II. Propositions-as-types

- Curry-Howard correspondence (1958,1969):
  - Formulae as types
  - Proofs as objects

formula	type	example
$P \supset Q$	$P \to Q$	If … then …
$\forall x{:}A.P(x)$	$\prod x{:}A.P(x)$	Every man is handsome.

Eg: $\lambda x{:}P.x : P \to P$

Connectives in predicative TTs

- Correspondences between logical connectives in predicative TTs (eg, Martin-Löf's TT)
  - Implication $P \supset Q$ – function type $P \to Q$
  - Conjunction $P \wedge Q$ – product type $P \times Q$
  - Disjunction $P \vee Q$ – disjoint union type $P+Q$
  - Universal quantification $\forall x{:}A.P(x)$ – $\prod$-type $\prod x{:}A.P(x)$
  - Existential quantification $\exists x{:}A.P(x)$ – $\sum$-type $\sum x{:}A.P(x)$
- Remarks:
  - "Correspondence" for $\exists$: $\sum$ not the same as the traditional $\exists$ ($\sum$ is "strong" with witnesses, while traditional $\exists$ is "weak").
  - Logical equality?

Curry-Howard correspondence: basic example

- Theorem.
  $\vdash^{L}$ for the implicational intutionistic logic and
  $\vdash$ for the simply typed λ-calculus.
  Then,
  - if $\Gamma \vdash M : A$, then $e(\Gamma) \vdash^{L} A$, where $e(\Gamma)$ maps x:A to A;
  - if $\Delta \vdash^{L} A$, then $\Gamma \vdash M : A$ for some Γ & M such that $e(\Gamma) \equiv \Delta$.

Implicational propositional logic

$$(Ax)\quad \frac{}{\Gamma,\ A \vdash A}$$

$$(\to I)\quad \frac{\Gamma,\ A \vdash B}{\Gamma \vdash A \to B}$$

$$(\to E)\quad \frac{\Gamma \vdash A \to B \quad \Gamma \vdash A}{\Gamma \vdash B}$$

where Γ is a set of formulas A.

Simply-typed λ-calculus (rules as before)

$$(Var)\quad \frac{}{\Gamma,\ x : A \vdash x : A}$$

$$(Abs)\quad \frac{\Gamma,\ x : A \vdash b : B}{\Gamma \vdash \lambda x : A.b : A \to B}$$

$$(App)\quad \frac{\Gamma \vdash f : A \to B \quad \Gamma \vdash a : A}{\Gamma \vdash f(a) : B}$$

where Γ is a set of assumptions of the form $x : A$.

Curry-Howard: further examples

- Predicative calculus with dependent types corresponds to the intuitionistic logic with 1st-order universal quantification.
- Impredicative calculus with dependent types corresponds to the higher-order intuitionistic logic.

First-order universal quantification

$$(\forall I) \qquad \frac{\Gamma \vdash B}{\Gamma \vdash \forall x.B} \quad (x \notin FV(\Gamma))$$

$$(\forall E) \qquad \frac{\Gamma \vdash \forall x.B}{\Gamma \vdash [a/x]B}$$

where Γ is a set of formulas.

Predicative dependent types

$$(\Pi_T) \qquad \frac{\Gamma \vdash A : Type \quad \Gamma,\ x : A \vdash B : Type}{\Gamma \vdash \Pi x : A.B : Type}$$

$$(\lambda) \qquad \frac{\Gamma,\ x : A \vdash b : B}{\Gamma \vdash \lambda x : A.b : \Pi x : A.B}$$

$$(app) \qquad \frac{\Gamma \vdash f : \Pi x : A.B \quad \Gamma \vdash a : A}{\Gamma \vdash f(a) : [a/x]B}$$

Higher-order logical systems

- Eg, Second-order propositional logic
  - Formulas: propositional variables X, implication φ→ψ, and quantification over all propositions ∀X.φ.
  - Rules: besides the usual rules for implication, we have

$$(\forall I) \qquad \frac{\Gamma \vdash \phi}{\Gamma \vdash \forall X.\phi} \quad (X \notin FV(\Gamma))$$

$$(\forall E) \qquad \frac{\Gamma \vdash \forall X.\phi}{\Gamma \vdash [\psi/X]\phi}$$

Impredicative types

- Impredicative type systems
  - F and F^{ω} (Girard 1972, Reynolds 1974)
  - CC (Coquand & Huet 1988)
  - ECC/UTT (Luo 1989/1994)
  - CIC (as implemented in Coq)
- Prop – impredicative universe
  - F (2nd-order) allows quantification over all propositions.
    - Eg, ∀X:Prop.X (the logical falsity)
  - F^{ω} (ω-order) allows quantification over connectives as well.
    - Eg, ∀X:Prop→Prop→Prop. … (for all binary connectives, …)
  - CC (ω-order + dependency) allows quantification over predicates as well.
    - Eg, ∀P:Nat→Prop. P(m)→P(n) (m and n are Leibniz equal)

Prop – universe of logical propositions

$$\frac{\Gamma\ valid}{\Gamma \vdash Prop : Type} \qquad \frac{\Gamma \vdash A : Prop}{\Gamma \vdash A : Type}$$

Intuitively,

Prop : Type and Prop ⊆ Type

Π-types/universal quantification with Prop

$$(\Pi_T) \qquad \frac{\Gamma \vdash A : Type \quad \Gamma,\ x : A \vdash B : Type}{\Gamma \vdash \Pi x : A.B : Type}$$

$$(\Pi_P) \qquad \frac{\Gamma \vdash A : Type \quad \Gamma,\ x : A \vdash P : Prop}{\Gamma \vdash \Pi x : A.P : Prop}$$

$$(\lambda) \qquad \frac{\Gamma,\ x : A \vdash b : B}{\Gamma \vdash \lambda x : A.b : \Pi x : A.B}$$

$$(app) \qquad \frac{\Gamma \vdash f : \Pi x : A.B \quad \Gamma \vdash a : A}{\Gamma \vdash f(a) : [a/x]B}$$

Π_T for Π-types and Π_P for universal quantification
(cf, previous rules for predicative Π-types)

Logical operators in, eg, UTT

$$\begin{aligned}
\forall x{:}A.P[x] &=_{df} \Pi x{:}A.P[x] \\
P_1 \supset P_2 &=_{df} \forall x{:}P_1.P_2 \\
\mathbf{true} &=_{df} \forall X{:}Prop.\ X \supset X \\
\mathbf{false} &=_{df} \forall X{:}Prop.X \\
P_1 \ \&\ P_2 &=_{df} \forall X{:}Prop.\ (P_1 \supset P_2 \supset X) \supset X \\
P_1 \vee P_2 &=_{df} \forall X{:}Prop.\ (P_1 \supset X) \supset (P_2 \supset X) \supset X \\
\neg P_1 &=_{df} P_1 \supset \mathbf{false} \\
\exists x{:}A.P[x] &=_{df} \forall X{:}Prop.\ (\forall x{:}A.(P[x] \supset X)) \supset X.
\end{aligned}$$

- Why are these definitions reasonable?
  - Usual introduction/elimination rules are all derivable.
- Examples
  - Conjunction
    - If P and Q are provable, so is P & Q.
    - If P & Q is provable, so are P and Q.
  - Falsity
    - false has no proof in the empty context (logical consistency).
    - false implies any proposition.

Logic-enriched type theories

- Curry-Howard naturally leads to *intuitionistic* logics.
  - What about, say, *classical* logics?
- But:
  - Type-checking and logical inference are orthogonal.
  - They can be independent with each other.
  - In particular, the embedded logic of a type theory is not necessarily intuitionistic.
  - Type theories are not just for constructive mathematics.
- A possible answer to the above question:
  - Logic-enriched type theories (LTTs)
  - Some recent work: Gambino & Aczel 2006, Luo 2006, Adams & Luo 2010.

LTTs in a logical framework

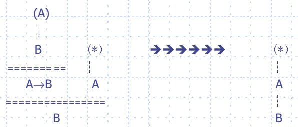
- An LTT consists of
  - Logic: Connectives & rules declared as constants for Prop.
    - Eg, $\supset$: Prop→Prop→Prop
      $\supset_I$: (P,Q:Prop) (P→Q)→(P⊃Q)
      $\supset_E$: (P,Q:Prop) (P⊃Q)→P→Q
    - Can be classical. Eg, Peirce : (P,Q:Prop) ((P→Q)→P)→P
    - Prop is a kind, not a type. In particular, no quantification over Prop.
  - Inductive data types: eg, Nat – elimination over Type, plus
  - Induction rule: one associated with each inductive type; eg,

$$\frac{P : (Nat)Prop \quad p_0 : P(0) \quad p_s : (x{:}Nat)P(x)\to P(x+1)}{Ind_{Nat}(P,p_0,p_s) : (x{:}Nat)P(x)}$$

- Formally formulated in LF/PAL$^+$

Reduction and proof normalisation

- Two aspects of Curry-Howard correspondence
  - Formulae-as-types (as explicated so far)
  - Proofs-as-objects ??
- Eg, β-reduction corresponds to proof normalisation in natural deduction, where an intro rule is immediately followed by an elim rule.



Revision Questions

- What is the Curry-Howard correspondence?
  - How to interpret the logical operators in a predicative TT?
  - What are the two aspects in Curry-Howard correspondence?
- How to define all logical operators by means of ∀ in an impredicative TT?
  - Why are such definitions reasonable?
  - What are the differences between the interpretations of logical operators in predicative/impredicative TTs?
- What is the basic idea behind a logic-enriched type theory?

Selected References

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