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SHORT NOTE

Identities of Regular Semigroup Rings

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Abstract

The author proves that, if S is an FIC-semigroup or a completely regular semigroup, and if RS is a ring with identity, then $R < E(S) >$ is a ring with identity.

Throughout this paper, R denotes as a ring with identity. Let S be a semigroup, $X \subseteq S$. The following notations are used in the paper:

$< X >$: the subsemigroup of S generated by X;

$|X|$: the cardinal number of X;

$E(S)$: the set of idempotents of S;

RS: the contracted semigroup ring of S over R;

$supp(A) = \{s \in S : r_s \neq 0\}$ for $0 \neq A = \sum r_s s \in RS$ with $s \in S$ and $r_s \in R$;

I_{RS}: the identity (if it exists) of RS.

The following Problem is raised in [1]: Problem 1 Which semigroup rings are rings with identity? It is known that for a semigroup S, the semigroup ring $R[S]$ possesses an identity iff both RS and R do; if RS is a ring with identity, then so is R (see[1]). So it is enough to discuss the existence of identity of RS only. Li Fang investigated the existence of identity of orthodox semigroup rings. The following problem is raised in [2]: Problem 2 Let S be a regular semigroup. If RS is a ring with identity, is $R < E(S) >$ a ring with identity? In the following we shall use the terminology, notation and basic results of [3].

Definition 1. A regular semigroup S is called an FIC-semigroup, if for any sequence $\{e_i\}_{i=1}^{\infty}$ of $E(S), e_1 \geq e_2 \geq e_3 \geq ... \geq e_n \geq \cdots$, there exists N, such that $e_N = e_{N+1} = \cdots = e_{N+m} = \cdots$.

Lemma 2. *Let $S = M^{[0]}(G; I, \Lambda; P)$ be a completely [0-] simple semigroup, $0 \neq e \in E(S)$. If $ea \neq 0$, then there exists $f \in E(S)$ such that $a = fea$.*

Proof. Let $e = (p_{\lambda i}^{-1}, i, \lambda), a = (g, j, \mu)$. By $ea \neq 0, p_{\lambda j} \neq 0$, thus $(p_{\lambda j}^{-1}, j, \lambda)ea = (p_{\lambda j}^{-1}, j, \lambda)(p_{\lambda i}^{-1} p_{\lambda j} g, i, \mu) = (p_{\lambda j}^{-1} p_{\lambda i} p_{\lambda i}^{-1} p_{\lambda j}, j, \mu) = (g, j, \mu) = a$. ■

Lemma 3. ([3]) *A [0-] simple semigroup is completely [0-] simple iff it contains a primitive idempotent.* ■

Lemma 4. ([1]) *Let S be a semigroup, let I be the identity of RS, and let $s \in S$. Then there exist $e, f \in E(S) \cap supp(I)$, such that $es = sf = s$.* ■

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Suppose ρ is a congruence on a semigroup S. Then the natural semigroup homomorphism $\Psi : S \to S/\rho$ induces the surjective ring homomorphism

$$\Psi RS \to R(S/\rho); \sum r_s s \to \sum r_s \bar{s}.$$

So we have the following.

Lemma 5. *Let RS be a ring with identity I, then $R(S/\rho)$ has the identity $\bar{I} = \Psi(I)$.* ■

Our main theorem for this paper is now

Theorem 6. *Let R be a ring with identity, S be an FIC-semigroup, in particular, S be a finite regular semigroup. Then RS is a ring with identity iff$R < E(S) >$ is a ring with identity.*

Proof. If $R < E(S) >$ is a ring with identity, then by Lemma 4, it is easy to show that RS is a ring with identity.

Conversely, suppose that there exists an FIC-semigroup S such that RS is a ring with identity, but $R < E(S) >$ is a ring without identity. Then the set

$A = \{T : T$ is an FIC-semigroup, RT contains an identity, $R < E(T) >$ contains no identity$\} \neq \emptyset$

Let $B = \{|supp(I_{RT})| : T \in A\}$, k be the minimal number of B. Let $S \in A$ such that $|supp(I_{RS})| = k$.

Let $C = \{D : D$ is an ideal of S, $D \cap supp(I_{RS}) = \emptyset\}$. Let $\bar{S} = S/M$ $if C \neq \emptyset$,where M is the union of all ideals of S such that $D \cap supp(I_{RS}) = \emptyset$, and $\bar{S} = S$ if $C = \emptyset$. By Lemma 4 and Lemma 5, it is easy to see that $\bar{S} \in A$, and $|supp(I_{R\bar{S}})| = k$. Let $I = I_{R\bar{S}} = \sum_{i=1}^{n} r_i b_i + \sum_{j=1}^{m} s_j a_j$, where $b_i \in < E(\bar{S}) >, r_i \in R, i = 1, 2, 3...n; a_j \notin < E(\bar{S}) >, s_j \in R, j = 1, 2, ..., m; n + m = k$.

Thus for any non-zero ideal D of $\bar{S}$, $D \cap supp(I) \neq \emptyset$.By Lemma 5, $\bar{I} = I_{R(\bar{S}/D)} = \sum_{i=1}^{n} r_i \bar{b}_i + \sum_{j=1}^{m} s_j \bar{a}_j$, therefore $|supp(\bar{I})| < |supp(T)|$. By the hypothesis and Lemma 4, it is easy to show that $\bar{I} = \sum_{i=1}^{n} r_i \bar{b}_i$, thus $a_j \in D (j = 1, 2, .., n)$. So the intersection K of all non-zero ideals of $\bar{S}$ is nonempty, therefore $a_j \in K (j = 1, 2, ..., n)$. Then K is a 0-minimal ideal or a minimal ideal of S.

Since S is regular, we have $K^2 \neq 0$,therefore K is [0-] simple. Since S is an FIC-semigroup, $\bar{S}$ is an FIC-semigroup, thus K is an FIC-semigroup. By Lemma 3, K is a completely [0-] simple semigroup. For any $e \in E(\bar{S})$, if $e(\sum_{j=1}^{m} s_j a_j) \neq 0$, then $eI = e = e(\sum_{j=1}^{n} r_i b_i) + e(\sum_{j=1}^{m} s_j a_j)$, there must exist a_j such that $0 \neq ea_j \in supp[e - e(\sum_{i=1}^{n} r_i b_i)]$, thus $0 \neq ea_j \in < E(\bar{S}) > \cap K$. There exists an element x of K such that $ea_j = ea_j x ea_j, x = x ea_j x$. Thus $0 \neq ea_j = ea_j x ea_j = (ea_j xe)a_j$. It is easy to verify that $ea_j xe \in E(K)$. So by Lemma 2, there exists $f \in E(K)$ such that $a_j = f(ea_j xe)a_j = f(ea_j xe)ea_j \in < E(\bar{S}) >$. This is a contradiction to the hypothesis $a_j \notin < E(\bar{S}) >$. Hence $e(\sum_{j=1}^{m} s_j a_j) = 0$,whence $e(\sum_{j=1}^{n} r_i b_i) = e$. By Lemma 4, for any $t \in \bar{S}$, there exists $f \in E(\bar{S})$, such that $t = tf$, thereby

$$t(\sum_{i=1}^{n} r_i b_i) = tf(\sum_{i=1}^{n} r_i b_i) = tf = t.$$

So

$$\begin{aligned}(\sum_{i=1}^{n} r_i b_i) &= I(\sum_{i=1}^{n} r_i b_i) = (\sum_{i=1}^{n} r_i b_i + \sum_{j=1}^{m} s_j a_j)(\sum_{i=1}^{n} r_i b_i) \\ &= \sum_{i=1}^{n} r_i b_i(\sum_{i=1}^{n} r_i b_i) + \sum_{j=1}^{m} s_j a_j(\sum_{i=1}^{n} r_i b_i) = \sum_{i=1}^{n} r_i b_i + \sum_{j=1}^{m} s_j a_j = I,\end{aligned}$$

a contradiction. Therefore, if RS is a ring with identity, so is $R < E(S) >$. ∎

It is easy to show the following.

Lemma 7. *Let S be a completely regular semigroup. If the intersection of all non-zero ideals of S is nonempty, and is denoted by K, and if $K \neq 0$, then K is a completely* [0-] *simple semigroup.* ∎

By Lemma 7 and the proof of Theorem 6, we have the following.

Theorem 8. *Let R be a ring with identity, S be a completely regular semigroup. Then RS is a ring with identity iff $R < E(S) >$ is a ring with identity.* ∎

References

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