Parameterized Complexity of Local Search

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Local Search

Local search is a meta-heuristic widely applied in practice for solving (hard) combinatorial problems.

- Meta-heuristic = applicable to (almost) any problem.

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Local search is a meta-heuristic widely applied in practice for solving combinatorially hard problems.

- Meta-heuristic = applicable to (almost) any problem.

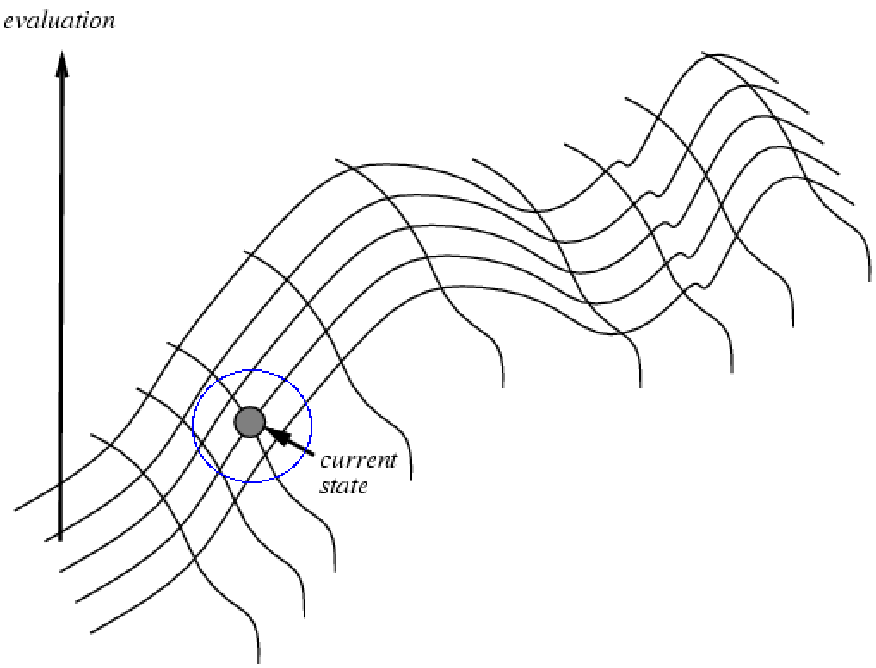
Extremely popular because:

- Easy to understand
- Easy to implement
- Generic
- Actually works (sometimes...)
  - Linear Programming (simplex)
  - Traveling Salesman
  - ...

Local Search

Basic idea:

1. Start with some solution S.
2. Local improvement:
   Search for a better solution S' in the "neighborhood" $N_k(S)$, where k is a predetermined "radius".
3. If such an S' exists, repeat 1+2 with S'.
4. Otherwise, return S'.



Analyzing Local Search

- How many local improvement steps are necessary?
  - PLS-completeness *[Johnson, Papadimitriou, and Yannakakis, 1988]*

Analyzing Local Search

- How many local improvement steps are necessary?
  - PLS-completeness *[Johnson, Papadimitriou, and Yannakakis, 1988]*

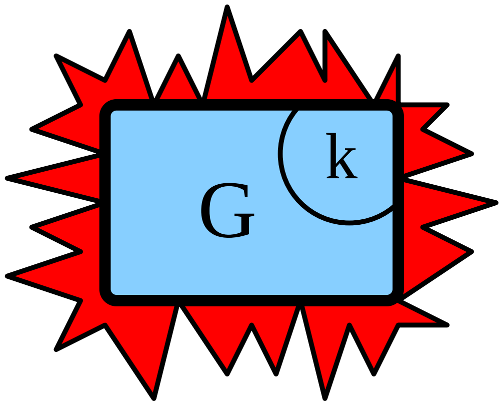
- How easy is the local improvement step?
  - Typically, the size of a k-radius neighborhood = $O(n^k)$.
  - Brute-force = $O(n^k)$ time.
  - Can we do better? — $f(k)n^{O(1)}$ time.
  - Parameterized Complexity

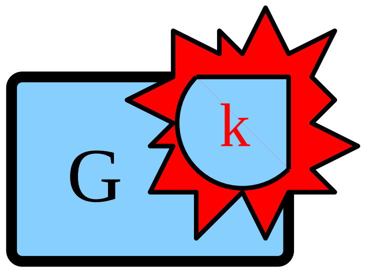
Parameterized Complexity

Definition Parameterized local improvement for optimization problems

Input An instance G, a solution S^*, a distance function d of solutions, and an integer k.

Output A better solution S and $d(S, S^*) \leq k$





fixed-parameter tractable—$f(k)|G|^{O(1)}$ vs. W[1]-hard

Parameterized Complexity

- Traveling Salesman *[E. Balas, 1999] [D. Marx, 2008]*
  [Guo, Hartung, Niedermeier, Súchy, 2011]
- r-Center, Vertex Cover, Odd Cycle Transversal, Max Cut, Min Bisection
  [Fellows, Fomin, Lokshtanov, Rosamond, Saurabh, Villanger, 2009]
- Feedback Arc Set on Tournaments
  [Fomin, Lokshtanov, Raman, Saurabh, 2010]
- Stable Marriage *[Marx and Schlotter, 2001]*
- Boolean Constraint Satisfiability *[Krokhin and Marx, 2012]*
- Satisfiability *[S. Szeider, 2011]*
- Cluster problems *[Dörnfelder, Guo Komusiewicz, Weller, 2011]*
- String problems *[Guo, Hermelin, Komusiewicz, 2012]*

Traveling Salesman

Definition Given two permutations π, π' of $\{1, 2, \ldots, n\}$, the shift distance between π and π' is defined as

$$d(\pi, \pi') = \max_{1 \leq i \leq n} |\pi(i) - \pi'(i)|,$$

where $\pi(i)$ denotes the position of i in π.

Local Improvement of Traveling Salesman

Input: A set of n cities $\{1, \ldots, n\}$ with pairwise distance $c(i, j)$ between the cities, a tour π^*, and an integer k

Output: A tour π with $c(\pi) < c(\pi^*)$ and $d(\pi, \pi^*) \leq k$.

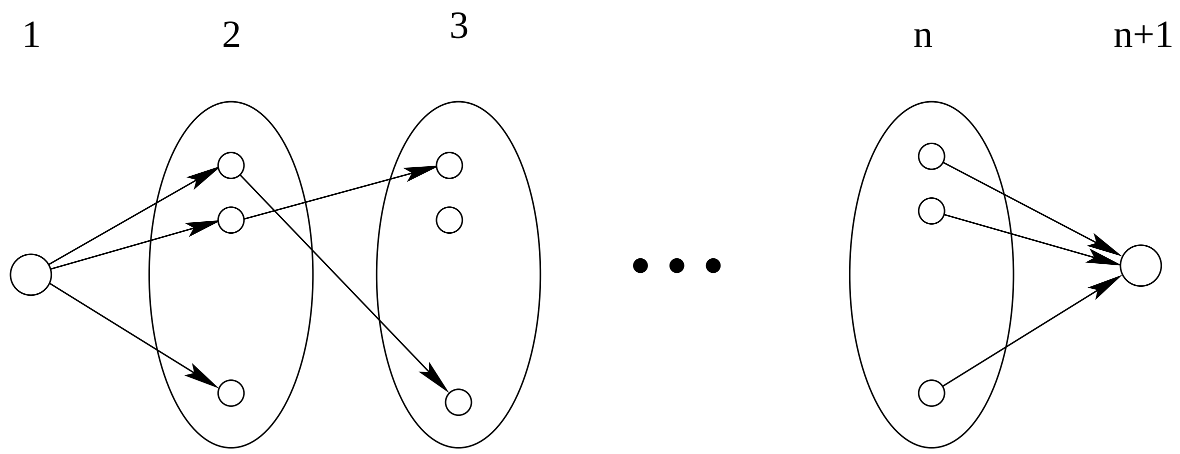
Assume π^* starts and ends at 1 and is equal to the identity permutation.

Traveling Salesman

Theorem *[E. Balas, 1999]*

Local Improvement of Traveling Salesman can be solved in $O(k^2 2^{2(k-1)} n)$ time.

Proof Reduction to the Shortest Path problem on a graph with $O(n \cdot (k+1)2^{2(k-1)})$ vertices and a maximum out-degree of $2k$.



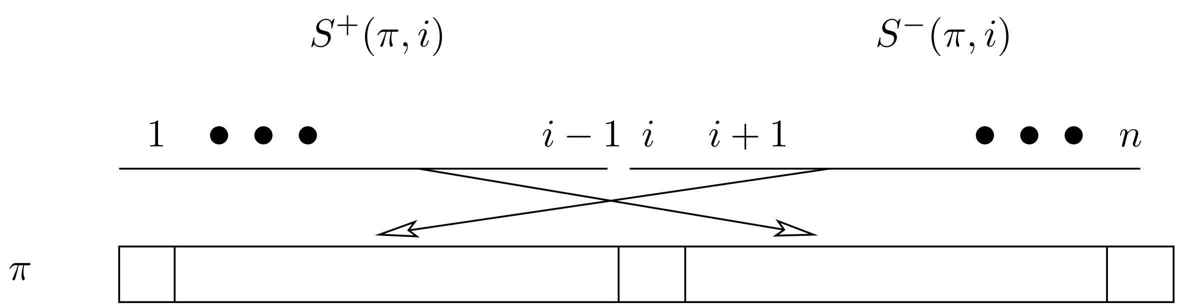
Traveling Salesman

Definition Given a permutation π of $\{1, 2, \ldots, n\}$ and $1 \leq i \leq n$, define

$$S^-(\pi, i) := \{l \geq i \mid \pi(l) < i\}$$

and

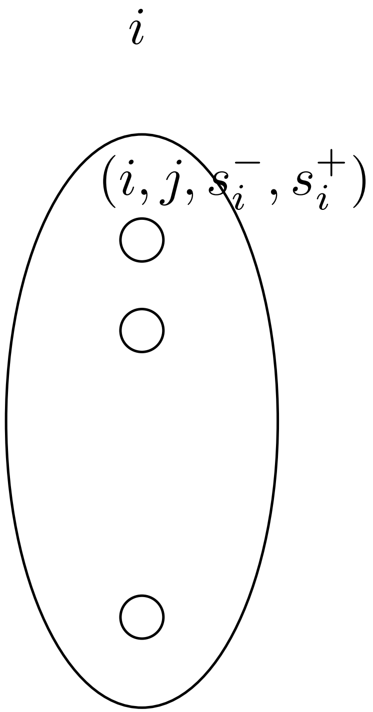
$$S^+(\pi, i) := \{l < i \mid \pi(l) \geq i\}.$$

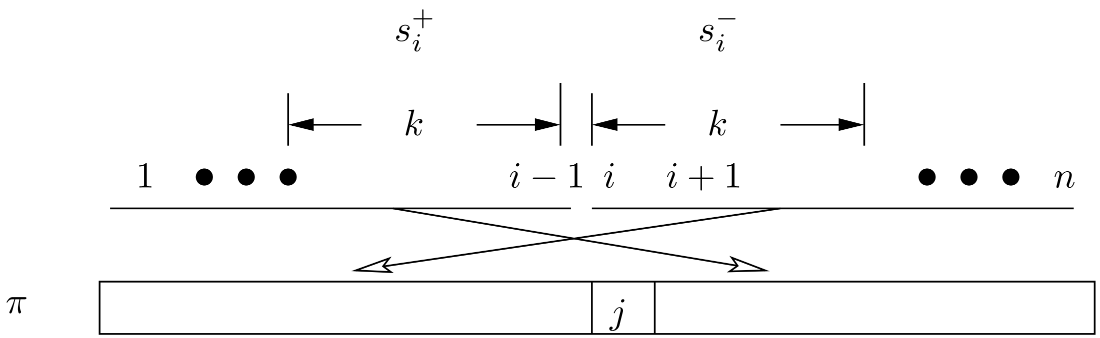


Traveling Salesman

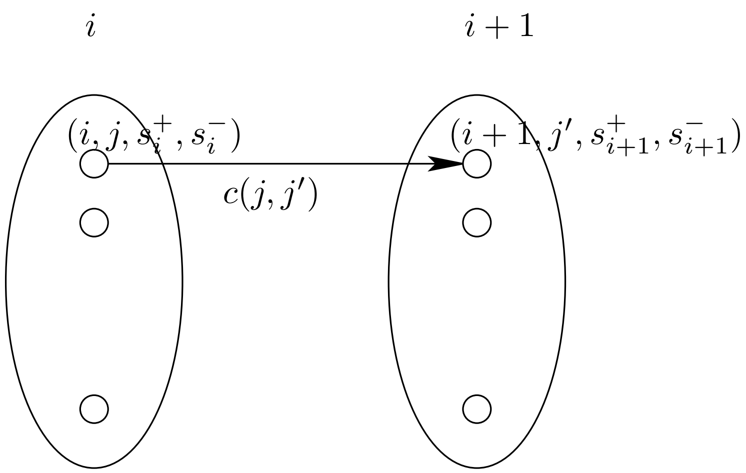
Observation For all feasible π and all $1 \leq i \leq n$,

$$|S^{-}(\pi, i)| = |S^{+}(\pi, i)| \leq k.$$





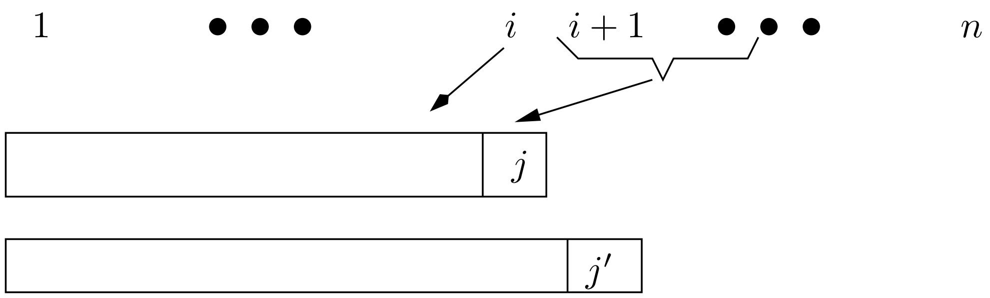
Traveling Salesman



"compatible":

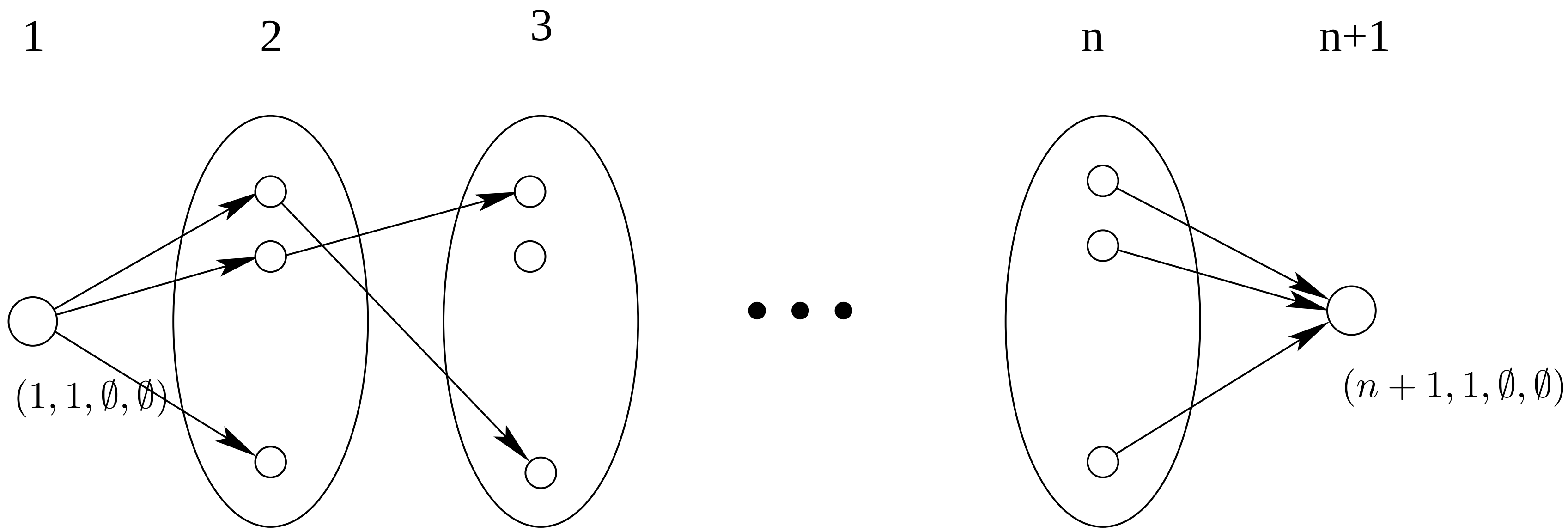
- $j \neq j'$,
- several cases ...

Case: $j > i$ and $i \in s_i^-$:



- $s_{i+1}^- = s_i^- \cup \{j\} \setminus \{i\}$
- $s_{i+1}^+ = s_i^+$

Traveling Salesman



Maximum out-degree $\leq 2k$.

Closest String

Definition Given two length-n strings S and S', the Hamming distance $d_H(S, S')$ is equal to the number of different positions in S and S'.

Closest String

Input A set of length-n strings $T_1, T_2, \ldots, T_m$

Output A length-n string S minimizing

$$\max_{1 \leq i \leq m} d_H(S, T_i).$$

Example:

$T_1 :=$	0	1	0	0
$T_2 :=$	1	1	1	0
$T_3 :=$	0	0	0	1
$T_4 :=$	1	1	0	0
$S :=$	1	1	0	1

Closest String

Hamming neighborhood: $N_k(S) := \{S' | d_H(S, S') \leq k\}$.

Example: $k = 1$

$T_1 :=$	0	1	0	0
$T_2 :=$	1	1	1	0
$T_3 :=$	0	0	0	1
$T_4 :=$	1	1	0	0
$S^* :=$	1	1	1	1
$S :=$	1	1	0	1

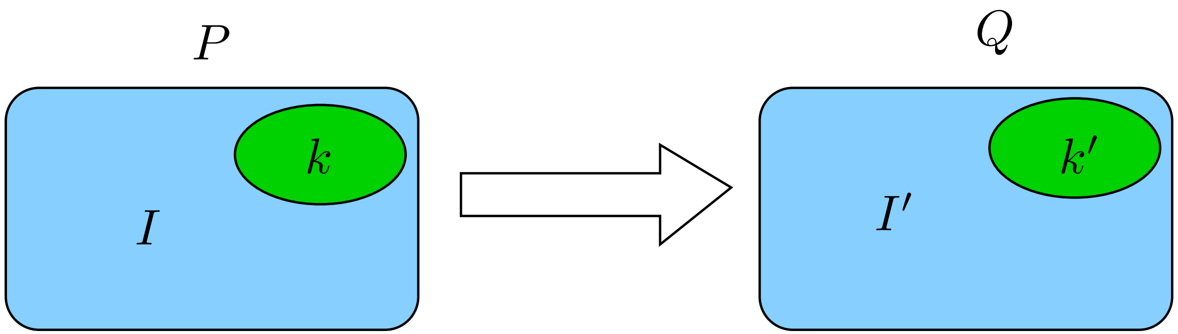
Local Improvement of Closest String

Input A set of length-n strings $T_1, T_2, \ldots, T_m$, a string S^*, and an integer k

Output A string S with $S \in N_k(S^*)$ and $\max_{1 \leq i \leq m} d_H(S, T_i) < \max_{1 \leq i \leq m} d_H(S^*, T_i)$

Closest String

Parameterized reductions:



- $(I, k) \in P \iff (I', k') \in Q$
- $f(k) \cdot |I|^{O(1)}$
- $k' = g(k)$

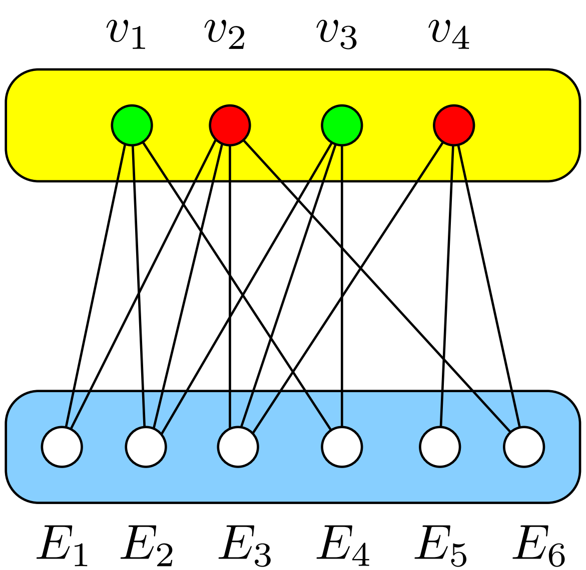
P W[1]-hard $\Longrightarrow$ Q W[1]-hard.

Closest String

Multicolored Hitting Set (MHS)

Input A set $V = \{v_1, v_2, \ldots, v_n\}$ colored by k colors, and a family of subsets $E_1, \ldots, E_m$ of V

Output A subset $H \subseteq V$ of k distinctly colored elements such that $H \cap E_i \neq \emptyset$ for all i's.



Closest String

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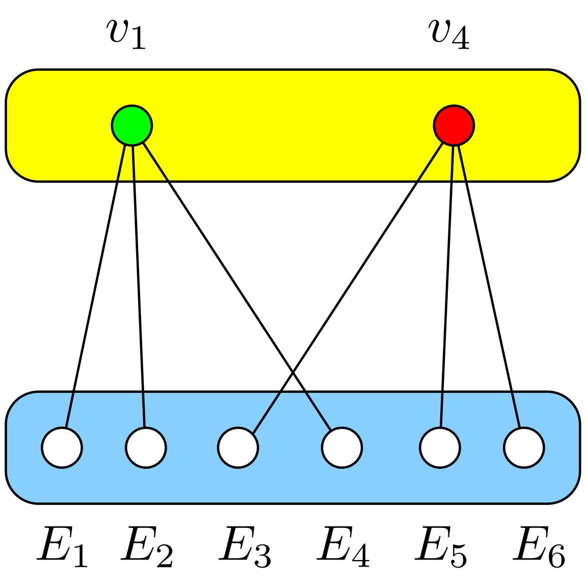
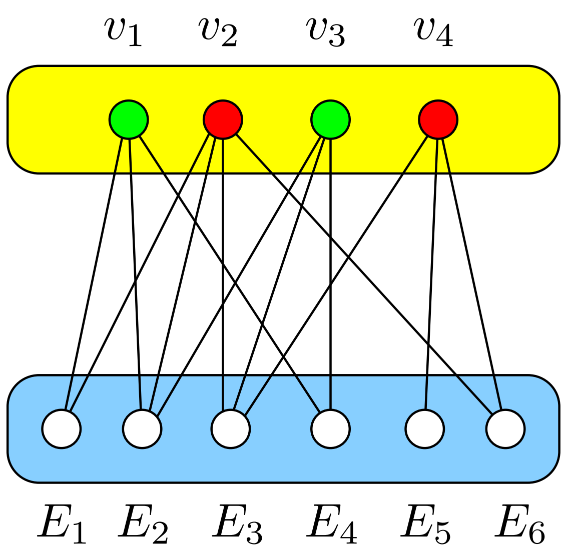


Closest String

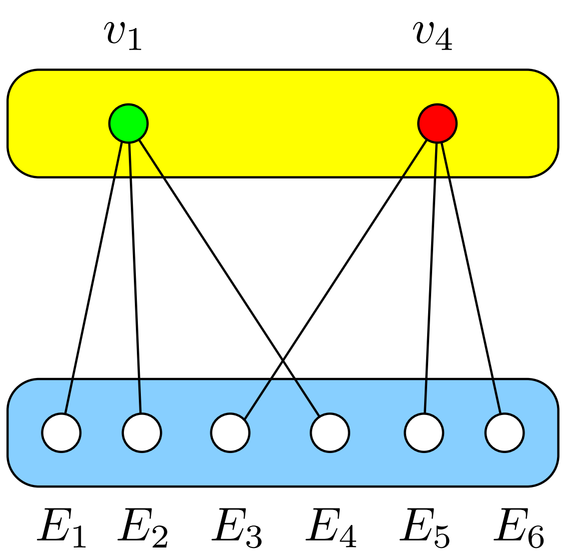
Warm up: $k+2$ letters:



$T_1 :=$	G	R	0	0
$T_2 :=$	G	R	G	0
$T_3 :=$	0	R	G	R
$T_4 :=$	G	0	G	0
$T_5 :=$	0	0	0	R
$T_6 :=$	0	R	0	R
$T_{\mathrm{G}} :=$	G	G	G	G
$T_{\mathrm{R}} :=$	R	R	R	R
$S^* :=$	1	1	1	1

Closest String

Warm up: $k+2$ letters:



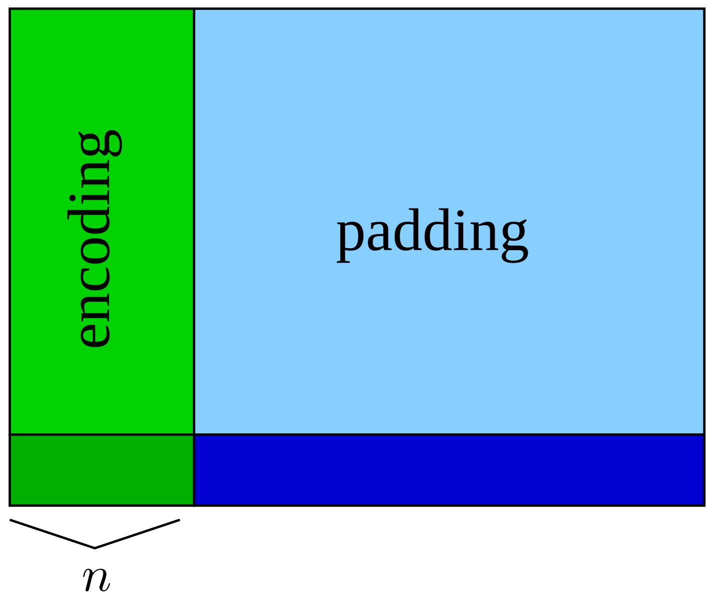
$T_1 :=$	G	R	0	0
$T_2 :=$	G	R	G	0
$T_3 :=$	0	R	G	R
$T_4 :=$	G	0	G	0
$T_5 :=$	0	0	0	R
$T_6 :=$	0	R	0	R
T_G: =	G	G	G	G
T_R: =	R	R	R	R
$S^* :=$	1	1	1	1
$S :=$	G	1	1	R

Closest String

Binary strings

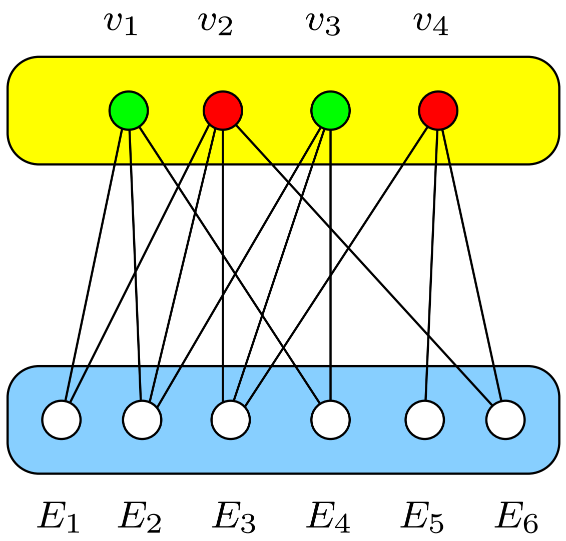
Using long strings with two parts: encoding and padding

- $S^* =$ all-0 string,
- Any column in the padding part will contain at most one 1.
- Will force changes to be done only in the encoding part of S^*.



Closest String

Binary strings

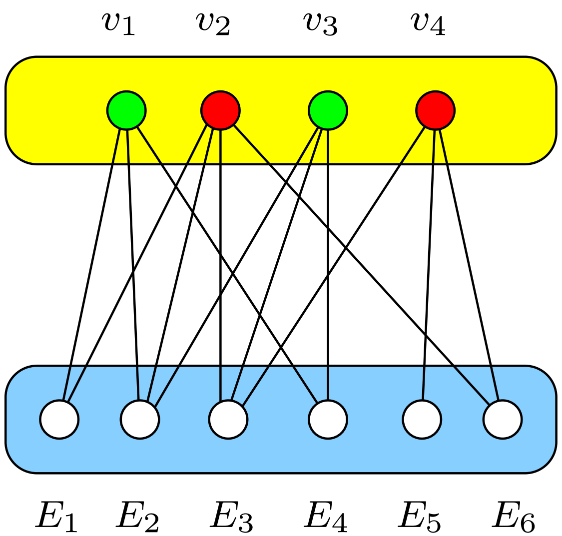


$T_1 :=$	1	1	0	0	...
$T_2 :=$	1	1	1	0	...
$T_3 :=$	0	1	1	1	...
$T_4 :=$	1	0	1	0	...
$T_5 :=$	0	0	0	1	...
$T_6 :=$	0	1	0	1	...

- m strings to encode E_i's.
- Pad so that each has exactly $n-k$ 1's.

Closest String

Binary strings

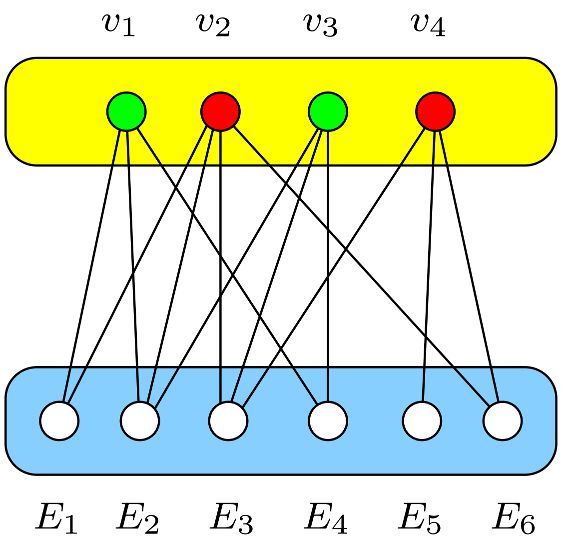


$T_\emptyset :=$	1	1	1	1	...
$T_{\{G\}} :=$	0	1	0	1	...
$T_{\{R\}} :=$	1	0	1	0	...

- $2^k - 1$ strings to encode each of the proper subset of colors.
- Pad so that T_C has exactly $n - |C|$ 1's.

Closest String

Binary strings



T_1 :=	1	1	0	0	...
T_2 :=	1	1	1	0	...
T_3 :=	0	1	1	1	...
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T_6 :=	0	1	0	1	...
$T_\emptyset$:=	1	1	1	1	...
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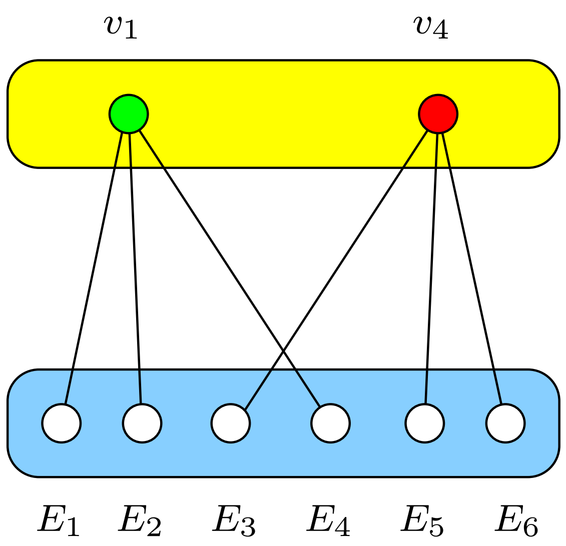
$T_{\{R\}}$:=	1	0	1	0	...
S^* :=	0	0	0	0	...

Closest String

Binary strings



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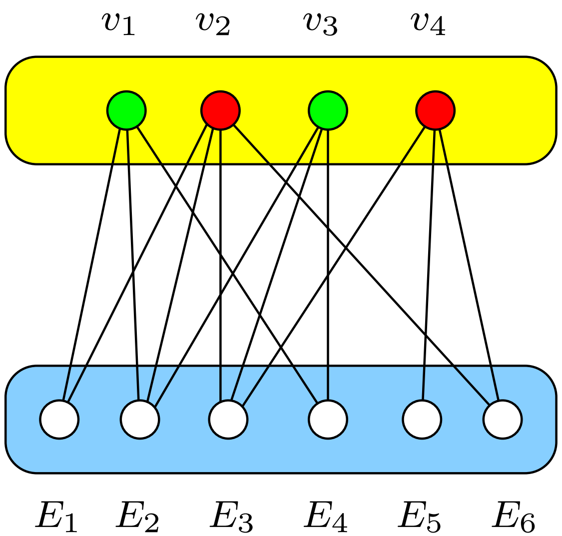
$S^* :=$	0	0	0	0	...
$S :=$	1	0	0	1	...

Closest String

Binary strings



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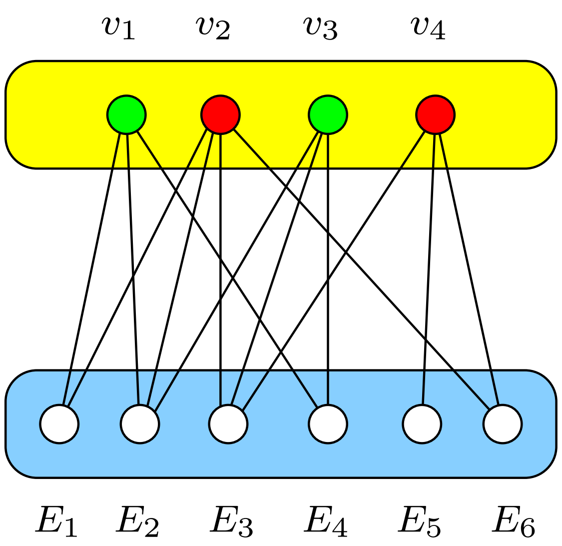
$S :=$	1	0	0	1	...

- $d_H(S, T_C) < n \Longrightarrow$ selected vertices have a color not in C.
- Selected vertices have exactly one of each color.

Closest String

Binary strings



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$T_6 :=$	0	1	0	1	...
$S^* :=$	0	0	0	0	...
$S :=$	1	0	0	1	...

- $d_H(S, T_{E_i}) < n \Longrightarrow$ selected vertices hit E_i.
- Selected vertices form a multicolored hitting set.

Closest String

Theorem *[Guo, Hermelin, Komusiewicz, 2012]*

The local improvement of Closest String is W[2]-hard even for binary alphabets.

Exponential Time Hypothesis (ETH): n-variable 3-SAT not solvable in $2^{o(n)}$ time.

Corollary The local improvement of Closest String cannot be solved in $n^{o(k)} \cdot \text{poly}(n, m)$, assuming ETH.

Brute-force is essentially optimal!.

Conclusion

- Until now, the local improvement of most problems W[1]-hard
  - W[1]-hard
    - TSP with edge exchange/swap/reveral/..., string problems with Hamming distance, subgraph problems with set difference, clustering problem with KT-distance, ...
  - FPT
    - TSP with shift distance, subgraph problems on special graphs, ...
- Ways out of the dilemma:
  - the starting solution S^*
  - new distance measures
  - ...
- Experimental studies *[Simonetti and Balas, 1996]*