Collapsibility for Algebras

- let $f: A^k \to A$ be an operation.

let $A_1 = (A_1^1, \ldots, A_n^1)$

$\vdots \qquad \vdots$

$A_k = (A_1^k, \ldots, A_n^k)$

be adversaries.

We say that the adversary

$$(f(A_1^1, \ldots, A_1^k), \ldots, f(A_n^1, \ldots, A_n^k))$$

is <u>f-composable</u> from $A_1, \ldots, A_k$.

- An adversary is <u>w-simple</u> ($w \geq 1$) if all coordinates are equal to $\{a\}$ (for some $a \in A$) except for w coordinates.

- Let $\mathbb{A}$ be an algebra.

Say $\mathbb{A}$ is <u>collapsible</u> if $\exists w \geq 1$ st

$\forall n \geq 1$, the adv. $A^n = (A, \ldots, A)$ is

f-composable from w-simple adversaries

for some term operation f of $\mathbb{A}$.

- <u>Thm.</u> Let Γ be a constr. lang., let $\mathbb{A}_\Gamma = (A, \mathrm{IdempPol}(\Gamma))$ be its algebra. If $\mathbb{A}_\Gamma$ is collapsible, then $QCSP_c(\Gamma)$ reduces to $CSP_c(\Gamma)$.

G-sets

- Def. An algebra (A, F) is a G-set if its universe is not 1-elt. + every op. $f \in F$ of form
$$f(x_1, \dots, x_k) = \pi(x_i)$$
for $i \in \{1, \dots, k\}$ and permutation $\pi : A \to A$.

- Thm. [BJK'05] Let Γ be a constr. lang.
If A_Γ has a G-set as factor, then
  - $CSP_c(\Gamma)$ NP-complete,
  - $QCSP_c(\Gamma)$ NP-hard.

- Conjecture [BJK'05] If A_Γ has no G-set factor, $CSP_c(\Gamma)$ in P.

- Thm. [Bulatov '02] Let Γ be constr. lang. over 3-elt. domain.
If A_Γ has no G-set factor, $CSP_c(\Gamma)$ in P.
(Otherwise, $CSP_c(\Gamma)$ NP-complete.)

Sinks

- Know:
  - A_Γ collapsible $\Rightarrow$ $QCSP_c(\Gamma) \leq CSP_c(\Gamma)$
  - A_Γ has G-set factor $\Rightarrow$ $QCSP_c(\Gamma)$ NP-hard

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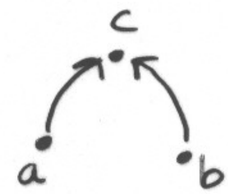
- Does one of these two cases always hold?

NO: $A = \{a, b, c\}$

$$S_{abc}(x,y) = \begin{cases} x & \text{if } x = y \\ c & \text{if } x \neq y \end{cases}$$



(semilattice operation)

Algebra $(A, \{S_{abc}\})$ not collapsible, has no G-set factor

- <u>Def</u>. An algebra is a <u>sink algebra</u> if it is not collapsible, has no G-set as factor, ...

- <u>Thm</u>. If A does not contain a sink nor a G-set as factor, then A collapsible.

- Thm. reduces $QCSP_c(\Gamma)$ classification to
  1) study of sinks
  2) $CSP_c(\Gamma)$ classification

3-element sinks

- Fact: No 1-elt. or 2-elt. sinks.
- Thm. Let A be a 3-elt. sink. Then, A has the semilattice op. S_{abc} as term operation.

$\Rightarrow$ $(\{a,b,c\}, \{S_{abc}\})$ "minimal" 3-elt. sink

- Implies classification for 3-elt. $QCSP_c(\Gamma)$ up to "excluded polymorphism":

Cor. Let Γ be a constr. lang. over 3-elt. domain D s.t. for any labelling of D as $\{a,b,c\}$,

Γ does not have S_{abc} as polymorphism.

Then, $QCSP_c(\Gamma)$ is

- in P if A_Γ has no G-set factor
- NP-hard otherwise

Proof idea. If no S_{abc} polymorphism, A_Γ has no sink factor.

G-set factor $\Rightarrow$ $QCSP_c(\Gamma)$ NP-hard.

No G-set factor: by thm. on sinks, A_Γ collapsible + $QCSP_c(\Gamma) \leq CSP_c(\Gamma)$.

Have $CSP_c(\Gamma)$ in P by Bulatov's theorem.

3-element sinks: complexity

- All 3-elt. sinks have S_{abc} polymorphism.
- Can we prove that the S_{abc} polymorphism implies QCSP tractability?

  - Would imply that all 3-elt. sinks are in P.
- NO: $\exists$ constr. lang. Γ with S_{abc} polymorphism such that $QCSP_c(\Gamma)$ coNP-hard [Chen '04]
- So, not all 3-elt. sinks are in P
- Can we prove that all sinks are hard?
- NO: there exist tractable 3-elt. sinks [Chen + Dalmau]
- NEEDED: complexity classification of 3-elt. sinks

$\exists$-restricted QCSP

- What is the complexity of S_{abc} polymorphism?

  i.e. of $QCSP(\Gamma_s)$ where

  $\Gamma_s = \{$all relations having S_{abc} as polymorphism$\}$

- Know that $CSP(\Gamma_s)$ tractable [JCG '97]

  - S_{abc} semilattice operation

- Thm. In $QCSP(\Gamma_s)$, only the restriction on the $\exists$-variables matters

- Say that QCSP instance Φ is in $QCSP^{\exists}(\Gamma)$ if,

  for each constraint C of Φ,

  fixing the $\forall$-variables of C in any way

  yields a constraint over Γ.

- Thm. (stated formally) $QCSP(\Gamma_s) \equiv^m_p QCSP^{\exists}(\Gamma_s)$.

- Note: class of formulas $QCSP^{\exists}(\Gamma)$ actually more general

$\exists$-restricted QCSP: examples

- Boolean domain $\{0, 1\}$

- Extended quantified 2-SAT formulas
  - ≤ 2 $\exists$-variables per clause

$$\forall y_1 \exists x_1 \forall y_2 \ldots \left((\bar{y}_1 \vee y_4 \vee \bar{x}_1 \vee \bar{x}_2) \wedge (x_1 \vee y_2 \vee \bar{x}_3 \vee \bar{y}_5 \vee \bar{y}_8) \wedge \ldots \right)$$

- Extended quantified Horn formulas
  - first defined + studied by [Kleine Büning et.al.'95]
  - the $\exists$-vars. of each clause are a Horn clause
  - Horn clause $\equiv$ ≤ 1 positive literal

$$\forall y_1 \exists x_1 \forall y_2 \ldots \left((x_1 \vee \bar{x}_2 \vee y_1 \vee \bar{y}_2 \vee \bar{x}_3 \vee \bar{x}_4) \wedge (\bar{y}_1 \vee \bar{x}_1 \vee x_2) \wedge \ldots \right)$$

- <u>Thm.</u> EQ Horn formulas are equivalent in complexity to $QCSP^{\exists}(\Gamma_s)$!

$\exists$-restricted QCSP: complexity

- What is the complexity of just-given formulas?

- First, let's think about $\forall\exists$-formulas

  - in coNP: for any instantiation of $\forall$-variables, what remains is tractable CSP

  - coNP-hard - exercise!

- Our main result: machinery for showing $QCSP^{\exists}(\Gamma) \in coNP$ under bounded alternation

- What about unbounded alternation?

  - EQ Horn Π_2^P-hard [Chen]
  - EQ Horn PSPACE-complete [Bulatov]

  - Complexity open for other Γ!