REMARK A1.13 (Typed terms). The proofs of the above Church-Rosser theorems are valid also for typed terms. To see this, all we need to check is that if P has type α and a redex R in P is contracted, then the resulting P' also has type α; and this property comes from the definitions of redexes in the typed systems (Definitions 13.9 - 11, 13.17 - 19, 18.4 - 5).

REMARK A1.14. Church-Rosser theorems have been proved and used in other contexts besides λ and CL. In general, their proofs have two parts; a 'global' topological part like the last part of the proof of Theorem A1.2, and a 'local' part like Lemma A1.8. In λ and CL the local lemma is so strong that the global part needed is almost trivial, but this is not always so.

Some examples of (CR) in non-λ-CL-contexts are in Downey and Sethi 1976, Knuth and Bendix 1967, Huet 1981, and the surveys Huet and Oppen 1980, Book 1983 and the references they cite.

Proofs for an infinitary λ-system are in Maass 1974 and Mitschke 1980.

Abstract discussions of (CR) are in Staples 1975, Rosen 1973, Sethi 1974, Huet 1980, Hindley 1969b, 1974, and Klop 1980, Chapters 2 and 3.

A P P E N D I X T W O

THE STRONG NORMALIZATION THEOREM FOR TYPES

As we have seen in Chapter 13, the main property of typed systems not possessed by untyped systems is that all reductions are finite, and hence each typed term has a normal form. In this appendix we shall prove this theorem for the various typed systems considered earlier.

The proofs will be variations on a method due to Tait 1967. (Cf. also Troelstra 1973 §§2.2.1 - 2.3.13.) Simpler methods are known for pure λ and CL, but Tait's has so far been the easiest to extend to new systems.

We begin with two definitions which have meaning for any reduction-concept defined by sequences of replacements (cf. Definition 3.14).

DEFINITION A2.1 (Strongly normalizable (SN) terms). A typed or untyped CL- or λ-term X is strongly normalizable with respect to a given reduction concept, iff all reductions starting at X are finite. (Cf. before Theorem 14.42, SN w.r.t. to weak reduction.) It is normalizable iff it reduces to a normal form. (Of course this does not always imply SN.)

DEFINITION A2.2 (Strongly computable (SC) terms). For typed CL- or λ-terms, strong computability is defined by induction on the number of occurrences of '→' in the term's type:

(a) a term of atomic type is SC iff it is SN;

(b) a term $X^{\alpha\to\beta}$ is SC iff, for every SC term Y^{α}, the term $(XY)^{\beta}$ is SC.

THEOREM A2.3 (Theorem 13.12). *In the system of typed λ-terms, there are no infinite β-reductions.*

Essential Notions & Steps & Proof Points
in SN proof of Tait's computability method:

① Def. of computability predicates over types ($\mathcal{C} : Type \to 2^{\Lambda}$) by ind on types (structural ind.)

② Def. of saturatedness (Lemma 1 (a) & Lemma 2), $\underline{Sat}$

③ $\forall \sigma \in Type . \mathcal{C}(\sigma) \in Sat.$ (so, $\mathcal{C}(\sigma) \subseteq SN$)

④ $\forall$ well-typed term M^{σ}, $M^{\sigma} \in \mathcal{C}(\sigma)$

Note that "Note 1" is calculus-dependent.

Proof. Let 'term' mean 'typed λ-term'; then the theorem says that all terms are SN (with respect to $\triangleright_\beta$). We shall prove that

(i) each SC term is SN,

(ii) every term is SC. (well-typed)

The proof will consist of five simple notes and three lemmas, and the last lemma will be equivalent to the theorem.

Note 1. Each type α can be written in a unique way in the form $\alpha_1 \to \ldots \to \alpha_n \to \theta$, where θ is atomic.

Note 2. It follows immediately from A2.2(b) that if

$$\alpha \equiv \alpha_1 \to \ldots \to \alpha_n \to \theta$$

where θ is an atomic type, then X^α is SC iff, for all SC terms

$$X_1^{\alpha_1}, \ldots, X_n^{\alpha_n},$$

$(XX_1 \ldots X_n)^\theta$ is SC. And by A2.2(a), $(XX_1 \ldots X_n)^\theta$ is SC iff it is SN.

α-conv. Note 3. If X^α is SC, then every term which differs from X^α only by changes of bound variables is SC. And the same holds for SN.

Note 4. By A2.2(b), if $X^{\alpha\to\beta}$ is SC and Y^α is SC, then $(XY)^\beta$ is SC.

Note 5. If X^α is SN, then every subterm of X^α is SN, because any infinite reduction from a subterm of X gives rise to an infinite reduction from X.

LEMMA 1. *Let* α *be any type:*

(a) *Every term* $(aX_1 \ldots X_n)^\alpha$, *where* a *is an atom and* $X_1, \ldots, X_n$ *are all* SN, *is* SC.

(b) *Every* SC *term of type* α *is* SN.

Proof. By induction on the number of occurrences of '→' in α. on type structure

Basic step: α is an atomic type.

(a) Since $X_1, \ldots, X_n$ are SN, $(aX_1 \ldots X_n)$ must be SN. Hence it is SC by Definition A2.2(a).

(b) By Definition A2.2(a).

Induction step: $\alpha \equiv \beta \to \gamma$.

(a) Let Y^β be SC. By the induction hypothesis (b), Y^β is SN. Using the induction hypothesis (a), we get that the term $(aX_1 \ldots X_n Y)^\gamma$ is SC. Hence, so is $(aX_1 \ldots X_n)^\alpha$, by A2.2(b).

(b) Let X^α be SC, and let x^β not occur (free or bound) in X^α. By the induction hypothesis (a) with $n = 0$, x is SC. Hence, by Note 4 above, $(Xx)^\gamma$ is SC. By the induction hypothesis (b), $(Xx)^\gamma$ is SN. But then by Note 5, X^α is SN as well. □

LEMMA 2. *If* $[N^\alpha/x^\alpha]M^\beta$ *is* SC, *then so is* $(\lambda x^\alpha . M^\beta)N^\alpha$; *provided that* N^α *is* SC *if* x^α *is not free in* M^β. ($x \notin FV(M)$: $[N/x]M^\beta \in SC$, $N \in SC$ ⟹ $(\lambda x^\alpha.M^\beta)N^\alpha \in SC$)

Remark. This lemma says that if the contractum of a typed redex R is SC, and if all terms (if any) that are cancelled when R is contracted are SC, then R is SC.

Proof. Let $\beta \equiv \beta_1 \to \ldots \to \beta_n \to \theta$, where θ is atomic, and let

$$M_1^{\beta_1}, \ldots, M_n^{\beta_n}$$

be SC terms. Since $[N^\alpha/x^\alpha]M^\beta$ is SC, it follows by Note 2 that

(1) $$(([N/x]M)M_1 \ldots M_n)^\theta$$

is SN. The lemma will follow by Note 2, if we can prove that

(2) $$((\lambda x.M)NM_1 \ldots M_n)^\theta$$

is SN.

Now since (1) is SN, so are all of its subterms; these include $[N/x]M$, $M_1, \ldots, M_n$. Also, by hypothesis and the preceding lemma, N is SN if it does not occur in $[N/x]M$. Therefore, an infinite reduction from (2) cannot consist entirely of contractions in $M, N, M_1, \ldots, M_n$. Hence, such a reduction must have the form

$$(\lambda x.M)NM_1 \ldots M_n \triangleright_\beta (\lambda x.M')N'M'_1 \ldots M'_n \qquad (M \triangleright M', \text{ etc.})$$
$$\triangleright_{1\beta} ([N'/x]M')M'_1 \ldots M'_n$$
$$\triangleright_\beta \ldots .$$

From the reductions $M \triangleright M'$ and $N \triangleright N'$ we get $[N/x]M \triangleright [N'/x]M'$; hence, we can construct an infinite reduction from (1):

$$([N/x]M)M_1 \ldots M_n \triangleright_\beta ([N'/x]M')M'_1 \ldots M'_n$$
$$\triangleright_\beta \ldots .$$

This contradicts the fact that (1) is SN. Hence, (2) must be SN. □

LEMMA 3. *For every typed term* M^β:

(a) M^β *is* SC;

(b) *For all* $x_1^{\alpha_1}, \ldots, x_n^{\alpha_n}$ *and all* SC *terms* $N_1^{\alpha_1}, \ldots, N_n^{\alpha_n}$, *the term* $M^{*\beta} \equiv [N_1/x_1] \ldots [N_n/x_n]M$ *is* SC.

Proof. Part (a) is all that is needed to prove the theorem, but the extra strength of (b) is needed to make the proof of the lemma work. (Part (a) is a special case of (b), namely $N_i \equiv x_i$).

We prove (b) by induction on the construction of M.

Case 1. M is a variable x_i and $\beta \equiv \alpha_i$. Then M^* is N_i and the result is immediate.

Case 2. M is an atom distinct from $x_1, \ldots, x_n$. Then

$$M^* \equiv M,$$

which is SC by Lemma 1.

Case 3. $M \equiv M_1M_2$. Then

$$M^* \equiv M_1^*M_2^*.$$

By the induction hypothesis, M_1^* and M_2^* are SC, and so M^* is SC by Note 4.

Case 4. $M^\beta \equiv (\lambda x^\gamma.M_1^\delta)$, where $\beta \equiv \gamma \to \delta$. Then

$$M^* \equiv \lambda x.M_1^*$$

if we neglect changes in bound variables.

To show that M^* is SC, we must prove that for all SC terms N^γ, the term M^*N is SC. But

$$M^*N \equiv (\lambda x.M_1^*)N$$
$$\triangleright_{1\beta} [N/x]M_1^*,$$

which is SC by the induction hypothesis applied with $n+1$ instead of n. Then M^*N is SC by Lemma 2. □

This completes the proof of Theorem A2.3. With a minor change, we can extend the proof to $\lambda\beta\eta_t$, as follows.

THEOREM A2.4 (Remark 13.14). *In the system of typed λ-terms, there are no infinite βη-reductions.*

Proof. The same as Theorem A2.3, except that in Lemma 2, near the end of the proof, we need to allow for the possibility that an infinite reduction of $(\lambda x.M)NM_1 \ldots M_n$ has the form

$$(\lambda x.M)NM_1 \ldots M_n \triangleright_{\beta\eta} (\lambda x.M')N'M'_1 \ldots M'_n \qquad (M \triangleright M', \text{ etc.})$$
$$\equiv (\lambda x.Px)N'M'_1 \ldots M'_n \qquad (x \notin FV(P))$$
$$\triangleright_{1\eta} PN'M'_1 \ldots M'_n$$
$$\triangleright_{\beta\eta} \ldots .$$

But in this case we can construct an infinite reduction from $([N/x]M)M_1 \ldots M_n$ as follows:

$$([N/x]M)M_1 \ldots M_n \triangleright_{\beta\eta} ([N'/x]M')M'_1 \ldots M'_n$$
$$\equiv PN'M'_1 \ldots M'_n$$
$$\triangleright_{\beta\eta} \ldots .$$

And this contradicts the fact that (1) is SN. □

Now let us take up CLw_t.

THEOREM A2.5 (Theorem 13.22). *In the system of typed* CL-*terms, there are no infinite weak reductions.*

Proof. Modify the proof of Theorem A2.3 as follows. In Lemma 1 (a), we need to specify that a is not an $S_{\alpha,\beta,\gamma}$ or $K_{\alpha,\beta}$. The proof of Lemma 1 is unchanged. Lemma 2 must be replaced by the following.

LEMMA 2′.

(a) *If* $X^{\alpha\to\beta\to\gamma} Z^{\alpha} (Y^{\alpha\to\beta} Z^{\alpha})$ *is* SC, *then so is* $S_{\alpha,\beta,\gamma}XYZ$.

(b) *If* X^{α} *and* Y^{β} *are* SC, *then so is* $K_{\alpha,\beta}XY$.

Proof of Lemma 2′.

(a) The type of XZ(YZ) is γ. Let

$$\gamma \equiv \gamma_1 \to \dots \to \gamma_n \to \theta,$$

where θ is atomic, and let

$$U_1^{\gamma_1}, \dots, U_n^{\gamma_n}$$

be SC terms. Since XZ(YZ) is SC, it follows by Note 2 that

$$XZ(YZ)U_1 \dots U_n$$

is SN, and hence X, Y, Z, $U_1, \dots, U_n$ are all SN.

We prove that $SXYZU_1 \dots U_n$ is SN. If there were an infinite reduction from $SXYZU_1 \dots U_n$, it would have to have the form

$$\begin{aligned} SXYZU_1 \dots U_n &\rhd_w SX'Y'Z'U_1' \dots U_n' \qquad (X \rhd X', \text{ etc.}) \\ &\rhd_{1w} X'Z'(Y'Z')U_1' \dots U_n' \\ &\rhd_w \dots . \end{aligned}$$

But this gives rise to an infinite reduction from $XZ(YZ)U_1 \dots U_n$, contrary to assumption.

(b) Let $\alpha \equiv \alpha_1 \to \dots \to \alpha_n \to \theta$, where θ is atomic, and let

$$U_1^{\alpha_1}, \dots, U_n^{\alpha_n}$$

be SC terms; since X is SC, it follows by Note 2 that

$$XU_1 \dots U_n$$

is SN, and hence X, $U_1, \dots, U_n$ are SN. Also, Y is SC, so Y is SN by Lemma 1(b).

The proof that $KXYU_1 \dots U_n$ is SN is like (a). This completes the proof of Lemma 2′. □

To complete the proof of Theorem A2.5, we must prove Lemma 3(a). (Part (b) turns out not to be needed for CL-terms.)

The proof is as before, except that Case 4 is now not needed and Case 2 needs the following two extra subcases.

Subcase 2a. $M^{\beta} \equiv S_{\gamma,\delta,\varepsilon}$ and $\beta \equiv (\gamma\to\delta\to\varepsilon)\to(\gamma\to\delta)\to(\gamma\to\varepsilon)$. For any SC terms $X^{(\gamma\to\delta\to\varepsilon)}$, $Y^{(\gamma\to\delta)}$, Z^{γ}, the term XZ(YZ) is SC by Note 4. Hence $S_{\gamma,\delta,\varepsilon}XYZ$ is SC by Lemma 2′. Since X, Y, and Z are arbitrary SC terms of the given types, $S_{\gamma,\delta,\varepsilon}$ is SC by A2.2(b).

Subcase 2b. $M^{\beta} \equiv K_{\gamma,\delta}$ and $\beta \equiv (\gamma\to\delta\to\gamma)$. For any SC terms X^{γ} and Y^{δ}, the term $K_{\gamma,\delta}XY$ is SC by Lemma 2′. Hence $K_{\gamma,\delta}$ is SC by A2.2(b). This completes the proof of Theorem A2.5. □

Let us now turn to $(CL+R)_t$ (Definition 18.4).

THEOREM A2.6 (Implies 18.8). *In the system of typed* CL-*terms of Definitions* 18.1–4, *there are no infinite* wR-*reductions.*

Proof. Modify the proof of Theorem A2.3 as follows.

In Lemma 1(a), specify that a is not an $S_{\alpha,\beta,\gamma}$ or $K_{\alpha,\beta}$ or R_{α} (although a is permitted to be $\bar{0}^N$ or $\bar{\sigma}^{N\to N}$). The proof of the lemma remains unchanged.

Lemma 2 must be replaced by two lemmas; Lemma 2′ above, and the following one.

LEMMA 2″. *Let* X^{α}, $Y^{N\to\alpha\to\alpha}$ *and* Z^{N} *be* SC:

(a) *If* Z *does not reduce to a numeral, then* $R_{\alpha}XYZ$ *is* SC.

(b) *If* $Z \rhd_{wR} \bar{0}$, *then* $R_{\alpha}XYZ$ *is* SC.

(c) *If* $Z \rhd_{wR} \overline{k+1}$ *and* $Y\bar{k}(R_{\alpha}XY\bar{k})$ *is* SC, *then so is* $R_{\alpha}XYZ$.

Proof of Lemma 2″. Let $\alpha \equiv \alpha_1 \to \ldots \to \alpha_n \to \theta$, where θ is atomic. (In fact, $\theta \equiv N$ because N is the only atomic type, but the proof would work for systems in which other atomic types are postulated.) Let

$$U_1^{\alpha_1}, \ldots, U_n^{\alpha_n}$$

be any SC terms. Note that X, Y, Z, $U_1, \ldots, U_n$, being SC, are SN by Lemma 1.

(a) Just as in the proof of Lemma 2, it is enough to prove that

$$R_{\alpha}XYZU_1 \ldots U_n$$

is SN. But Z does not reduce to a numeral, so the only possible reductions in this term are inside X, Y, Z, $U_1, \ldots, U_n$. The result follows.

(b) It is enough to prove that

$$R_{\alpha}XYZU_1 \ldots U_n$$

is SN, and this follows by the argument of Lemma 2.

(c) We must prove that

$$R_{\alpha}XYZU_1 \ldots U_n$$

is SN, given that $Z \rhd_{wR} \overline{k+1}$, and that

$$Y\bar{k}(R_{\alpha}XY\bar{k})U_1 \ldots U_n$$

is SN. But an infinite reduction from $R_{\alpha}XYZU_1 \ldots U_n$ would have to have the form

$$\begin{aligned} RXYZU_1 \ldots U_n &\rhd_{wR} RX'Y'\overline{k+1}\,U_1' \ldots U_n' \qquad (X \rhd X', \text{ etc.}) \\ &\rhd_{1wR} Y'\bar{k}(RX'Y'\bar{k})U_1' \ldots U_n' \\ &\rhd_{wR} \ldots . \end{aligned}$$

This would imply that $Y\bar{k}(RXY\bar{k})U_1 \ldots U_n$ is not SN, contrary to hypothesis. This completes the proof of Lemma 2″. □

To complete the proof of Theorem A2.6, we must prove Lemma 3(a). The proof is the same as before, except that Case 1 is omitted, subcases 2a and 2b (see earlier) are added to prove that $S_{\alpha,\beta,\gamma}$ and $K_{\alpha,\beta}$ are SC, and the following new subcase is added to prove that each R_{α} is SC.

Subcase 2c. $M^{\beta} \equiv R_{\alpha}$, where $\beta \equiv \alpha \to (N \to \alpha \to \alpha) \to N \to \alpha$. Let X^{α}, $Y^{N\to\alpha\to\alpha}$ and Z^{N} be SC. We must prove that $R_{\alpha}XYZ$ is SC.

If Z does not reduce to a numeral, the result follows from Lemma 2″(a). If Z reduces to a numeral $\bar{k}$, we proceed by induction on k.

Basis (k = 0): The result follows by Lemma 2″(b).

Induction step (k to k + 1): By the induction hypothesis, $R_{\alpha}XYW$ is SC for all SC terms W^{N} that reduce to $\bar{k}$. In particular, $R_{\alpha}XY\bar{k}$ is SC. Hence the result follows by Lemma 2″(c).

This completes the proof of Lemma 3, and hence of Theorem A2.6. □

REMARK A2.7. This proof has been written out for the system in Chapter 18, in which N is the only atomic type and reduction is defined in CL. But it would also apply if there were other atomic types besides N, and it can easily be modified to apply to systems defined in λ-calculus (obtained by adding R and its reduction-axioms to $\lambda\beta_t$ or $\lambda\beta\eta_t$). See Troelstra 1973 §§2.2.27 - 31.

REMARK A2.8 (Combinatory strong reduction, cf. 8.16). This is not covered by the above theorems. No doubt a non-trivial definition of SN could be given, using the axioms in Hindley 1967, but no SN theorem has yet been attempted. In contrast, the normal-form or normalization theorem (Theorem 14.42) was probably the earliest to be published. (Curry et al. 1958 §9F, Corollary 9.2.)

REMARK A2.9 (Normalization theorems). A normalization theorem says that every typed or stratified term has a normal form. Historically, such theorems came before SN theorems: for pure λβ, Turing sketched a proof in 1942, see Gandy 1980a (cf. also Andrews 1971, Proposition 2.7.3); for pure combinatory strong reduction, see Curry et al. 1958 §9F, as above. Weak reduction came later; two different proofs are in Sanchis 1967 (with R included), and Seldin 1977b (without R).

The proofs for $\lambda\beta$, $\lambda\beta\eta$ and CL_w determine a specific order for contracting redexes in a term, which guarantees to reach a normal form. For a comparison of proofs, see Troelstra 1973 §2.2.35.

REMARK A2.10 (Arithmetizability). As noted in Remark 18.9, no proof of the normalization theorem for $(CL+R)_t$ can be translated into a valid proof in first-order arithmetic. Hence the proof of the SN theorem, A2.6 above, is also not arithmetizable. One non-arithmetical part of that proof is the definition of SC, A2.2; see Troelstra 1973 §2.3.11. Another is the statement of the SN theorem itself, because infinite reductions are not a first-order concept.

For systems without R, we can avoid these problems by proving the normalization theorem directly, using the non-R proofs in Remark A2.9, which can all be arithmetized. The SN theorem can then be deduced, if desired, by Gandy 1980b §§1-3. But that theorem now says that for each typed term X there is a number n(X) such that all finite reductions starting at X have less than n(X) steps. (This indirect method was not used for Theorems A2.3-5, because the Tait method is shorter and more adaptable to stronger systems.)

Systems with R have the property that a term $R_\alpha XYZ$ which is not a redex (because Z is not a numeral) may become a redex when Z is reduced, and this fouls up the definition of the order for contracting redexes given in the non-R normal-form proofs, when we try to apply them directly to R.

The unarithmetizability of Theorem A2.6's proof can be characterized in terms of ordinal numbers. Let ε_0 be the least ordinal ε for which $\omega^\varepsilon = \varepsilon$; ε_0 is countable, so there are ways of coding the ordinals $\leq \varepsilon_0$ as natural numbers. Also there are proofs of the normalization theorem that use induction up to ε_0 and no stronger principles. (Eg. Schütte 1977 Chapter VI §16 has a proof for a system equivalent to the present one.) Hence there is no way of coding the ordinals which makes the principle of induction up to ε_0 provable in first-order arithmetic. On the other hand, Gentzen has shown that induction up to α can be formalized in arithmetic for each ordinal $\alpha < \varepsilon_0$, so induction up to ε_0 is a 'minimal' unarithmetizable principle.

For more on arithmetizability, see Troelstra 1973, §§2.3.11-13.

APPENDIX THREE

CARE OF YOUR PET COMBINATOR

Contributed by C.J. Hindley

Combinators make ideal pets.

Housing They should be kept in a suitable axiom-scheme, preferably shaded by Böhm trees. They like plenty of scope for their contractions and a proved extensionality is ideal for this.

Diet To keep them in strong normal form a diet of mixed free variables should be given twice a day. Bound variables are best avoided as they can lead to contradictions. The exotic R combinator needs a few Church numerals added to its diet to keep it healthy and active.

House-training If they are kept well supplied with parentheses, changed daily (from the left), there should be no problems.

Exercise They can safely be let out to contract and reduce if kept on a long corollary attached to a fixed point theorem, but do watch that they don't get themselves into a logical paradox whilst playing around it.

Discipline Combinators are generally well behaved but a few rules of inference should be enforced to keep their formal theories equivalent.

Health For those feeling less than weakly equal a check up at a nearby lemma is usually all that is required. In more serious cases a theorem (Church-Rosser is a good general one) should be called in. Rarely a trivial proof followed by a short remark may be needed to get them back on their feet.

Travel If you need to travel any distance greater than the length of M with your combinators try to get a comfortable Cartesian Closed Category. They will feel secure in this and travel quite happily.