Typed Operational Semantics Rules for Inductive Types – $N$ and $\Sigma$

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1. Schemata formulation for UTT inductive types

In UTT, inductive data types could be abstractly given from the following definition of inductive schemata:

Definition 1.1. (Inductive Schemata) Let $\Gamma$ be a valid context and $\Theta \equiv \langle \Theta_1, ..., \Theta_n \rangle (n \in \omega)$ a sequence of inductive schemata in $\Gamma$. Then, $\Theta$ generates a $\Gamma$-type which is introduced by declaring the following contrast expressions w.r.t. $\Theta$:

\[
\begin{align*}
\mathcal{M}[\Theta] & : \text{Type} & (1) \\
\iota_i[\Theta] & : \Theta_i[\mathcal{M}[\Theta]] & (i = 1, \ldots, n) & (2) \\
\mathcal{E}[\Theta] & : (C:(\mathcal{M}[\Theta])\text{Type}) \quad (f_i:\Theta_i^\circ[\mathcal{M}[\Theta]], C, \iota_i[\Theta]]) \\
& \quad (f_n:\Theta_n^\circ[\mathcal{M}[\Theta]], C, \iota_n[\Theta]]) \\
& \quad (z:\mathcal{M}[\Theta])C(z) & (3)
\end{align*}
\]

where $\Theta_i^\circ$ is the kind constructor corresponding to inductive schemata sequence $\Theta_i$.

First example of inductive data type, the Natural Numbers:

\[
\begin{align*}
N & : \text{Type} \\
0 & : N \\
succ & : (N)N \\
\text{Rec}_N & : (C:(N)\text{Type})(c:C(0))(f:(x:N)(C(x))C(succ(x)))) \\
& \quad (n:N)C(n)
\end{align*}
\]

where the eliminator $\text{Rec}_N$ could be defined via the elimination constant $\mathcal{E}_N$, in other words $\text{Rec}_N = df \mathcal{E}_N[\Theta_N]$:

\[
\begin{align*}
\mathcal{E}_N & : (C:(N)\text{Type}) (C(0)) (n:N)(C(n))(C(succ(n)))) (z:N)C(z) \\
\text{Rec}_N & = df [C:\text{Type}] [c:C] [f:(N)(C)C] \mathcal{E}_N([n:N]C, c, f)
\end{align*}
\]
is defined in terms of the schemata formulation Definition 1.1:

**Definition 1.2. (Natural Numbers via Inductive Schemata)** With $\bar{\Theta}_N \equiv (X, (X)X)$, the natural number type $N =_{df} M[X, (X)X]$ is defined as:

$$
N =_{df} M[\bar{\Theta}_N] : Type \\
0 =_{df} \iota_1[\Theta_N] : N \\
succ =_{df} \iota_2[\Theta_N] : (N)N
$$

$$
E_N : (C:(N)Type) (C(0)) ((n:N)(C(n))(C(succ(n)))) (z:N)C(z) \\
Rec_N =_{df} [C:Type] [c:C] [f:(N)C] E_N([m:N]C, c, f)
$$

Second example, the $\Sigma$-type is defined as following:

**Definition 1.3. ($\Sigma$-type via Inductive Schemata)** With $\bar{\Theta}_{\Sigma(A,B)} \equiv (x:A)(B(x))X$, the strong sum type $\Sigma =_{df} [A:Type][B:(A)Type], M[(x:A)(B(x))X]$ is defined as:

$$
\Sigma =_{df} M[\bar{\Theta}_{\Sigma(A,B)}] : (A:Type)(B:(A)Type)Type \\
pair =_{df} \iota_1[\Theta_{\Sigma(A,B)}] : (A:Type)(B:(A)Type)((x:A)(B(x)))\Sigma(A, B)
$$

$$
E_{\Sigma} : (A:Type)(B:(A)Type)(C:(\Sigma(A,B))Type) \\
\quad ((g:(x:A)B(x))C(pair(A, B, g))) (z:\Sigma(A,B))C(z) \\
\pi_1 =_{df} [A:Type][B:(A)Type][z:\Sigma(A,B)] E_{\Sigma}(A, B, [z:\Sigma(A,B)]A, [x:A][y:B(x)]x, z) \\
\pi_2 =_{df} [A:Type][B:(A)Type][z:\Sigma(A,B)] E_{\Sigma}(A, B, [z:\Sigma(A,B)][B(\pi_1(A, B, z)), [x:A][y:B(x)]y, z)
$$

2. TOS rules for inductive types

The typed operational semantics [Goguen94] for inductive schemata is:

$SCH^\bar{\Phi}_{X}(\bar{\Theta})$ means that $\vdash_S \Gamma, PSCH_X(\bar{\Theta})$ and $\Gamma, X:Type \vdash_S \Theta_1 \rightarrow^n \Theta'_1$ for $1 \leq i \leq n$.

($PSCH_X(\bar{\Theta})$ means that $\bar{\Theta}$ well-formed schemata in UTT.)
\[ \frac{SCH_{T,X}^{S}(\Theta)}{\Gamma \vdash S \cdot \mathcal{M}^{X}[\Theta] \rightarrow_{nf} \mathcal{M}^{X}[\Theta'] : Type} \quad (S - M) \]

\[ \frac{SCH_{T,X}^{S}(\Theta)}{\Gamma \vdash i_{X}[\Theta] \rightarrow_{nf} i_{X}[\Theta'] : \Theta'(\mathcal{M}^{X}[\Theta'])} \quad (1 \leq i \leq n) \quad (S - i) \]

\[ \frac{SCH_{T,X}^{S}(\Theta)}{\Gamma \vdash E^{X}[\Theta] \rightarrow_{wh} E^{X}[\Theta'] : (\mathcal{C} : (\mathcal{M}^{X}[\Theta'] : Type) \times (\Phi_{1}^{X}[\mathcal{M}^{X}[\Theta'], C, i_{X}[\Theta']]) \times \cdots \times (\Phi_{n}^{X}[\mathcal{M}^{X}[\Theta'], C, i_{X}[\Theta']]) : \mathcal{B})} \quad (S - E) \]

\[ \frac{\Gamma \vdash \Theta \downarrow \Theta'[\Theta'']}{\Gamma \vdash \mu^{X}[\Theta] \rightarrow_{nf} \mu^{X}[\Theta'] : Type} \quad (\Gamma', A) \in TYPE_{S'}(\Theta') \implies A \equiv T(a) \quad (S - \mu) \]

Fig. 1. TOS-UTT : Canonical Forms for Schemata

For the weak-head reductions \( E \) the rule is:

\[ \frac{\Gamma \vdash E^{X}[\Theta_{1}](C, f) \rightarrow_{nf} E^{X}[\Theta_{3}](C, f') : (\mathcal{M} : \mathcal{M}^{X}[\Theta_{3}]) \mathcal{D}}{\Gamma \vdash i_{X}[\Theta_{2}](a) \rightarrow_{nf} i_{X}[\Theta_{3}](a') : \mathcal{M}^{X}[\Theta_{3}]} \]

\[ \frac{\Gamma \vdash C(i_{X}[\Theta_{2}](a)) \rightarrow_{nf} B}{\Gamma \vdash E^{X}[\Theta_{1}](C, f, i_{X}[\Theta_{2}](a)) \rightarrow_{wh} E^{X}[\Theta_{3}](C, f, a_{1}, \ldots) : \mathcal{B}} \quad (W - E^{X}[\Theta]) \]

Fig. 2. TOS-UTT : Weak Head Reduction for Schemata
So for $N$, the corresponding rules are:

\[
\frac{\Gamma \vdash S \Gamma}{\Gamma \vdash S N \rightarrow^{nf} N : Type} \quad (S - N)
\]

\[
\frac{\Gamma \vdash S \Gamma}{\Gamma \vdash S 0 \rightarrow^{nf} 0 : N} \quad (S - 0)
\]

\[
\frac{\Gamma \vdash S \Gamma}{\Gamma \vdash S \text{succ} \rightarrow^{nf} \text{succ} : N(N)} \quad (S - \text{succ})
\]

\[
\frac{\Gamma \vdash S \Gamma}{\Gamma \vdash S \text{E}N \rightarrow^{wh} \text{E}N : (C : (N)Type)}
\]

\[
\quad \quad ((n : N)(C(n))(C(\text{succ}(n)))) \quad (z : N)C(z)
\]

Fig. 3. TOS Canonical Forms for Natural Numbers

\[
\frac{\Gamma \vdash S \Gamma}{\Gamma \vdash S \text{E}N \rightarrow^{wh} \text{E}N : (C : (N)Type)}
\]

\[
\quad \quad (C(0))
\]

\[
\quad \quad ((n : N)(C(n))(C(\text{succ}(n)))) \quad (z : N)C(z)
\]

Fig. 4. TOS Weak Head Reduction for Natural Numbers

Note that for $N = df M[X, (X)X]$, $SCH_{F,X}^{S}(\Theta_N)$ actually means that $X \rightarrow^{nf} X'$ and $X(X) \rightarrow^{nf} X'(X')$, and by the rules set of TOS we know that $X \equiv X'$ and $X(X) \equiv X'(X')$ so among the premises some are omitted due to this admissibility.
For $\Sigma$.

\[
\begin{align*}
\Gamma \vdash S : \text{Type} & \quad \text{(S - } \Sigma) \\
\Gamma \vdash S \text{ pair} : \text{Type} & \quad \text{(S - pair)} \\
\Gamma \vdash S \text{ E} : \text{Type} & \quad \text{(S - E}\Sigma) \\
\end{align*}
\]

Fig. 5. TOS Canonical Forms for Strong Sum Type

\[
\begin{align*}
\Gamma \vdash S : \text{Type} & \quad \text{(W - } E\Sigma - 1) \\
\Gamma \vdash S : \text{Type} & \quad \text{(W - } E\Sigma - 2) \\
\end{align*}
\]

Fig. 6. TOS Weak Head Reduction for Strong Sum Type
3. Abbreviated TOS forms

In [Goguen99], an abbreviated format of TOS rules is proposed, we can call it TOS’, as:

Three judgement forms:

- $\Gamma \vdash_S \Delta$ meaning that $\Gamma$ has canonical (normal) form $\Delta$;
- $\Gamma \vdash_S A \rightarrow B$ meaning that $A$ is well-formed in $\Gamma$ and has canonical form $B$;
- $\Gamma \vdash_S M \rightarrow N \rightarrow P : A$ meaning that $M$ weak head reduces to $N$ and has canonical for $P$ of kind $A$, and $M$, $N$, $P$ are all well-formed in $\Gamma$.

With four variations of the judgement forms:

- $\Gamma \vdash_S \text{ok}$ when $\Delta$ is ignored;
- $\Gamma \vdash_S M \rightarrow_w N : A$ meaning that $M$ weak head reduces to $N$ of kind $A$ (the normal form $P$ is ignored);
- $\Gamma \vdash_S M \rightarrow_n P : A$ meaning that $M$ has normal form $P$ of kind $A$ (the weak head normal form $N$ is ignored);
- $\Gamma \vdash_S M : A$ when $N$ and $P$ are both ignored.

Different rule forms for $N$ and $\Sigma$:

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Fig. 7. Abbreviated TOS’ Rules for Natural Numbers

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Fig. 8. Abbreviated TOS’ Rules for Strong Sum Type

REFERENCES

