SOOCHOW JOURNAL OF MATHEMATICS
Volume 22, No. 3, pp. 405-410, July 1996

THE JOIN OF THE VARIETIES OLBG AND ORBG*

BY

HAIXUAN YANG

Abstract. In this paper the varieties OLBG and ORBG are investigated. And the join of the varieties of OLBG and ORBG is described.

1. Introduction

Completely regular semigroups are semigroups that are unions of their subgroups. They may be regarded as universal algebras with an associative binary operation (multiplication) and a unary operation (inversion). As universal algebras, completely regular semigroups form a variety determined by the identities

$$x = xx^{-1}x, xx^{-1} = x^{-1}x, (x^{-1})^{-1} = x.$$

Let CR denote this variety and L_{CR} denote the lattice of subvarieties of CR.

In 1975, M. Petrich [1] showed that $B \vee G = OBG$, where B is the variety of bands, G is the variety of groups and OBG is the variety of orthodox bands of groups. In 1980, T.E. Hall and P.R. Jones [2] showed that $B \vee CS = POBG$, where CS is the variety of completely simple semigroups and $POBG$ is the variety of pseudo-orthodox bands of groups. In 1985, N.R. Reilly [3] showed that $O \vee CS = O^+ \cap PO$, where $O^+ = \{S \in CR : S/\mu \in O\}$, μ is the maximum

Received August 28, 1995.

AMS Subject Classification. 20M07.

*This work is supported by National Natural Science Foundation and Doctoral Research Program Foundation of Chinese Education Committee. The author wishes to thank Professor Y. Q. Guo for his help.

idempotent-separating congruence, O is the variety of orthodox completely regular semigroups. In this paper, we investigate the variety OLBG[ORBG] of orthodox completely regular semigroups on which Green's relation H is a left [right] congruence, In Theorem 3.4, the join of OLBG and ORBG is described.

We shall use the following notation.

L^0: the largest congruence contained in L;

R^0: the largest congruence contained in R;

μ: the maximum idempotent-separating congruence;

τ: the maximum idempotent pure congruence;

I_s: the equality relation on a semigroup S;

$[u_\alpha = v_\alpha]_{\alpha\in A}$: the variety of completely regular semigroups determined by the identities $u_\alpha = v_\alpha (\alpha \in A)$.

LBG[RBG]: the variety of completely regular semigroups on which Green's relation H is a left [right] congruence.

Let $a^0 = aa^{-1} = a^{-1}a$. Then $B = [x^2 = x]$, $O = [(x^0y^0)^0 = x^0y^0]$, and $SLG = [x^0y = yx^0]$ is the variety of Clifford semigroups.

For undefined notation or terminology see [4].

2. OLBG

Lemma 2.1. *Let $S \in CR$. Then $S \in O$ if and only if $S/\tau \in SLG$.*

Proof. Let $S \in O, v = \{(a,b) \in S \times S : V(a) = V(b)\}$. Then S/v is an inverse semigroup (see [4]), hence S/v is a Clifford semigroup, i.e. $S/v \in SLG$. Obviously v is idempotent pure and thus $v \subseteq \tau$. Therefore $S/\tau \in SLG$.

Conversely, Let $S/\tau \in SLG$, e,$f \in E(S)$. Then $ef\tau \in E(S/\tau)$. By Lallement's Lemma (see [4]), there exists $g \in E(S)$ such that $(ef, g) \in \tau$. Since τ is idempotent pure, $ef \in E(S)$. Therefore S is orthodox.

Obviously we have

Lemma 2.2. *Let S be a semigroup. Then $\mu \subseteq L^0$.*

Now we can prove the following theorem:

Theorem 2.3. *The following conditions on a completely regular semigroup S are equivalent.*

(I) $S \in OLBG$.

(II) $S/L^0 \in B, S/\tau \in SLG$.

(III) $S/L^0 \in LBG, S/\tau \in SLG$.

(IV) $S/\mu \in LBG, S/\tau \in SLG$.

(V) *S satisfies the identities* $(xy)^0 = (xy^0)^0, (x^0y^0)^0 = x^0y^0$.

(VI) *S satisfies the identity* $(x^0y)^0 = x^0y^0$.

Proof. (I) implies (IV). It follows directly from Lemma 2.1.

(IV) implies (III). It follows directly from Lemma 2.2.

(III) implies (II). Let $S/L^0 \in$ LBG. Let L_1^0 be the largest congruence contained in the Green's L-relation on S/L^0, H be the Green's H-relation on S/L^0. Then H is a left congruence and thus $H \subseteq L_1^0$. We may define a function from S to $S/L^0/L_1^0$ by

$$\Phi : a \to \overline{a}L_1^0 \quad (a \in S)$$

where $\overline{a} = aL^0$. Then Φ is a surjective homomorphism. Since $H \subseteq L_1^0, S/L^0/L_1^0 \in B$. Let ρ be the kernel of Φ. Let $(a, b) \in \rho$, i.e. $(\overline{a}, \overline{b}) \in L_1^0$. Then there exists $\overline{x} \in S/L^0$ such that $\overline{a} = \overline{x}\overline{b}$, i.e. $(a, xb) \in L^0$, and thus there exists $y \in S$ such that $a = yxb \in Sb$. Similarly $b \in Sa$, hence aLb. So $\rho \subseteq L$ and thus $\rho \subseteq L^0$. Therefore $S/L^0 \in B$ since $S/\rho \in B$.

(II) implies (V). The hypothesis implies that yL^0y^0, hence xyL^0xy^0. Since $xyRxy^0$, $xyHxy^0$, i.e. $(xy)^0 = (xy^0)^0$.

Since $S/\tau \in SLG$, $S \in O$ by Lemma 2.1, and thus $(x^0y^0)^0 = x^0y^0$.

(V)implies (VI). This is obvious.

(VI)implies (I). Let $e, f \in E(S)$, then $ef = (ef)^0 \in E(S)$, hence $S \in O$.

Let $x, y, z \in S, xHy$. Then $(z^0x)^0 = z^0x^0 = z^0y^0 = (z^0y)^0$. Hence $zx = zz^0x = zz^0x(z^0x)^0 = zz^0x(z^0y)^0 = zz^0x(z^0y)^{-1}z^{-1}zy \in Szy$. Similarly $zy \in$

Szx and thus $zxLzy$. Obviously $zxRzy$ and thus $zxHzy$. Therefore H is a left congruence, as required.

In view of Theorem 2.3, OLBG is a variety and OLBG$= [(x^0y)^0 = x^0y^0]$. Dually ORBG $= [(xy^0)^0 = x^0y^0]$.

3. OLBG ∨ ORBG

Notation 3.1. ([6]) Let $V \in L_{CR}$, define

$$V^{T1} = \{S \in CR : S/L^0 \in V\},$$
$$V^{Tr} = \{S \in CR : S/R^0 \in V\}.$$

Lemma 3.2. ([5, lemma 2.4] [6]) *Let $V \in L_{CR}$, and $V = [u_\alpha = v_\alpha]_{\alpha\in A}$. Then*

$$V^{Tr} = [(u_\alpha x)^0 = (v_\alpha x u_\alpha x)^0, (v_\alpha x)^0 = (u_\alpha x v_\alpha x)^0]_{\alpha\in A},$$
$$V^{T1} = [(x u_\alpha)^0 = (x u_\alpha x v_\alpha)^0, (x v_\alpha)^0 = (x v_\alpha x u_\alpha)^0]_{\alpha\in A}.$$

Lemma 3.3. *Let S be a completely regular semigroup. Then $\tau \cap H = Is$.*

Proof. Let $a, b \in He$, $e \in E(S)$, and $(a, b) \in \tau$. Then $(aa^{-1}, ba^{-1}) \in \tau$. Since τ is idempotent pure, $ba^{-1} \in E(S)$. Since $a, b \in He$, $ba^{-1} \in He$, and thus $ba^{-1} = e$. Therefore $a = b$, as required.

Theorem 3.4. *The following conditions on a completely regular semigroup S are equivalent.*

(I) $S \in OLBG \vee ORBG$.

(II) *S is a subdirect product of a semigroup in OLBG and a semigroup in ORBG.*

(III) $S \in O \cap LBG^{Tr} \cap RBG^{T1}$.

(IV) $S/\tau \in SLG$, $S/R^0 \in LBG$, $S/L^0 \in RBG$.

Proof. (I) implies (III). By implication (I)⇒(II) of Theorem 2.3 and its dual, it is easily checked that OLBG$\subseteq O \cap \mathrm{LBG}^{Tr} \cap \mathrm{RBG}^{T1}$, ORBG $\subseteq$

$O \cap \mathrm{LBG}^{Tr} \cap \mathrm{RBG}^{T1}$. Therefore OLBG $\vee$ ORBG$\subseteq O \cap \mathrm{LBG}^{Tr} \cap \mathrm{RBG}^{T1}$, as required.

(III) implies (IV). This follows directly from Notation 3.1 and Lemma 2.1.

(IV) implies (II). By Lemma 3.3, $\tau \cap R^0 \cap L^0 = Is$, and thus S is a subdirected product of $S/\tau \cap R^0$ and S/L^0. Since $S/\tau \cap R^0$ is a subdirect product of S/τ and S/R^0, $S/\tau \cap R^0 \in$ SLG $\vee$ LBG $=$ LBG. By Lemma 2.1, $S \in O$, hence $S/\tau \cap R^0 \in$ OLBG, $S/L^0 \in$ ORBG. Therefore S is a subdirect product of a semigroup in OLBG and a semigroup in ORBG, as required.

(II) implies (I). This is obvious.

4. $V\cap$ (OLBG $\vee$ ORBG)

Theorem 4.1. *Let $V \in L_{CR}$. Then*

$$V \cap (OLBG \vee ORBG) = (V \cap OLBG) \vee (V \cap ORBG).$$

Proof. Obviously $V\cap$ OLBG$\subseteq V \cap$ (OLBG $\vee$ ORBG), $V \cap$ ORBG $\subseteq V \cap$ (OLBG $\vee$ ORBG), and thus

$$(V \cap \mathrm{OLBG}) \vee (V \cap \mathrm{ORBG}) \subseteq V \cap (\mathrm{OLBG} \vee \mathrm{ORBG}).$$

Conversely, let $S \in V \cap (\mathrm{OLBG} \vee \mathrm{ORBG}) \subseteq O$. Then by Theorem 3.4, $S/\tau \in V \cap SLG$, $S/R^0 \in V \cap$ ORBG. By Lemma 3.3, $\tau \cap R^0 \cap L^0 = Is$, and thus S is a subdirect product of S/τ,S/L^0 and S/R^0, hence $S \in (V \cap SLG) \vee (V \cap \mathrm{OLBG}) \vee (V \cap \mathrm{ORBG}) = (V \cap \mathrm{OLBG}) \vee (V \cap \mathrm{ORBG})$. The required conclusion now follows.

Theorem 4.2. *Let $V \in L_{CR}$, and $SLG \subseteq V$. Then*

$$(V \cap OLBG) \vee (V \cap ORBG) = O \cap (V \cap OLBG)^{Tr} \cap (V \cap ORBG)^{T1}.$$

Proof. By the implication (I)$\Rightarrow$(II) of Theorem 2.3 and its dual, it is easily checked that $V\cap$ OLBG, $V \cap$ ORBG $\subseteq O \cap (V \cap \mathrm{OLBG})^{Tr} \cap (V \cap \mathrm{ORBG})^{T1}$, and thus $(V \cap \mathrm{OLBG}) \vee (V \vee \mathrm{ORBG}) \subseteq O \cap (V \cap \mathrm{OLBG})^{Tr} \cap (V \cap \mathrm{ORBG})^{T1}$.

Conversely, Let $S \in O \cap (V \cap \text{OLBG})^{Tr} \cap (V \cap \text{ORBG})^{T1}$. Then by Lemma 2.1 and Notation 3.1, $S/\tau \in SLG$, $S/R^0 \in V \cap \text{OLBG}$, $S/L^0 \in V \cap \text{ORBG}$. By Lemma 3.3, $\tau \cap R^0 \cap L^0 = Is$, and thus S is a subdirect product of S/τ, S/L^0 and S/R^0, and thus $S \in (V \cap \text{OLBG}) \vee (V \cap \text{ORBG}) \vee SLG = (V \cap \text{OLBG}) \vee (V \cap \text{ORBG})$ since $SLG \subseteq V$, as required.

If $SLG \subseteq V \subseteq \text{OLBG} \vee \text{ORBG}$, then by Theorem 4.1 and Theorem 4.2, $V = (V \cap \text{OLBG}) \vee (O \cap \text{ORBG}) = O \cap (V \cap \text{OLBG})^{Tr} \cap (V \cap \text{ORBG})^{T1}$.

And if $V \cap \text{OLBG} = [u_\alpha = v_\alpha]_{\alpha \in A}$, $V \cap \text{ORBG} = [s_\beta = t_\beta]_{\beta \in B}$, then by Lemma 3.2, the identity of V could be given.

References

[1] M. Petrich, *Varietiesoforthodoxbandsofgroups*, Pacific J. Math., 91(1980), 209-217.

[2] T.E. Hall and P.R. Jones,*Onthelatticeofvarietiesofbandsofgroups*, 91(1980), 327-337.

[3] N.R. Reilly, *Varietiesofcompletelyregularsemigroups*, J. Austral. Math. Soc. (series A) 38(1985), 372-393.

[4] J.M. Howis, An introduction to semigroup theory, Academic Press, London/ New York, 1976.

[5] Shuhua Zhang, *CertainoperatorsrelatedtoMalφcevProductsonvarietiesofcompletelyregularsemi-groups*, J. Algebra 168(1994), 249-272.

[6] M. Petrich and N.R. Reilly, *Operatorsrelatedtoidempotentgeneratedandmonoidcompletelyreg-ularsemigroups*, J. Austral. Math . Soc., 49(1990), 1-29.

Department of Mathematics, Lanzhon University, Lanzhon 730000, China.