Mathematics of Quantified Constraint Satisfaction

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CSP + QCSP

- There are different formulations of CSP

  e.g. instance is a set of constraints

  $\{N(v_1, v_2), R(v_2, v_3, v_4), S(v_1, v_4)\}$

  - N, R, S relations
  - $\{v_1, \dots, v_4\}$ variables

- Can view each instance as a logical sentence

  $$\exists v_1 \exists v_2 \exists v_3 \exists v_4 (N(v_1, v_2) \wedge R(v_2, v_3, v_4) \wedge S(v_1, v_4))$$

  (primitive positive)

- View naturally gives way to quantified CSP (QCSP) where $\forall$-quantification permitted

  $$\forall v_1 \exists v_2 \forall v_3 \exists v_4 (N(v_1, v_2) \wedge R(v_2, v_3, v_4) \wedge S(v_1, v_4))$$

- QCSP has higher expressivity that comes at price of complexity:

  - CSP NP-complete (in general)
  - QCSP PSPACE-complete (in general)

Constraint Languages

- Study complexity of restricted cases of QCSP
  - one hope: discover tractable cases

- ~~This talk: restrict constraint language~~

- Constraint language ≡ set of relations $\Gamma = \{R_1, R_2, \ldots\}$

- For every constraint language Γ, define

  $QCSP(\Gamma) \equiv$ QCSP restricted to instances where all relations come from Γ

- Research problem: identify constraint languages Γ such that $QCSP(\Gamma)$ is poly-time tractable

- Here: study problems $QCSP_c(\Gamma)$ — where constants allowed in constraints

  constraint: $R(w_1, \ldots, w_k)$ where each w_i a variable or constant

This talk

- Algebraic tools for studying problems $QCSP_c(\Gamma)$
  - Collapsibility - lets us show $QCSP_c(\Gamma) \leq CSP_c(\Gamma)$ for certain Γ

  - Sink algebras (= "bad" algebras for QCSP)
  - 3-element case
- A different model: $\exists$-restricted QCSP ($QCSP^{\exists}(\Gamma)$)

An algebraic approach

- Algebraic approach for studying: based on *polymorphisms*

- Let $R \subseteq D^m$ be a relation

- An operation $f : D^k \to D$ on D is a *polymorphism* of R if for any choice of k tuples

$$\overline{t_1}, \ldots, \overline{t_k} \in R$$

from R, the tuple

$$f(\overline{t_1}, \ldots, \overline{t_k})$$

also in R

- What is $f(\overline{t_1}, \ldots, \overline{t_k})$? Apply f *point-wise.*

- Let R be relation $\{(1,0), (0,1), (1,1)\}$.

  Let maj : $\{0,1\}^3 \to \{0,1\}$ be the majority function on $\{0,1\}$

  - equal to 0 if 2 or 3 of arguments equal to 0
  - equal to 1 if 2 or 3 of arguments equal to 1

- maj is a polymorphism of R

  Consider $\overline{t_1} = (1,0)$, $\overline{t_2} = (0,1)$, and $\overline{t_3} = (1,1)$.

  We have

$$\begin{aligned} \mathrm{maj}(\overline{t_1}, \overline{t_2}, \overline{t_3}) &= \mathrm{maj}((1,0),(0,1),(1,1)) \\ &= (\mathrm{maj}(1,0,1), \mathrm{maj}(0,1,1)) \\ &= (1,1) \end{aligned}$$

An algebraic approach

- Polymorphisms of a constraint language Γ defined as $\mathsf{Pol}(\Gamma) = \{f \mid f \text{ is polymorphism of all } R \in \Gamma\}$
- We have a way to pass from

  constraint language $\Gamma \longrightarrow$ set of operations $\mathsf{Pol}(\Gamma)$

- Notice $\mathsf{Pol}(\cdot)$ order-reversing:

  More relations in $\Gamma \longleftrightarrow$ Less ops. in $\mathsf{Pol}(\Gamma)$

  Less relations in $\Gamma \longleftrightarrow$ More ops. in $\mathsf{Pol}(\Gamma)$

- Mapping $\mathsf{Pol}(\cdot)$ and related map $\mathsf{Inv}(\cdot)$ give *Galois connection*
- Key fact: the complexity of $\mathsf{QCSP}(\Gamma)$ depends only on $\mathsf{Pol}(\Gamma)$!

  Theorem: If $\mathsf{Pol}(\Gamma_1) = \mathsf{Pol}(\Gamma_2)$, then $\mathsf{QCSP}(\Gamma_1)$, $\mathsf{QCSP}(\Gamma_2)$ reduce to each other

Revisiting Schaefer's Theorem

- Schaefer's theorem with constants – algebraic formulation!

Let Γ be constraint language w/ relations over domain $\{0, 1\}$.

The problem $\mathsf{CSP}_c(\Gamma)$ is poly-time tractable if:

1. The operation $\wedge$ is a polymorphism of Γ.
2. The operation $\vee$ is a polymorphism of Γ.
3. The operation $f(x, y, z) = x \oplus y \oplus z$ is a polymorphism of Γ.
4. The operation maj is a polymorphism of Γ.

Otherwise, $\mathsf{CSP}_c(\Gamma)$ is NP-complete

QCSP$_c$(Γ), two-elements

- Classification of QCSP$_c$(Γ) for two-elt. domains

  (e.g. [Creignou et al., 2001])

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  The problem QCSP$_c$(Γ) is poly-time tractable if:

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  3. The operation $f(x, y, z) = x \oplus y \oplus z$ is a polymorphism of Γ.
  4. The operation maj is a polymorphism of Γ.

  Otherwise, QCSP$_c$(Γ) is PSPACE-complete

- Same statement... except hard problems are PSPACE-complete

- Easy problems stay easy, hard problems get harder!

Tractable cases

- Tractable cases of two-elt. $QCSP_c(\Gamma)$:
  - QUANTIFIED HORN SAT [Karpinski et al. '87]
  - QUANTIFIED XOR-SAT [Creignou et al. '01]
  - QUANTIFIED 2-SAT [Aspvall et al. '79]
- Originally proved tractable via disparate proof techniques!
  - QUANTIFIED HORN SAT - *developing and studying sound & complete proof system*
  - QUANTIFIED XOR-SAT - *innermost quantifier elimination*
  - QUANTIFIED 2-SAT - *analysis of "implication graphs"*