Collapsibility

- Will present new technique for deriving tractability results, *collapsibility*
- Technique reconciles & reveals common structure among tractable classes just given
- Technique allows us to show for certain Γ that

  $$QCSP_c(\Gamma) \text{ reduces to } CSP_c(\Gamma)$$

  - ...via a single, uniform reduction
  - Reduction conceptually simple
- We then use technique along w/ $CSP_c(\Gamma)$ tractability results to obtain $QCSP_c(\Gamma)$ tractability results

Game View

Can view instance of QCSP as a game!

Universal player

sets ∀-variables — tries to falsify

$$\forall y_1 \exists x_1 \forall y_2 \exists x_2 \cdots \forall y_n \exists x_n \; \mathcal{C}$$

sets ∃-variables — tries to satisfy

Existential player

- Existential player wins if, after all variables set, <u>all</u> constraints in $\mathcal{C}$ satisfied

- Formula true iff existential player can always win

Adversaries

- Consider QCSP instance

$$\phi = \forall y_1 \exists x_1 \ldots \forall y_n \exists x_n C$$

- Define versions of the game where universal player restricted
- *Adversary* is a tuple $(A_1, \ldots, A_n)$ where $A_i \subseteq D$
- Universal player must set y_i to value in A_i
- Original game corresponds to adversary $(D, \ldots, D)$ ("largest" adversary)

Strategies

- Have QCSP instance

$$\phi = \forall y_1 \exists x_1 \ldots \forall y_n \exists x_n C$$

- An *adversary* is a tuple $(A_1, \ldots, A_n)$ where $A_i \subseteq D$

- Consider maps $\tau : \{y_1, \ldots, y_n\} \to D$
with $\tau(y_i) \in A_i$

- A *strategy* is a sequence of mappings

$$\sigma = \{\sigma_i : D^i \to D\}_{i \in [n]}$$

- Let (σ, τ) denote the map from $\{y_1, x_1, \ldots, y_n, x_n\} \to D$
that results when variables set according to σ and τ

- Say that adversary $(A_1, \ldots, A_n)$ is *winnable* if
exists strategy σ such that for all such τ,
the assignment (σ, τ) satisfies C

Collapsibility

- We want to show (for certain Γ)

$\mathrm{QCSP_c}(\Gamma)$ reduces to $\mathrm{CSP_c}(\Gamma)$

- How to do this:
  - Define *simple* adversaries
    Winnability can be formulated as $\mathrm{CSP_c}(\Gamma)$
  - Show that simple adversaries winnable

    $\Leftrightarrow$

    largest adversary $(D, \ldots, D)$ winnable
    (i.e., if formula true)
  - Then, formula true $\Leftrightarrow$ some instance of $\mathrm{CSP_c}(\Gamma)$ true
- If largest adversary winnable, simple adversaries winnable
- Want to show if simple adversaries winnable, then largest winnable
- Idea: give general theorem for inferring the winnability of adversary from simpler adversaries
  - Will use induction to go from winnability of simpler adversaries → winnability of largest adversary

Composing adversaries

- Let $f : D^2 \to D$ be polymorphism of the QCSP formula

$$\phi = \forall y_1 \exists x_1 \ldots \forall y_n \exists x_n C$$

- Theorem: If adversaries

$$(A'_1, \ldots, A'_n)$$

$$(A''_1, \ldots, A''_n)$$

  winnable, then adversary

$$(f(A'_1, A''_1), \ldots, f(A'_n, A''_n))$$

  also winnable

- Here, $f(A, B)$ denotes $\{f(a, b) : a \in A, b \in B\}$

- Theorem holds for higher rank polymorphisms

Proof idea...

- Have formula $\phi = \forall y_1 \exists x_1 \ldots \forall y_n \exists x_n \mathcal{C}$, f a polymorphism of $\mathcal{C}$

- For an elt. $a \in f(A'_i, A''_i)$, let $a' \in A'_i$, $a'' \in A''_i$ denote elements such that $a = f(a', a'')$

- Proof idea: use strategies σ', σ'' for

$$(A'_1, \ldots, A'_n), (A''_1, \ldots, A''_n)$$

to construct strategy for

$$(f(A'_1, A''_1), \ldots, f(A'_n, A''_n))$$

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	y_1	x_1	y_2	x_2	y_3	x_3	$\ldots$
σ :	a_1	c_1	a_2	c_2	a_3	c_3	$\ldots$
	$\downarrow$	$\uparrow f$	$\downarrow$	$\uparrow f$	$\downarrow$	$\uparrow f$	$\ldots$
σ' :	a'_1	b'_1	a'_2	b'_2	a'_3	b'_3	$\ldots$
σ'' :	a''_1	b''_1	a''_2	b''_2	a''_3	b''_3	$\ldots$

- At the end of the day, know that assignments produced by σ', σ'' satisfy $\mathcal{C}$

- But, assignment of σ is f applied to those assignments!

Collapsibility, deployed

- Want to apply this theorem to show winnability of complex adversaries from simpler ones

- Say polymorphism $f : D^2 \to D$ has a unit element u

$$f(u, d) = f(d, u) = d$$

- Take all adversaries where one coordinate equal to D, rest $\{u\}$

$$(D, \{u\}, \{u\}, \{u\}, \ldots, \{u\})$$
$$(\{u\}, D, \{u\}, \{u\}, \ldots, \{u\})$$
$$(\{u\}, \{u\}, D, \{u\}, \ldots, \{u\})$$
$$\vdots$$

- Compose first two with f:

$$(D, D, \{u\}, \{u\}, \ldots, \{u\})$$

- Then compose this one with third one:

$$(D, D, D, \{u\}, \ldots, \{u\})$$

- Continue iteratively to get

$$(D, \ldots, D)$$

Collapsibility, deployed

- If have polymorphism $f : D^2 \to D$ with unit element

$$f(u, d) = f(d, u) = d$$

then

  - simple adversaries $\leftrightarrow$ largest adversary, and
  - $\mathrm{QCSP_c}(\Gamma)$ reduces to $\mathrm{CSP_c}(\Gamma)$

- Holds when we have $\wedge$ or $\vee$ as polymorphism!

$$d \wedge 1 = 1 \wedge d = d$$

$$d \vee 0 = 0 \vee d = d$$

Collapsibility, deployed

- Can extend this argument to higher-rank polymorphisms
- Say have polymorphism $f : D^3 \rightarrow D$ and element $a \in D$ st

$$f(\{a\}, D, D) = D$$
$$f(D, \{a\}, D) = D$$
$$f(D, D, \{a\}) = D$$

- Similar argument holds, to give reduction $\mathrm{QCSP_c}(\Gamma) \leq \mathrm{CSP_c}(\Gamma)$
- Applies to maj, with $a = 0$ or $a = 1$

$$f(a, d, d) = f(d, a, d) = f(d, d, a) = d$$

- Applies to $f(x, y, z) = x \oplus y \oplus z$

$$f(a, a, d) = f(a, d, a) = f(d, a, a) = d$$

- Applies to all 2-elt. tractable cases