Horn</u> WRT the R_j
    (arbitrary in input predicates)

- <u>Thm.</u> (Grädel) On ordered finite structures, SO-HORN captures P.

- <u>Def.</u> SO-$\exists$HORN - same as SO-HORN, but C_i Horn WRT $\exists$-quantified R_j.

- <u>Thm.</u> On ordered finite structures, SO-$\exists$HORN captures coNP.

Interesting 1/2 of proof: Let $\psi \in$ SO-$\exists$HORN.
Given finite structure $\mathcal{U}$, can decide if $\mathcal{U} \models \psi$ by reduction to deciding a EQ Horn formula w/ bounded # of alternations.

Summary

Two technologies for establishing <u>positive</u>
complexity results on QCSP:

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1) Collapsibility – "eliminates" $\forall$-quantification

Shows $QCSP_c(\Gamma) \leq CSP_c(\Gamma)$

(and hence $QCSP_c(\Gamma) \in NP$)

2) Fingerprint schemes – exploits structure among $\exists$-variables

Shows bounded-alternation $QCSP^{\exists}(\Gamma) \in coNP$

Help from the CSP

- Collapsibility shows (for certain Γ)

$$QCSP_c(\Gamma) \equiv_p^m CSP_c(\Gamma).$$

- Suggests that CSP classification will help us to achieve QCSP classification

- To what extent do we <u>need</u> CSP classification?

- To the fullest extent possible!

- <u>Thm.</u> For every Γ, $\exists$ collapsible Γ' st

$$CSP_c(\Gamma) \equiv CSP_c(\Gamma') \equiv QCSP_c(\Gamma').$$

- <u>Cor</u>. If $CSP_c(\Gamma)$ has problems of intermediate complexity, $QCSP_c(\Gamma)$ does too.

References

- H. Chen. The Complexity of Quantified Constraint Satisfaction: Collapsibility, Sink Algebras, and the Three-Element Case. (To appear soon on H. Chen's home page)
- H. Chen. Existentially Restricted Quantified Constraint Satisfaction. (CoRR cs.CC/0506059, also available on H.Chen's home page).