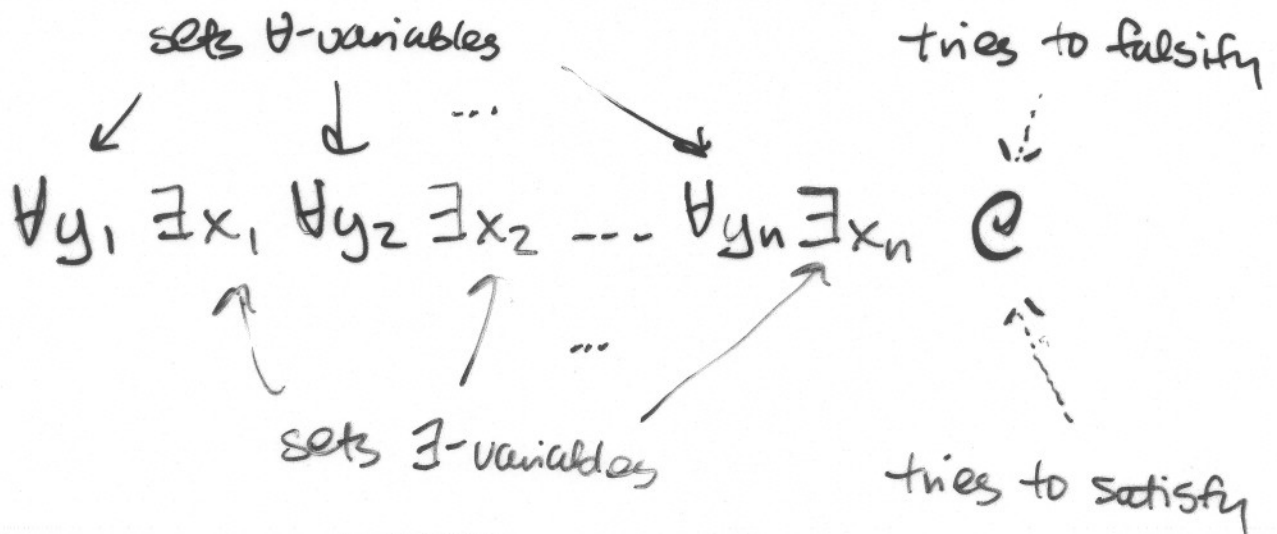

Collapsibility

- Will present new technique for deriving tractability results, *collapsibility*
- Technique reconciles & reveals common structure among tractable classes just given
- Technique allows us to show for certain Γ that
$$\text{QCSP}_c(\Gamma) \text{ reduces to } \text{CSP}_c(\Gamma)$$
 - ...via a single, uniform reduction
 - Reduction conceptually simple
- We then use technique along w/ $\text{CSP}_c(\Gamma)$ tractability results to obtain $\text{QCSP}_c(\Gamma)$ tractability results

Game View

Can view instance of QCBP as a game!

Universal player



Existential player

- Existential player wins if,
after all variables set,
all constraints in \mathcal{C} satisfied
- Formula true iff existential player
can always win

Adversaries

- Consider QCSP instance

$$\phi = \forall y_1 \exists x_1 \dots \forall y_n \exists x_n \mathcal{C}$$

- Define versions of the game where universal player restricted
- *Adversary* is a tuple (A_1, \dots, A_n) where $A_i \subseteq D$
- Universal player must set y_i to value in A_i
- Original game corresponds to adversary (D, \dots, D)
("largest" adversary)

Strategies

- Have QCSP instance

$$\phi = \forall y_1 \exists x_1 \dots \forall y_n \exists x_n C$$

- An *adversary* is a tuple (A_1, \dots, A_n) where $A_i \subseteq D$
- Consider maps $\tau : \{y_1, \dots, y_n\} \rightarrow D$
with $\tau(y_i) \in A_i$

- A *strategy* is a sequence of mappings

$$\sigma = \{\sigma_i : D^i \rightarrow D\}_{i \in [n]}$$

- Let (σ, τ) denote the map from $\{y_1, x_1, \dots, y_n, x_n\} \rightarrow D$
that results when variables set according to σ and τ
- Say that adversary (A_1, \dots, A_n) is *winnable* if
exists strategy σ such that for all such τ ,
the assignment (σ, τ) satisfies C

Collapsibility

- We want to show (for certain Γ)

QCSP_c(Γ) reduces to CSP_c(Γ)

- How to do this:

- Define *simple* adversaries
Winnability can be formulated as CSP_c(Γ)
- Show that simple adversaries winnable

\Leftrightarrow

largest adversary (D, \dots, D) winnable
(i.e., if formula true)

- Then, formula true \Leftrightarrow some instance of CSP_c(Γ) true
- If largest adversary winnable, simple adversaries winnable
- Want to show if simple adversaries winnable, then largest winnable
- Idea: give general theorem for inferring the winnability of adversary from simpler adversaries
 - Will use induction to go from winnability of simpler adversaries \rightarrow winnability of largest adversary

Composing adversaries

- Let $f : D^2 \rightarrow D$ be polymorphism of the QCSP formula

$$\phi = \forall y_1 \exists x_1 \dots \forall y_n \exists x_n C$$

- Theorem: If adversaries

$$(A'_1, \dots, A'_n)$$

$$(A''_1, \dots, A''_n)$$

winnable, then adversary

$$(f(A'_1, A''_1), \dots, f(A'_n, A''_n))$$

also winnable

- Here, $f(A, B)$ denotes $\{f(a, b) : a \in A, b \in B\}$
- Theorem holds for higher rank polymorphisms

Proof idea...

- Have formula $\phi = \forall y_1 \exists x_1 \dots \forall y_n \exists x_n C$, f a polymorphism of \mathcal{C}
-

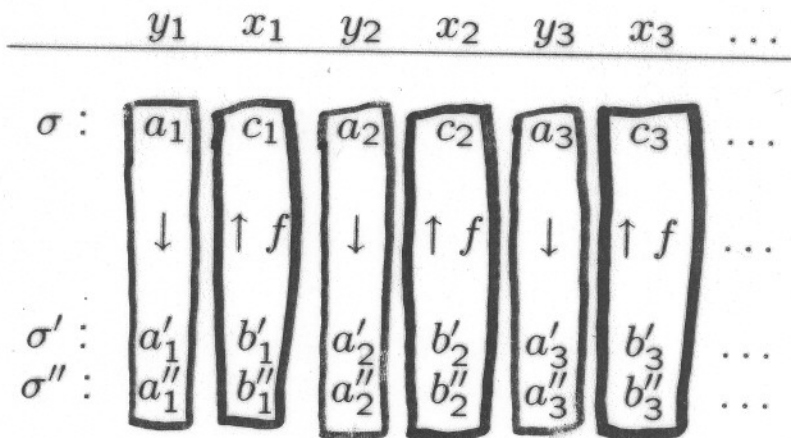
- For an elt. $a \in f(A'_i, A''_i)$, let $a' \in A'_i$, $a'' \in A''_i$ denote elements such that $a = f(a', a'')$

- Proof idea: use strategies σ', σ'' for

$$(A'_1, \dots, A'_n), (A''_1, \dots, A''_n)$$

to construct strategy for

$$(f(A'_1, A''_1), \dots, f(A'_n, A''_n))$$



- At the end of the day, know that assignments produced by σ', σ'' satisfy C
- But, assignment of σ is f applied to those assignments!

Collapsibility, deployed

- Want to apply this theorem to show winnability of complex adversaries from simpler ones
-

- Say polymorphism $f : D^2 \rightarrow D$ has a unit element u

$$f(u, d) = f(d, u) = d$$

- Take all adversaries where one coordinate equal to D , rest $\{u\}$

$$(D, \{u\}, \{u\}, \{u\}, \dots, \{u\})$$

$$(\{u\}, D, \{u\}, \{u\}, \dots, \{u\})$$

$$(\{u\}, \{u\}, D, \{u\}, \dots, \{u\})$$

\vdots

- Compose first two with f :

$$(D, D, \{u\}, \{u\}, \dots, \{u\})$$

- Then compose this one with third one:

$$(D, D, D, \{u\}, \dots, \{u\})$$

- Continue iteratively to get

$$(D, \dots, D)$$

Collapsibility, deployed

- If have polymorphism $f : D^2 \rightarrow D$ with unit element

$$f(u, d) = f(d, u) = d$$

then

- simple adversaries \leftrightarrow largest adversary, and
 - $\text{QCSP}_c(\Gamma)$ reduces to $\text{CSP}_c(\Gamma)$
- Holds when we have \wedge or \vee as polymorphism!

$$d \wedge 1 = 1 \wedge d = d$$

$$d \vee 0 = 0 \vee d = d$$

Collapsibility, deployed

- Can extend this argument to higher-rank polymorphisms
- Say have polymorphism $f : D^3 \rightarrow D$ and element $a \in D$
st

$$f(\{a\}, D, D) = D$$

$$f(D, \{a\}, D) = D$$

$$f(D, D, \{a\}) = D$$

- Similar argument holds, to give reduction $\text{QCSP}_c(\Gamma) \leq \text{CSP}_c(\Gamma)$
- Applies to maj, with $a = 0$ or $a = 1$

$$f(a, d, d) = f(d, a, d) = f(d, d, a) = d$$

- Applies to $f(x, y, z) = x \oplus y \oplus z$

$$f(a, a, d) = f(a, d, a) = f(d, a, a) = d$$

- Applies to all 2-elt. tractable cases