Model Checking Recursive Programs with Numeric Data Types

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Abstract. Pushdown systems (PDS) naturally model sequential recursive programs. Numeric data types also often arise in real-world programs. We study the extension of PDS with unbounded counters, which naturally model numeric data types. Although this extension is Turing-powerful, reachability is known to be decidable when the number of reversals between incrementing and decrementing modes is bounded. In this paper, we (1) pinpoint the decidability/complexity of reachability and linear/branching time model checking over PDS with reversal-bounded counters (PCo), and (2) experimentally demonstrate the effectiveness of our approach in analysing software. We show reachability over PCo is NP-complete, while LTL is coNEXP-complete (coNP-complete for fixed formulas). In contrast, we prove that EF-logic over PCo is undecidable. Our NP upper bounds are by a direct poly-time reduction to satisfaction over existential Presburger formulas, allowing us to tap into highly optimized solvers like Z3. Although reversal-bounded analysis is incomplete for PDS with unbounded counters in general, our experiments suggest that some intricate bugs (e.g. from Linux device drivers) can be discovered with a small number of reversals. We also pinpoint the decidability/complexity of various extensions of PCo, e.g., with discrete clocks.

1 Introduction

Pushdown systems (PDS) are natural abstractions of sequential programs with unbounded recursions whose verification problems have been extensively studied (cf. [6, 13, 32]). In addition to recursions, numerical data types commonly arise in real-world programs. A standard approach to these potentially infinite domains is to map them into abstract domains that are more amenable to analysis (e.g. finite ones like {Pos, Neg, Zero}, or infinite ones expressed by intervals, difference bound matrices, polyhedra, etc.). For other types of program analysis, it is also common to simply ignore numerical data types. For a comprehensive treatment of these techniques, and others, the reader is referred to the survey [12].

In this paper, we study a different approach. Motivated by the success of pushdown systems (or similar models like boolean programs) in software model checking (cf. [2, 3, 27]), we aim to investigate extensions of pushdown systems with numerical data types and preserve nice properties, such as decidability and good complexity. A clean and simple approach is to enrich pushdown systems with unbounded counters, which can be incremented/decremented and tested for zero. Unfortunately, this model is Turing-powerful, even without the stack.

One way to retain decidability of reachability is to impose an upper bound r on the number of reversals between incrementing and decrementing modes for each counter (cf. [11, 18]). In fact, decidability holds even if discrete clocks are added to the model [11]. On the other hand, the complexity of reachability over these models is still open; a simple analysis of [11, 18] yields at least double exponential-time complexity for their algorithms. Recently, the authors of [?] observed that replacing the use of Parikh's Theorem [?] in [18] by a recently improved version of [31] immediately yields an NP procedure for this reachability problem (without clocks) for *fixed* numbers of reversals and counters, yielding only an NEXP procedure in general. Furthermore, the decidability of linear/branching time model checking over these models is also unknown.

There are at least two potential applications of reversal-bounded model checking (cf. [11, 19]). First, it could be used as a sound but *incomplete* verification technique for the case of unbounded reversals. Despite this incompleteness, results in bounded model checking suggest that "shallow" bugs are common in practice (cf. [12]). Clearly, reversal-bounded model checking is an infinite-state generalization of bounded model checking over counter systems. A second application is the use of reversal-bounded counters for tracking the number of times certain actions have been executed to reach the current configuration. For example, we can check the existence of a computation path in a recursive program where the number of invocations for the functions f_1, f_2, f_3, and f_4 are the same. Similar counting properties (and their model checking problems) have been studied in many other contexts (cf. [20] and references therein).

Contributions. We begin by studying pushdown systems enriched with reversal-bounded counters, which can be compared against and incremented by constants given in binary, but without clocks (PCo). This model is more general than the model studied in [?, 18], which allows only counters to be compared against 0 and incremented by $\{-1, 0, 1\}$, though at the cost of an exponential blow-up (cf. [19]) our model can be translated into their model. Our main contributions are (1) to pinpoint the decidability/complexity of reachability and linear/branching time model checking over PCo, and (2) experimentally demonstrate the effectiveness of our approach in the analysis of software.

We show that reachability over PCo is NP-complete, while LTL model checking is coNEXP-complete (coNP-complete for fixed formulas). In contrast, we prove that model checking EF-logic over PCo is undecidable. All of our lower bounds hold already for PDS with one 1-reversal counter wherein numeric constants, which can be either be compared against or used to increment/decrement counters, are restricted to 0 or 1. Our NP upper bounds are established by a direct poly-time reduction to satisfaction over existential Presburger formulas, allowing us to tap into highly optimized solvers like Z3 [?]. This reduction also permits additional constraints on the number of actions executed and the values of the counters at the end of the run (also specified in existential Presburger arithmetic) without further computation overhead in the reduction. We have implemented our algorithm and use it to analyse several examples, including some derived from memory management issues in Linux device drivers. Although reversal-bounded analysis is only complete up to the bound on the number of reversals, our experiments suggest that many subtle bugs manifest themselves even within a small number of reversals, which our tool can detect reasonably fast. Without increasing complexity, our algorithm can also check whether a given PDS $\mathcal{P}$ with unbounded counters is r-reversal bounded, for a given input $r \in \mathbb{N}$; note that this is not the same as deciding whether $\mathcal{P}$ is reversal-bounded, which is undecidable [**?**]. In the case when $\mathcal{P}$ is r-reversal bounded, our technique gives a complete coverage of the infinite state space, which suggests the usefulness of our technique in proving correctness as well as finding bugs.

We then study the extension of PCo with discrete clocks (PCC). We show that LTL model checking over PCC is still coNEXP-complete, but hardness holds even for a fixed formula. Similarly, we show that reachability over PCC is NEXP-complete. We also show that, without reversal-bounded counters, model checking EF-logic over PCC is EXPSPACE-complete even for a fixed formula.

Related work. The complexities of most standard model checking problems over pushdown systems are well-understood. In relation to our results, we mention that LTL model checking over PDS is EXP-complete and is P-complete for fixed formulas [6] (the latter is also the complexity for reachability), whereas model checking EF-logic is PSPACE-complete [6, 32]. Therefore, adding reversal-bounded counters yields computationally harder problems in both cases.

Over reversal-bounded counter systems (without stack), reachability is NP-complete but becomes NEXP-complete when the number of reversals is given in binary [17]. On the other hand, when the numbers of reversals and counters are fixed, the problem is solvable in P [14]. The techniques developed by [14, 17], which reason about the maximal counter values, are very different to our techniques, which exploit the connection to Parikh images of pushdown automata (first explicated in Ibarra's original paper [18] though not in a way that gives optimal complexity or practical algorithm). For LTL model checking, the problem is solvable in EXP even in the presence of discrete clocks [28], whereas EF-logic model checking is still decidable but becomes undecidable for CTL [28].

For discrete-timed systems, reachability is known to be PSPACE-complete [1], where hardness holds already for three clocks [10]. Using region graph constructions [1], LTL model checking and EF-logic can also be easily shown to be PSPACE-complete. Note that we do not consider timed logics (cf. [7]). The complexities of pushdown systems with clocks have also been studied, e.g., [9].

Organization. §2 contains preliminaries. In §3, we define the basic model PCC that we study. §4 and §5, contain upper and lower bounds for model checking PCo. In §6, we extend our results to PCC. Experimental results are given in §7. Other extensions and future work are given in §8. Due to the space limit, some proofs are in the full version from the project page http://www.cs.ox.ac.uk/recount.

2 Preliminaries

Transition systems. An *action alphabet* ACT is a finite nonempty set of *actions.* A *transition system* over ACT is a tuple $\mathfrak{S} = \langle S, \{\to_a\}_{a \in \mathsf{ACT}} \rangle$, where S

is a set of *configurations* and each $\rightarrow_a \subseteq S \times S$ is an a-labeled *transition relation* containing the set of all a-labeled *transitions*. We use $\rightarrow$ to denote the union of all $\rightarrow_a$. A *(computation) path* in $\mathfrak{S}$ is a finite or infinite sequence $\pi = \alpha_0 \rightarrow_{a_1} \alpha_1 \rightarrow_{a_2} \ldots$ such that $\alpha_i \in S$, $\alpha_i \rightarrow_{a_{i+1}} \alpha_{i+1}$ and $a_i \in \mathsf{ACT}$ for each i. If π is finite, let $\mathbf{last}(\pi)$ denote the last configuration in π. In this case, $a_1 a_2 \ldots$ is said to be a *(finite) trace in* $\mathfrak{S}$ *from* α_0 *to* $\boldsymbol{last}(\pi)$. **Automata** We assume familiarity with automata theory. In particular, nondeterministic Büchi automata (NBWA), cf. [29], and pushdown automata (PDA), cf. [26]. For an NBWA $\mathcal{A}$, we denote by $\mathcal{L}(\mathcal{A})$ the language recognized by $\mathcal{A}$. Similarly, given a PDA we also write $\mathcal{L}(\mathcal{P})$ for the language $\mathcal{P}$ recognizes.
Parikh images Given an alphabet $\Sigma = \{a_1, \ldots, a_k\}$ and a word $w \in \Sigma^*$, we write $\mathbb{P}(w)$ to denote a tuple with $|\Sigma|$ entries where the ith entry counts the number of occurrences of a_i in w. Given a language $\mathcal{L} \subseteq \Sigma^*$, we write $\mathbb{P}(\mathcal{L})$ to denote the set $\{\mathbb{P}(w) : w \in \mathcal{L}\}$. We say that $\mathbb{P}(\mathcal{L})$ is the *Parikh image* of $\mathcal{L}$.
Logic The syntax of LTL (cf. [16, 28, 29]) over ACT is given by: $\varphi, \varphi' := a \ (a \in \mathsf{ACT}) \mid \neg\varphi \mid \varphi \vee \varphi' \mid \varphi \wedge \varphi' \mid \mathbf{X}\varphi \mid \varphi \mathbf{U} \varphi'$. Given an ω-word $w \in \mathsf{ACT}^\omega$ and an LTL formula φ over ACT, we define the satisfaction relation $w \models \varphi$ in the standard way. Write $[\![\varphi]\!]$ for all $w \in \mathsf{ACT}^\omega$ such that $w \models \varphi$. EF-logic (over ACT) is a fragment of CTL (cf. [16, 28]) with the syntax $\varphi, \psi := \top \mid \neg\varphi \mid \varphi \vee \psi \mid \langle a \rangle \varphi \ (a \in \mathsf{ACT}) \mid \mathsf{EF}\varphi$. Given an EF formula φ, a transition system $\mathfrak{S} = \langle S, \{\rightarrow_a\}_{a \in \mathsf{ACT}} \rangle$ and $s \in S$, we may define $\mathfrak{S}, s \models \varphi$ in the standard way.
Presburger formulas Presburger formulas are first-order formulas over natural numbers with addition. Here, we use extended existential Presburger formulas $\exists x_1, \ldots, x_k.\varphi$ where φ is a boolean combination of expressions $\sum_{i=1}^{k} a_i x_i \sim b$ for constants $a_1, \ldots, a_k, b \in \mathbb{Z}$ and $\sim \in \{\leq, \geq, <, >, =\}$ with constants represented in binary. It is known that satisfiability of existential Presburger formulas is NP-complete even with these extensions (cf. [25]).

3 Pushdown systems with counters and clocks

The model. An *atomic clock constraint* on clock variables $Y = \{y_1, \ldots, y_m\}$ is simply an expression of the form $y_i \sim c$ or $y_i - y_j \sim c$, where $\sim \in \{<, >, =\}$, $1 \leq i, j \leq m$ and $c \in \mathbb{Z}$. An *atomic counter constraint* on counter variables $X = \{x_1, \ldots, x_n\}$ is simply an expression of the form $x_i \sim c$, where $c \in \mathbb{Z}$. A *clock-counter (CC) constraint* θ on (X, Y) is simply a boolean combination of atomic counter constraints on X and atomic clock constraints on Y. Given a valuation $\nu : X \cup Y \rightarrow \mathbb{N}$ to the counter/clock variables, we can determine whether $\theta[\nu]$ is true or false by replacing a variable z by $\nu(z)$ and evaluting the resulting arithmetic expressions in the obvious way. Let $Const_{X,Y}$ denote the set of all CC constraints on (X, Y). Intuitively, these formulas will act as "enabling conditions" (or "guards") to determine whether certain transitions can be fired.

A *pushdown system with* n *counters and* m *discrete clocks* is a tuple $\mathcal{P} = (Q, \mathsf{ACT}, \Gamma, \delta, X, Y)$ where (1) Q is a finite set of states, (2) ACT is a set of action labels, (3) Γ is a stack alphabet, (4) $X = \{x_1, \ldots, x_n\}$ is a set of counter variables, (5) $Y = \{y_1, \ldots, y_m\}$ is a set of clock variables, and (6) δ is a transition relation of $\mathcal{P}$, which is defined to be a finite subset of $(Q \times \Gamma^* \times Const_{X,Y}) \times \mathsf{ACT} \times (Q \times \Gamma^* \times 2^Y \times \mathbb{Z}^n)$. A *configuration* of $\mathcal{P}$ is a tuple $(q, u, \mathbf{u}, \mathbf{v}) \in Q \times \Gamma^* \times \mathbb{N}^n \times \mathbb{N}^m$. Given $a \in \mathsf{ACT}$ and two configurations $\alpha_1 = (q, u, \mathbf{u}, \mathbf{v})$ and $\alpha_2 = (q', u', \mathbf{u}', \mathbf{v}')$ of $\mathcal{P}$, $\alpha_1 \to_{a,\mathcal{P}} \alpha_2$ iff there exists $\langle (q, w, \theta), a, (q', w', Y', \mathbf{w}) \rangle \in \delta$ such that

- θ holds under the valuation $(\mathbf{u}, \mathbf{v})$,
- for some $v \in \Gamma^*$, $u = wv$ and $u' = w'v$,
- if $Y' = \emptyset$, then each clock progresses by one time unit, i.e., $\mathbf{v}' = \mathbf{v} + \mathbf{1}$; if $Y' \neq \emptyset$, then the clocks in Y' are reset, while other clocks do not change, i.e., for each $y_i \in Y$, set $v'_i := 0$ and, for each $y_i \notin Y$, set $v'_i := v_i$.
- $\mathbf{u}' := \mathbf{u} + \mathbf{w}$ (note, all elements of $\mathbf{u}'$ must be non-negative).

The transition system generated by $\mathcal{P}$ is $\mathfrak{S}_{\mathcal{P}} = \langle S, \{\to_a\}_{a \in \mathsf{ACT}} \rangle$, where S denotes the set of configurations of $\mathcal{P}$ and $\to_a$ is defined to be $\to_{a,\mathcal{P}}$.

Notice that in this model, which is similar to the model given in [11], clocks are updated via transitions. This is slightly different from the usual definition of discrete timed systems (cf. [1]) in which (1) clocks may progress within any particular state of the system without taking any transitions as long as some *invariants* are satisfied, and (2) transitions are *instantaneous*. However, by introducing self-looping transitions with no reset and adding an extra "dummy" clock which always resets for old transitions, we can easily construct a weakly bisimilar system in our model. See [11] for more details.

Let us now define the r-reversal-bounded variant of this model for each $r \in \mathbb{N}$. Syntactically, a *pushdown system with n r-reversal bounded counters and m discrete clocks* is simply a pair $(r, \mathcal{P})$ of number r and a pushdown system $\mathcal{P}$ with n counters and m clocks. Together with a given initial configuration α of $\mathcal{P}$, the system $(r, \mathcal{P})$ generates a transition system $\mathfrak{S}^{\alpha}_{(r,\mathcal{P})} = \langle S, \{\to_a\}_{a \in \mathsf{ACT}} \rangle$ defined as follows. Let $\mathfrak{S}_{\mathcal{P}} = \langle S', \{\to'_a\}_{a \in \mathsf{ACT}} \rangle$ be the transition system generated by $\mathcal{P}$. A path π in $\mathfrak{S}_{\mathcal{P}}$ from α is said to be *r-reversal-bounded* if each counter of $\mathcal{P}$ changes from a non-incrementing mode to non-decrementing mode (or vice versa) at most r times in the path π. For example, if the values of a counter x in a path π from α are $1, 1, 1, 2, 3, 4, 4, \overline{4, 3}, 2, \overline{2, 3}$, then the number of reversals of x is 2 (reversals occur in between the overlined positions). This sequence has three *phases* (i.e. subpaths interleaved by consecutive reversals or end points): non-decrementing, non-incrementing, and finally non-decrementing. This intuition suffices for understanding the main ideas in this paper, though we provide the detailed definitions in the full version. The set S is then defined to be the set of all finite r-reversal-bounded paths from α. Given two such paths π and π' such that $\pi' = \pi, \alpha'$, we define $\pi \to_a \pi'$ iff $\mathbf{last}(\pi) \to'_a \alpha'$. Notice that $\mathfrak{S}^{\alpha}_{(r,\mathcal{P})}$ is a tree.

We write (r, n, m)-PCC to denote the set of all pushdown systems with n r-reversal-bounded counters and m discrete clocks. We write PCC to denote the union of all (r, n, m)-PCC for all $r, n, m \in \mathbb{N}$. Similarly, we use (r, n)-PCo to denote $(r, n, 0)$-PCC and (r, m)-PCl to denote $(r, 0, m)$-PCC. We use PCo and PCl as well in the same way.

Unless stated otherwise, we make the following conventions: (1) the number r of reversals is given in unary, and (2) numeric constants in CC constraints *and counter increments are given in binary.* In the sequel, we will deal with (control-state) reachability, model checking LTL and EF over PCC (and their variants) defined as follows:

- *Reachability*: given a PCC $\mathcal{P}$ and two configurations α, α' of $\mathcal{P}$ (in binary representation), decide whether α' is reachable in $\mathfrak{S}_{\mathcal{P}}^{\alpha}$.
- *Control-state reachability*: given a PCC $\mathcal{P}$ and two states q, q' of $\mathcal{P}$, decide whether there exist stack contents $u, u' \in \Gamma$ and counter values $\mathbf{c}, \mathbf{c}' \in \mathbb{N}^k$ such that $(q', u', \mathbf{c}')$ is reachable in $\mathfrak{S}_{\mathcal{P}}^{(q,u,\mathbf{c})}$.
- *LTL model checking*: given a PCC $\mathcal{P}$, a configuration α of $\mathcal{P}$ (in binary), and an LTL formula φ, decide whether $\mathfrak{S}_{\mathcal{P}}^{\alpha}, \alpha \models \varphi$.
- *EF model checking*: given a PCC $\mathcal{P}$, a configuration α of $\mathcal{P}$ (in binary), and an EF formula φ, decide whether $\mathfrak{S}_{\mathcal{P}}^{\alpha}, \alpha \models \varphi$.

4 Upper bounds for model checking PCo

In this section, we show that reachability over PCo is in NP by providing a direct poly-time reduction to satisfactions of existential Presburger formulas. As applications of our technique, we will provide: (1) an NP upper bound for control-state reachability with additional constraints on counter values at the beginning/end of the run and how many times actions are executed at the end of the run, and (2) a coNEXP upper bound for LTL model checking over PCo (coNP for fixed formulas). Lower bounds are proved in the next section.

An NP procedure for reachability over PCo. We prove the following.

Theorem 1. *Reachability over PCo is* NP*-complete. In fact, it is poly-time reducible to checking satisfactions of existential Presburger formulas.*

Given an (r, k)-PCo $\mathcal{P} = (Q, \mathsf{ACT}, \Gamma, \delta, X)$, and two configurations $\alpha = (q, u, \mathbf{c})$ and $\alpha' = (q', u', \mathbf{c}')$ of $\mathcal{P}$, our algorithm decides if α' is reachable from α in $\mathfrak{S}_{\mathcal{P}}$. Let $\mathbf{c} = (c_1, \ldots, c_k)$ and $\mathbf{c}' = (c'_1, \ldots, c'_k)$. We assume that $u = u' = \perp$ since, by hardwiring u and u' into the finite control of $\mathcal{P}$ the standard way, the PCo $\mathcal{P}$ may initialize the stack content to u and make sure the final stack content is precisely u'. In addition, let $d_1 < \ldots < d_m$ denote all the numeric constants appearing in an atomic counter constraint as a part of CC constraints in $\mathcal{P}$. Without loss of generality, we assume that $d_1 = 0$ for notational convenience. Let $\mathtt{REG} = \{\varphi_1, \ldots, \varphi_m, \psi_1, \ldots, \psi_m\}$ be a set of formulas defined as follows. Note that these formulas partition $\mathbb{N}$ into $2m$ pairwise disjoint regions.

$$\varphi_i(x) \equiv x = d_i, \quad \psi_i(x) \equiv d_i < x < d_{i+1} \ (1 \leq i < m), \quad \psi_m(x) \equiv d_m < x \ .$$

A vector $\mathbf{v}$ in $\mathbf{Modes} = \mathtt{REG}^k \times [0, r]^k \times \{\uparrow, \downarrow\}^k$ is said to be a *mode vector*. Given a path $\pi = \alpha_0\alpha_1 \ldots \alpha_h$ from α to α', we may associate a mode vector $\mathbf{v}_j$ to each configuration α_i in π that records for each counter: which region its value is in, how many reversals its used, and whether its phase is non-decrementing ($\uparrow$) or non-incrementing ($\downarrow$). Consider the sequence $\sigma = \{\mathbf{v}_j\}_{j=0}^{h}$ of mode vectors. A crucial observation is that each mode vector $\mathbf{v} \in \mathbf{Modes}$ in this sequence occurs in a contiguous block, i.e., if $0 \leq j \leq j' \leq h$ are such that $\mathbf{v}_j$ (resp. $\mathbf{v}_{j'}$) is the first (resp. last) time $\mathbf{v}$ appearing in σ, then $\mathbf{v}_l = \mathbf{v}$ for all $l \in [j, j']$. Intuitively, once a change occurs in σ, we cannot revert to the previous vector. This is because any such change will incur an extra reversal for at least one counter. There are at most $N_{\max} := |\texttt{REG}| \times (r+1) \times k = 2mk(r+1)$ distinct mode vectors in σ.

In outline, to avoid an exponential blow-up in our reduction, we will first construct a very rough "upperapproximation" of the PCo $\mathcal{P}$ as a PDA $\mathcal{P}'$. Intuitively, $\mathcal{P}'$ will simulate $\mathcal{P}$, while guessing and remembering *only* how many mode vector changes have occurred, but disregarding the counter information. In this case, there are runs in $\mathcal{P}'$ that are not valid in $\mathcal{P}$. Each time $\mathcal{P}'$ fires a transition t (derived from a transition t' of $\mathcal{P}$ by disregarding counters), it will also output information about the counter tests and in(de)crements associated with t', and how many changes in the mode counter have occurred thus far (recorded in the states of $\mathcal{P}'$). Since $\mathcal{P}'$ is of polynomial size, we apply on $\mathcal{P}'$ the linear-time algorithm of Verma *et al.* [31] to compute the Parikh images of CFGs (equivalently, PDA). Building on the output formula, we use further existential quantifications to guess the evolution of the mode vectors, on which we impose further constraints to eliminate invalid runs. We give the details below.

Building the PDA $\mathcal{P}$. Define $\mathcal{P}' = (Q', \mathsf{ACT}', \Gamma, \delta', (q,0), F')$ allowing transitions to execute a (finite) *sequence* of actions, instead of just one. These are for convenience and can be encoded in the states of $\mathcal{P}'$. Let $Q' = Q \times [0, N_{\max}-1]$ and define ACT' implicitly from δ'. In fact, ACT' is a (finite) subset of $\{(\texttt{ctr}_i, u, j, l) : i \in [1,k], u \in \mathbb{Z}, j \in [0, N_{\max}-1], l \in \{0,1\}\} \cup (Const_{X,\emptyset} \times [0, N_{\max}-1])$. Here, $l \in \{0,1\}$ signifies whether this action changes the mode vector. We define δ' by initially setting $\delta' = \emptyset$ and adding rules as follows. If $\langle (q, w, \theta), a, (q', w', \mathbf{u}) \rangle \in \delta$ where $\mathbf{u} = (u_1, \ldots, u_k)$ then for each $i \in [0, N_{\max}-1]$ we add

$$\langle ((q,i),w), (\theta,i)(\texttt{ctr}_1,u_1,i,0)(\texttt{ctr}_2,u_2,i,0)\ldots(\texttt{ctr}_k,u_k,i,0), ((q,i),w') \rangle$$

to δ'. If $i \in [0, N_{\max}-1)$, we also add the following rule:

$$\langle ((q,i),w), (\theta,i)(\texttt{ctr}_1,u_1,i,1)(\texttt{ctr}_2,u_2,i,1)\ldots(\texttt{ctr}_k,u_k,i,1), ((q,i+1),w') \rangle$$

In this way, $\mathcal{P}'$ makes "visible" the counter tests and the counter updates performed. Finally, the set F' of final states are defined to be $\{q'\} \times [0, N_{\max}-1]$ and the initial control state is $(q,0)$.

Constructing the formula. Fix an ordering on ACT', say $\alpha_1 < \ldots < \alpha_l$. For convenience, by f we denote a function mapping α_i to i for each $i \in [1,l]$. We apply the linear-time algorithm of [31] on $\mathcal{P}'$ above to obtain $\chi(\mathbf{z})$, where $\mathbf{z} = (z_1, \ldots, z_l)$, such that for each $\mathbf{n} \in \mathbb{N}^l$ we have $\mathbf{n} \in \mathbb{P}(\mathcal{L}(\mathcal{P}'))$ iff $\chi(\mathbf{n})$ holds. We impose constraint χ to eliminate vectors that do not correspond to traces of $\mathcal{P}'$. Currently, $\mathcal{P}'$ knows only the maximum number of allowed changes in mode vectors, but is not "aware" of other information about the counters. Therefore, the formula that we construct should assert the existence of a valid sequence of mode vectors that respect the counter tests and updates that $\mathcal{P}'$ outputs. We construct `HasRun` of the form

$$\exists \mathbf{z} \exists \mathbf{m}_0, \ldots, \mathbf{m}_{N_{\max}-1} \left(\begin{array}{c} \mathtt{Init}(\mathbf{m}_0) \wedge \mathtt{GoodSeq}(\mathbf{m}_0, \ldots, \mathbf{m}_{N_{\max}-1}) \wedge \chi(\mathbf{z}) \wedge \\ \mathtt{Respect}(\mathbf{z}, \mathbf{m}_0, \ldots, \mathbf{m}_{N_{\max}-1}) \wedge \mathtt{EndVal}(\mathbf{z}) \end{array} \right)$$

where $\mathbf{m}_0, \ldots, \mathbf{m}_{N_{\max}-1}$ are variables for representing a valid sequence of mode vectors occurring in the run of $\mathcal{P}$.

Let us now elaborate `HasRun`. First, since a mode vector is a member of $\mathbf{Modes} = \mathtt{REG}^k \times [0, r]^k \times \{\downarrow, \uparrow\}^k$, we set $\mathbf{m}_i$ to be a tuple of variables $\{reg^i_j, rev^i_j, arr^i_j : j \in [1, k]\}$, where:

- reg^i_j is a variable that will range over $[1, 2m]$ denoting which region the jth counter is in (a number of the form $2i+1$ refers to φ_i, while $2i$ refers to ψ_i).
- rev^i_j is a variable that will range over $[0, r]$ denoting the number of reversals that have been used thus far by the jth counter.
- arr^i_j is a variable that will range over $\{0, 1\}$ denoting the current arrow direction, e.g., $0/1$ for $\uparrow/\downarrow$ (non-decrementing/non-incrementing mode).

Using $\mathtt{Init}(\mathbf{m}_0)$, we assert that the initial mode vector needs to respect the given initial configuration $\alpha = (q, u, \mathbf{c})$, where $\mathbf{c} = (c_1, \ldots, c_k)$. More precisely,

$$\mathtt{Init}(\mathbf{m}_0) \equiv \bigwedge_{j=1}^{k} \left(rev^0_j = 0 \wedge \bigwedge_{i=1}^{m} \left(\begin{array}{c} reg^0_j = 2i - 1 \leftrightarrow \varphi_i(c_j) \bigwedge \\ reg^0_j = 2i \leftrightarrow \psi_i(c_j) \end{array} \right) \right).$$

In fact, since $\mathbf{c}$ is given, we could replace some of these variables by constants. However, being able to define this in the formula allows us to prove something more general, as we will see later.

Recall that the target configuration is $\alpha' = (q', u', \mathbf{c}')$, where $\mathbf{c}' = (c'_1, \ldots, c'_k)$. We assert that the end counter values match $\mathbf{c}'$. This definition is given as follows. Note, multiplications by constants are allowed within Presburger arithmetic.

$$\mathtt{EndVal}(\mathbf{z}) \equiv \bigwedge_{j=1}^{k} \left(\sum_{i=0}^{r} \sum_{(\mathtt{ctr}_j, d, i, l) \in \mathsf{ACT}'} d \times z_{f(\mathtt{ctr}_j, d, i, l)} \right) = c'_j.$$

We define $\mathtt{GoodSeq}(\mathbf{m}_0, \ldots, \mathbf{m}_{N_{\max}-1})$ to express that $\mathbf{m}_0, \ldots, \mathbf{m}_{N_{\max}-1}$ is a valid sequence of mode counters. The formula is a conjunction of smaller formulas defined below. One conjunct says that each rev^i_j must be a number in $[0, r]$. Likewise, we add conjuncts expressing that each reg^i_j (resp. arr^i_j) ranges over $[1, 2m]$ (resp. $\{0, 1\}$). We also need to state that changes in the direction arrows incur an extra reversal (otherwise, no reversal is incurred):

$$\bigwedge_{j=1}^{k} \bigwedge_{i=0}^{N_{\max}-1} 0 \leq rev^i_j \leq r. \wedge \bigwedge_{j=1}^{k} \bigwedge_{i=0}^{N_{\max}-2} \left(\begin{array}{c} arr^i_j \neq arr^{i+1}_j \rightarrow rev^{i+1}_j = rev^i_j + 1 \bigwedge \\ arr^i_j = arr^{i+1}_j \rightarrow rev^{i+1}_j = rev^i_j \end{array} \right).$$

Finally, the sequence $\{reg^i_j\}_{i=0}^{N_{\max}-1}$ must obey the changes in $\{arr^i_j\}_{i=0}^{N_{\max}-1}$:

$$\bigwedge_{j=1}^{k} \bigwedge_{i=0}^{N_{\max}-2} \left(reg^i_j < reg^{i+1}_j \rightarrow arr^{i+1}_j = 0 \bigwedge reg^i_j > reg^{i+1}_j \rightarrow arr^{i+1}_j = 1 \right).$$

I.e., since regions denote increasing intervals, going to higher (resp. lower) regions mean the counter mode must be non-decrementing (resp. non-incrementing).

Lastly, we give $\texttt{Respect}(\mathbf{z}, \mathbf{m}_0, \ldots, \mathbf{m}_{N_{\max}-1})$. Again, this is a conjunction. First, when the jth counter is non-incrementing (resp. non-decrementing), we allow only non-negative (resp. non-positive) counter increments:

$$\bigwedge_{j=1}^{k} \bigwedge_{i=1}^{N_{\max}-1} \begin{pmatrix} arr_j^i = 0 \to \bigwedge_{(\texttt{ctr}_k,d,i,l)\in\mathsf{ACT}',d<0} z_{f(\texttt{ctr}_k,d,i,l)} = 0 \bigwedge \\ arr_j^i = 1 \to \bigwedge_{(\texttt{ctr}_k,d,i)\in\mathsf{ACT}',d>0} z_{f(\texttt{ctr}_k,d,i,l)} = 0 \end{pmatrix}.$$

Secondly, the value of the jth counter at the beginning and end of each mode must respect the guessed mode vector. Let us first introduce some notations. Observe that the value of the jth counter at the *end* of ith mode vector can be expressed by $c_j + \left(\sum_{i'=0}^{i-1} \sum_{(\texttt{ctr}_j,d,i',l)\in\mathsf{ACT}'} d \times z_{f(\texttt{ctr}_j,d,i',l)}\right) + \sum_{(\texttt{ctr}_j,d,i,0)} d \times z_{f(\texttt{ctr}_j,d,i,0)}$. Let us denote this term by $\mathit{EndCounter}_j^i$. Similarly, the value at the *beginning* of the ith mode is $c_j + \sum_{i'=0}^{i-1} \sum_{(\texttt{ctr}_j,d,i',l)\in\mathsf{ACT}'} d \times z_{f(\texttt{ctr}_j,d,i',l)}$. We denote this term by $\mathit{StartCounter}_j^i$. Hence, this second conjunct is

$$\bigwedge_{j=1}^{k} \bigwedge_{i=0}^{N_{\max}-1} \bigwedge_{l=1}^{m} \begin{pmatrix} reg_j^i = 2l-1 \to (\varphi_l(\mathit{EndCounter}_j^i) \wedge \varphi_l(\mathit{StartCounter}_j^i)) \bigwedge \\ reg_j^i = 2l \to (\psi_l(\mathit{EndCounter}_j^i) \wedge \psi_l(\mathit{StartCounter}_j^i)) \end{pmatrix}.$$

Finally, we need to express that no invalid counter tests are executed in a given mode. To test whether a CC constraint θ is satisfied by the values $\mathbf{b} = (b_1, \ldots, b_k)$ of the counters, it is necessary and sufficient to test whether θ is satisfied by *some* vector $\mathbf{b}' = (b_1', \ldots, b_k')$, where each b_i lies in the same region in REG as b_i'. Therefore, the desired property can be expressed as:

$$\bigwedge_{i=0}^{N_{\max}-1} \bigwedge_{(\theta,i)\in\mathsf{ACT}'} z_{f(\theta,i)} > 0 \to \theta(\mathit{StartCounter}_1^i, \ldots, \mathit{StartCounter}_k^i).$$

Of course, we might want to make the formula smaller by associating new variables for all terms $\mathit{StartCounter}_j^i$ and $\mathit{EndCounter}_j^i$. Since the translation from PDA to CFGs produces an output of cubic size, it is easy to check that the size of $\texttt{HasRun}$ is cubic in $\|\mathcal{P}\|$, r, and k.

Applications. We start with a straightforward application of the above proof: Control-state reachability over PCo with additional existential Presburger constraints on counter values at the beginning/end of the run and on how many times actions are executed at the end of the run can be checked in NP. In fact, it is poly-time reducible to checking satisfactions over existential Presburger formulas. To see this, observe that the counter values in α and α', which were treated as constants in $\texttt{HasRun}$, could be treated as *variables*. Hence, we can add the additional constraint within the inner bracket of $\texttt{HasRun}$ as a conjunct, and quantify the new variables for counter values. Secondly, we have the following:

Theorem 2. *LTL model checking over PCo is* coNEXP*-complete. For fixed formulas, it is* coNP*-complete.*

For this, we begin with the standard Vardi-Wolper automata-theoretic approach [29] reducing the *complement* of this problem to *recurrent reachability* over PCo: given a PCo $\mathcal{P} = (Q, \mathsf{ACT}, \Gamma, \delta, X)$ and a subset $F \subseteq Q$, decide if there is a run of $\mathcal{P}$ visiting configurations of the form $(q, u, \mathbf{c})$ where $q \in F$ infinitely often. This reduction obtains an NBWA of exponential size from the negation of the given LTL formula, and builds the product of this NBWA and $\mathcal{P}$. Thus, the reduction is exponential in the LTL formula but polynomial in the PCo. Therefore, it suffices to show that recurrent reachability is in NP. Observe that all infinite runs π of $\mathcal{P}$ stabilise in some mode, expressed by a mode vector $\mathbf{m}$. We split π into a subpath from the initial configuration α to a configuration $\alpha' = (q', u', \mathbf{c}')$ in π in mode $\mathbf{m}$, and the rest of the path from α'. In fact, if $\mathbf{c}' = (c'_1, \ldots, c'_k)$, then α' could be chosen such that if the value of the jth counter after α' in π has a maximum c (e.g. when jth counter value stabilises in a region $< 2m$, which is bounded), then $c'_j = c$. By allowing only rules which do not decrement counters whose values do not stabilise, we may treat the path from α' as a path of a PDS with no counters. Hence, we use a well-known lemma of PDS (w.l.o.g. assume that transitions of PDS/PCo only check the top stack character):

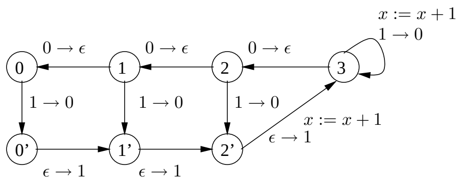
Lemma 3 ([6]). *Given a PDS $\mathcal{P} = (Q, \mathsf{ACT}, \Gamma, \delta)$, an initial configuration α, and a set $F \subseteq Q$, then there exists an infinite computation path of $\mathcal{P}$ from α visiting F infinitely often iff there exist configurations $(p, \alpha) \in Q \times \Gamma$, $(g, u), (p, \alpha v)$ such that $g \in F$ and (i) α can reach $(p, \alpha w)$ for some $w \in \Gamma^*$, and (ii) (p, α) can reach (g, u), from which $(p, \gamma v)$ is reachable.*

We construct a PCo $\mathcal{P}'$ that simulates $\mathcal{P}$ (for the initial subpath of the recurrent reachability witness). Eventually, it decides to stop simulating $\mathcal{P}$ at some configuration $(p, \alpha w, \mathbf{v})$. The state and the top-stack character are then made visible by $\mathcal{P}'$ with an action $\overline{(p, \alpha)}$. The PCo $\mathcal{P}'$ then empties the stack and continues the simulation of $\mathcal{P}$ from (p, α) except that (1) only rules which increment counters by non-negative values are allowed, but instead of actually modifying the counter values, actions of the form $+\mathtt{ctr}_j$ will be executed if the rule increments the jth counter by a positive value (2) counter tests θ are no longer performed, but instead we execute actions that signify θ. When a configuration of the form $(g, u, \mathbf{v})$ is reached with $g \in F$, it will make it visible by performing an action of the form $g!$. At any given time after this, it may output a character $\overline{(p', \alpha')}$, where p' is the current state and α' is the current top stack character, and go into the state $\mathtt{Finish}$. There exists a computation path of $\mathcal{P}$ from α that visits F infinitely often iff there exists a path from α to the $\mathtt{Finish}$ in $\mathcal{P}'$ such that: (a) the number executions of some $\overline{(p, \alpha)}$ is 2, (b) some $g!$ action has been executed, (c) if the region of the end value of the jth counter (corresponding to the jth counter value of the initial witnessing subpath of $\mathcal{P}$) is bounded (i.e. $< 2m$), then no action $+\mathtt{ctr}_j$ must have been executed, and (d) no counter test actions violating the regions of the end counter values have been executed. Using the techniques from Theorem 1, we may express these constraints as a poly-size existential Presburger formula. In conclusion, we have reduced recurrent reachability over PCo to control-state reachability over PCo with constraints, which we already saw to be in NP.

5 Lower bounds for model checking PCo

In this section, we prove coNEXP and coNP lower bounds for model checking LTL and reachability over PCo. We will also show that model checking EF-logic is undecidable even for a fixed formula. All our lower bounds hold for $(1, 1)$-PCo where counters can only be compared against 0 and incremented by $\{-1, 0, 1\}$.

Lower bound for LTL and reachability We start with coNP-hardness for a fixed LTL formula over $(1, 1)$-PCo. We show that the complement of this model checking problem is NP-hard. It will be clear later that the same proof can be used to show that reachability is NP-hard over $(1, 1)$-PCo. The idea is to reduce from the NP-complete `0-1 Knapsack` problem ([24]): given $a_1, \ldots, a_k, b \in \mathbb{N}$ in binary, decide whether $\sum_{i=1}^{k} a_i x_i = b$ for some $x_1, \ldots, x_k \in \{0, 1\}$. The reduction constructs a $(1, 1)$-PCo $\mathcal{P}$ which initializes the counter to 0 and repeats for each $i \in [1, k]$: guess x_i, add $a_i x_i$ to the counter. The PCo then checks whether the counter is b by substracting b and checking whether the result is 0. If the test is positive, execute a special action *success* and then loop silently. The LTL formula is $\mathsf{G}\neg success$. Note, this PCo makes one reversal. The problem with this reduction, however, is that the numbers $a_1, \ldots, a_k, b$ are given in binary, whereas each transition in $\mathcal{P}$ can add -1 or 1. Hence, we cannot naively hardwire the "intermediate" values of $a_1, \ldots, a_k, b$ during the computation in the finite control. Instead, we need to use the stack. We illustrate this technique by the example in the following figure on the right. This is a PCo with one counter x that increases x by the number represented by the topmost four bits on the stack (edge label $u \to v$ defines the stack operation). For example, if the PCo starts with the configuration $(3, 1101, 0)$, then it will end at the configuration $(0, 0, 13)$. Using this technique, we will only need at most $\sum_{i=1}^{k} \log(a_i) + \log(b)$ extra states and therefore avoiding an exponential blow-up.



For the coNEXP lower bound for non-fixed formulas, we reduce succinct `0-1 Knapsack`. We define `Succinct 0-1 Knapsack` as the `0-1 Knapsack` problem where the input is given as a boolean formula θ with variables $x_1, \ldots, x_{k+m}$ where $k, m \in \mathbb{Z}_{>0}$ are given in unary. Here θ represent the numbers $a_1, \ldots, a_{2^k-1}, b \in \mathbb{N}$ each with precisely 2^m bits (leading 0s permitted) as follows:

- The ith bit of $\mathbf{bin}(b)$ is defined to be $x \in \{0, 1\}$ iff the formula θ evaluates to x when $x_1, \ldots, x_{k+m}$ are evaluated to $0^k \mathbf{bin}^m(i)$.
- The ith bit of $\mathbf{bin}(a_j)$ is defined to be $x \in \{0, 1\}$ iff the formula θ evaluates to x when the inputs to $x_1, \ldots, x_{k+m}$ are $\mathbf{bin}^k(j)\mathbf{bin}^m(i)$.

The problem is to check whether $\sum_{i=1}^{2^k-1} a_i z_i = b$ for some $z_1, \ldots, z_{2^k-1} \in \{0, 1\}$. The problem of `Succinct 0-1 Knapsack` can be shown to be NEXP-complete in the same way that the problem `Succinct Knapsack`, where natural numbers can be assigned to z_i's (instead of only $\{0, 1\}$), is shown in [30] to be NEXP-complete.

We now show a NEXP lower bound for the complement of LTL model checking over $(1, 1)$-PCo by reducing from `Succinct 0-1 Knapsack`. The idea of the reduction is the same as for the case of fixed formulas, but we will have to use the LTL formula to count up to doubly exponential values.

The stack alphabet includes $\#$, $c_1^0, c_1^1, \ldots, c_m^0, c_m^1$. If the ith bit of a_j is 1 and z_j has been guessed to be 1, then we need to add 2^i to the counter. E.g., suppose we're on the $11\ldots1$th bit of the a_0. Using the techniques below, we calculate the value of this bit. If it is 1, we increment $2^{(2^m-1)}$ times. To do this, we push the following sequence on the stack, again using techniques described below.

$$0c_m^0 \ldots c_1^0\#0c_m^1c_{m-1}^0 \ldots c_1^0\# \ldots \;\ldots\; \ldots\#0c_m^1 \ldots c_1^1\# \qquad (*)$$

That is, a bit-string with 2^m many 0s, annotated with their bit positions (least significant bit on the top/left of the stack). From this stack configuration, the system counts up to $1c_m^0 \ldots c_1^0\#1c_m^1c_{m-1}^0 \ldots c_1^0\# \ldots \;\ldots\; \ldots\#1c_m^1 \ldots c_1^1\#$, i.e., a bit-string with 2^m many 1s, taking $2^{(2^m-1)}$ steps. We increment the counter $+1$ at each step.

To complete the proof, we need to know how to: (1) enumerate all assignments to $x_1, \ldots, x_{k+m}$ and evaluate θ, (2) initialise the large counter described above, and (3) increment the large counter from $00\ldots0$ to $11\ldots1$.
Problem 1: We store the current assignment on the bottom of the stack, using $x_1^0, x_1^1, \ldots, x_{k+m}^0, x_{k+m}^1$. Initially, we push $x_1^0, \ldots, x_{k+m}^0$, making the pushed characters visible using suitable action symbols. We then guess whether θ evaluates to 1. An LTL formula encoding θ asserts this guess is correct. To move to the lexicographically next assignment, we erase the stack, making the characters visible, and then guess the next assignment, using the LTL to check it.
Problem 2: Guess and push $0c_1^{y_1} \ldots c_m^{y_m}\#$, using the formula to check this matches the $x_{k+1}^{y_1}, \ldots, x_{k+m}^{y_m}$ on the stack. We then push $0c_1^{Y_1} \ldots c_m^{Y_m}\#$ (with the values of $Y_1, \ldots, Y_m \in \{0, 1\}$ guessed) to the stack. This is done arbitrarily many times and the PCo may stop at some point. To make sure we obtain (*) on the top of the stack, we assert the successor property using the LTL formula.
Problem 3: This uses the idea for the fixed formula case: pop from the stack until the first 0, then push a correct number of 1s annotated with the bit positions. For this, an LTL formula asserts the successor property, cf. [**?**].

Undecidability results for EF-logic We now turn our attention to model checking EF-logic over PCo. It turns out that the problem is already undecidable for a fixed formula with two operators. We reduce from the emptiness of linear bounded Turing machines (LTM), which is undecidable (cf. [26]). Given an LTM $\mathcal{M}$ that accept/rejects an input w using at most $c|w|$ space, we compute a $(1, 1)$-PCo $\mathcal{P}$, an EF formula φ, and a start state q_0 of $\mathcal{P}$ such that $\mathfrak{S}_{\mathcal{P}}, (q_0, \epsilon) \models \varphi$ iff $\mathcal{M}$ is nonempty. We make $\mathcal{P}$ guess a word w and an accepting computation path of $\mathcal{M}$, which is stored in the stack. The length $c|w|$ is stored in the counter. Once the path has been guessed, it suffices to show that: (P1) the length of *each* guessed configuration is $c|w|$, and (P2) *each* non-initial configuration is a successor of its previous configuration. The guessing and checking stages incur one alternation for the EF formula. Since checking P2 requires us to check at most four consecutive tape cells of $\mathcal{M}$ (once a cell is chosen), we can remember this location by decrementing the counter, moving to the previous configuration of $\mathcal{M}$ that is stored in the stack, and then further decrementing the counter making sure that the end value is 0. See the full version for the proof.

Theorem 4. *Model checking EF-logic over* $(1,1)$*-PCo is undecidable even for a fixed formula with two* EF *operators.*

6 Adding clocks

To extend our results to the case of PCC, we use the region construction of Alur and Dill [1] to reduce a PCC to a PCo of exponential size (in exponential time) such that every run of the PCC can be projected, state-for-state, onto a run of the PCo. From S4, we obtain a coNEXP (resp. NEXP) upper bound for LTL model checking (resp. reachability) over PCC. We next provide a lower bound.

Theorem 5. *Reachability is* NEXP*-hard for PDS with discrete clocks and one 1-reversal-bounded counter. When clock constraint constants are given in binary, only three clocks and one single-reversal counter are needed. Similarly, LTL model-checking is* coNEXP*-hard, even for a fixed formula.*

We first consider unary constants, and a non-fixed number of clocks. We adapt the previous reduction from `Succinct 0-1 Knapsack`, except the formula can no longer be used to evaluate the boolean formulas. Instead, we encode bits with two clocks, which are equal iff the bit is 1. We evaluate boolean formulas using the transition guards. To test all assignments to $x_1, \ldots, x_{k+m}$ we store the valuation in the counters (not the stack). This is straightforward. Then, to build the large counter on the stack, we use another set of clocks to store the bit position values of the last two blocks pushed on to the stack. These clocks can be used to ensure the successor relation between the two values. For binary clock constraints, we use Courcoubetis and Yannakakis [10] to reduce to three clocks.

Recall that model checking EF-logic over PCo is undecidable. It turns out that this problem is decidable over PCl. See the full version for a proof.

Theorem 6. *Model checking EF over PCl is* EXPSPACE*-complete. The lower bound holds for fixed formulas with two clocks with binary constraint constants, or with a non-fixed number of clocks when the constraints are in unary.*

7 Experimental Results

We provide a prototypical C++ implementation of an optimised version of the reduction in Section 4. In particular, from the Verma *et al.*, we can derive, at no cost, the number of times each rule of the PCo is fired. From this, we infer the number of action symbols output, and hence, do not need additional variables and transitions for these. In addition, the per mode information per counter uses a single variable. Finally, we look for pairs of pushdown rules $\langle (q_1, a_1, \top), \epsilon, (q_2, a_2, \emptyset, \emptyset^n)\rangle$ and $\langle (q_2, a_2, \top), \epsilon, (q_3, a_3, \emptyset, \emptyset^n)\rangle$ such that a_1 and a_2 are single characters and the pair q_2, a_2 does not appear together in any other rule. Furthermore, a_2 does not appear in a rule $\langle (q, w, \theta), a, (q', w'a_2w'', Y, \mathbf{w})\rangle$ where $|w'| > 0$. These rules can be replaced by $\langle (q_1, a_1, \top), \epsilon, (q_3, a_3, \emptyset, \emptyset^n)\rangle$. Then we remove unreachable productions from the generated CFGs. We used the 64-bit Linux binary of Z3 2.16 [?] on the Presburger formulas on a quad-core, 2.4Gz Intel® Xeon® machine with 12GB of RAM, running Fedora™12. We report the time from the Linux command `time` for the translation and the run of Z3.

Double free in dm-target.c. In version 2.5.71 of the Linux kernel, a double free was introduced to `drivers/md/dm-target.c` [?]. This was introduced to fix a perceived memory-leak. When registering a new target, memory was allocated and a check made to see if the target was already known. If so, it was freed and an "exists" flag set. Otherwise the target was added to the target list. Before returning, the exists flag was checked and the object was freed (again) if it was set. We created, by hand, a model of this file using a counter to store the length of the target list. The complete control flow of the file was maintained and data only tracked when relevant to the memory management. The model outputs special symbols to mark when memory is allocated or freed. We then look for a run where, either an item was removed from the empty list, the number of free calls was greater than the number of allocations, or, the code exited normally, but more memory was allocated than freed. We were able to verify that the code contained a memory error in version 2.5.71 and that the memory management was correct in earlier versions (for a bounded number of reversals) provided as many targets were registered as unregistered. Note, the counter is required to track the size of the list which ensures that the number of allocations matches the number of frees. The size of the `dm-target.c` is approximately 175 lines without comments. Detecting the bug took 2s, and proving correctness took 1.7s, 2.9s, 16s, 24s and 77s for 1, 2, 3, 4 and 5 reversals respectively.

Memory leak in aer_inject.c. In version 2.6.32 of the Linux kernel, the file `drivers/pci/pcie/aer/aer_inject.c` contains a memory leak that was patched in the next version [?]: two lists of allocated objects are maintained, but, when exiting, the code empties the items from the first list and frees them, then, empties and frees the first list again, instead of the second. We created a model of this driver with two counters to track the size of the lists and searched for memory errors as in the previous example. Only one reversal was required to detect the memory leak. We showed that the patch corrects the problem (up to one reversal). Note, without counters, it would always be possible for the number of allocations to differ from the number of frees. The file `aer_inject.c` is approximately 470 lines without comments. Detecting the bug required 220s and proving correctness for a single reversal took 508s.

Buffer overflow. Jhala and McMillan have a buffer overflow example which their technique, SatAbs and Magic, failed to verify [?]. There are three buffers, x, y and z of sizes $100, 100$ and 200 and two counters i and j. First, i is used to copy up to 100 positions of x into z. Then, counters i and j are used to copy up to 100 positions of y into the remainder of z. There is overflow if i, which indexes z, finishes greater than 199. This simply encodes into our model[1] and could be verified correct (since the example is trivially reversal bounded) in 1.6s.

David Gries's coffee can problem [?]. We have an arbitrary number of black and white coffee beans in a can. We pick two beans at random. If they are the same colour, they are discarded and an extra black bean is put in the tin. If they differ, the white is kept but the black discarded. The last bean in the can is black iff the number of white beans is odd. This problem can be modelled without abstraction by using the stack to count the number of black beans, and a single-reversal counter to track the number of white beans. We verified in 1.3s that the last bean cannot be white if the number of white beans is odd, and in 1.1s that the last bean may be white if the number of white beans is even.

8 Extensions and Future Work

We prove, in the full version of this paper, that (i) reachability and LTL for a prefix-recognisable version of PCo is NEXP-complete and coNEXP-complete respectively, even with one 1-reversal-bounded counter and a fixed formula; (ii) LTL model checking for a fixed formula is coNEXP when the number of reversals is given in binary, whereas reachability is in NEXP (a matching lower bound is in [17], even without the stack); and (iii) the reachability problem for second-order pushdown systems (cf. [16]) with reversal bounded counters is undecidable.

For future work we may investigate counter-example guided abstraction refinement. We would need counter-examples from the models of the existential Presburger formula, and refinement techniques to add counters and reversals as well as predicates. Furthermore, as we allow user defined numerical constraints on reachability, we may also restrict LTL model checking to runs satisfying additional numerical fairness constraints.

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A Definition of r-reversal-bounded paths

Let $\mathcal{P}$ be a pushdown system with k counters and m clocks, and let α be a configuration of $\mathcal{P}$. Suppose that π is a path in $\mathfrak{S}_{\mathcal{P}}$ from α and let x a counter variable in $\mathcal{P}$. If $\{i_j\}_{j=1}^{k}$ (where $k \leq \omega$) is the projection of π to the values of a counter x, define the "succesive difference" sequence $\sigma = \{d_j\}_{j=1}^{k-1}$ as $d_j := i_{j+1} - i_j$. Replacing each positive/negative/zero number in σ by $\uparrow/\downarrow/0$, respectively, and then viewing σ as a word over $\{\uparrow, \downarrow, 0\}$, we mark all substrings that match either the regular expression $\downarrow 0^* \uparrow$ or $\uparrow 0^* \downarrow$. If $\sigma[j, j']$ is one such substring, a *reversal* for counter x occurs in between $\pi[j'-1]$ and $\pi[j']$. The path π is said to be r-reversal-bounded if the number of reversals in π is at most r.

Another concept that is intimately connected to reversals in π is that of "phase". A *phase* for counter x, then, is any nonempty subpath of π in between (1) any two consecutive reversals, (2) $\pi[1]$ and the first reversal, (3) the final reversal and the end (or "limit") configuration in π, or (4) $\pi[1]$ and the end (or limit) configuration in π if no reversal occurs in π. A phase $\pi[j, j']$ for counter x is said to be *non-decrementing* (resp. *non-incrementing*) if no $\downarrow$ (resp. $\uparrow$) occurs in $\sigma[j, j'-1]$. Note that a phase could be both non-incrementing and non-decrementing when the values of the counters do not change throughout the path. It is easy to see that the number of reversals in π is one less than the number of phases in π.

B Missing proofs from Section 4

Lemma 7. *There are at most* $N_{max} := |\mathtt{REG}| \times (r+1) \times k = 2mk(r+1)$ *distinct mode vectors in* σ*.*

Proof. At any given time, each of the k counters could have incurred j reversals, where $j \in [0, r]$. At any given time, each of the k counters could have a value that is in any of the $|\mathtt{REG}|$ distinct regions. Since there is no way to revert to the previous mode vector in σ once a change occurs, there are at most $N_{\max}$ distinct mode vectors in σ. □

C Reduction from the complement of LTL model checking over PCo to recurrent reachability over PCo

Given the LTL formula φ, we compute the NBWA $\mathcal{A} = (\mathsf{ACT}, Q, \delta, q_0, F)$ of size $2^{O(|\varphi|)}$ such that $\mathcal{L}(\mathcal{A}) = [\![\neg\varphi]\!]$. Without loss of generality, we assume that $(q, \epsilon, q) \in \delta$ for each $q \in Q$. Given a PCo $\mathcal{P} = (P, \mathsf{ACT}, \Gamma, \Delta, X)$, we construct a new PCo $\mathcal{P}' = (P', \mathsf{ACT}, \Gamma, \Delta', X)$ such that $P' = Q \times P$, and that $(((q_1, p_1), v, \theta), a, ((q_2, p_2), w, i)) \in \Delta'$ iff $(q_1, a, q_2) \in \delta$ and $((p_1, v, \theta), a, (p_2, w, i)) \in \Delta$, for each $a \in \mathsf{ACT}_\epsilon$. Given any configuration $\alpha = (p, v, \mathbf{u})$ of $\mathcal{P}$, we have $\mathfrak{S}_{\mathcal{P}}^{\alpha}, \alpha \not\models \varphi$ iff there exists an infinite trace from $\alpha' = ((q_0, p), v, \mathbf{u})$ in $\mathfrak{S}_{\mathcal{P}'}^{\alpha'}$ visiting $(F \times P) \times \Gamma^* \times \mathbb{N}^k$ infinitely often.

D Proof of Theorem 4

Theorem 4. *Model checking EF-logic over* $(1,1)$*-PCo is undecidable even for a fixed formula with two* EF *operators.*

Proof. The proof is by a reduction from the emptiness of linear bounded Turing machines (LTM), which is undecidable (cf. [26]). Suppose $\mathcal{M}$ is an LTM with states Q, an accept state s_{acc}, an input alphabet Σ, a tape alphabet $\Omega = \Sigma \cup \{\Box\}$ for some blank symbol $\Box \notin \Sigma$, and a transition relation $\delta \subseteq (Q \times \Omega) \times (Q \times \Omega \times \{L, R\})$, where L (resp. R) indicates "left" (resp. "right"). The machine $\mathcal{M}$ also comes with a constant c such that, given an input word $w \in \Sigma$ of length n, $\mathcal{M}$ will never use more than cn space.

We now show how to compute a $(1,1)$-PCo $\mathcal{P}$, an EF formula φ, and a start state q_0 of $\mathcal{P}$ such that $\mathfrak{S}_{\mathcal{P}}, (q_0, \epsilon) \models \varphi$ iff $\mathcal{M}$ has a nonempty language. The idea is to make $\mathcal{P}$ guess a word w and an accepting computation path of $\mathcal{M}$ on w, which will be stored in the stack. The length of w (times the constant c) will be remembered in the counter. Once the computation path has been guessed, it suffices to show that:

(P1) the length of *each* guessed configuration is $c|w|$, and
(P2) *each* non-initial configuration is a successor of its previous configuration in the path.

The guessing corresponds to an EF formula of the form $\mathsf{EF}\varphi$, while the checking corresponds to a subformula of φ of the form $\mathsf{AG}\psi$.

Let us first define $\mathcal{P}$. The stack alphabet Γ of $\mathcal{P}$ is $Q \cup \Omega \cup \{\wr, \bot\}$, where $\wr \notin \Omega$ is a separator between two consecutive (guessed) configurations of $\mathcal{M}$, and $\bot$ is a stack-bottom symbol. Initially, starting in state q_0, the PCo $\mathcal{P}$ will push a stack-bottom symbol, followed by the start state symbol s_0 of $\mathcal{M}$, on to the stack. It then nondeterministically pushes a word $s_0 w \in \Sigma^*$ that witnesses nonemptiness of $\mathcal{M}$. In doing so, $\mathcal{P}$ will also increment the counter by c for each symbol in w that is guessed so that, when w has been guessed, the counter stores the length cn, where $n = |w|$. The PCo $\mathcal{P}$ will then simply nondeterministically push arbitrarily many blank symbols $\Box$ to the stack and at some point it may push the separator symbol $\wr$. After this, $\mathcal{P}$ will simply arbitrarily push symbols from $Q \cup \Omega\{\wr\}$, while making sure that exactly one symbol from Q appears between any two consecutive symbols $\wr$ in the stack. That is, if $\wr v \wr$ is any substring of the stack content at any given time in this guessing stage, if no $\wr$ occurs in v, then exactly one symbol from Q occurs in v. This can be easily implemented in $\mathcal{P}$ (e.g. using the finite control). At some point, $\mathcal{P}$ may guess the final configuration by pushing some word of the form $us_{acc}v$ and then move to state q_{check} in its finite control via the action *check*.

From state q_{check}, the PCo $\mathcal{P}$ will pop arbitrarily many symbols. At any point when the symbol $\wr$ is seen, $\mathcal{P}$ may switch to the state q_{check1} or the state q_{check2}. Intuitively, in state q_{check1} (resp. q_{check2}) the PCo $\mathcal{P}$ will attempt to verify Property (P1) (resp. (P2)) with respect to the configuration C that is immediately below the topmost character in the stack, which is $\wr$.

Let us elaborate first on how to implement the first functionality. In state q_{check1}, the PCo $\mathcal{P}$ will continue popping symbols until the next $\wr$ is seen. In doing so, the counter is decremented by 1 for each symbol $\notin Q$ that is popped. At the time when the next $\wr$ is seen, if the counter is 0, then we may deduce that Property (P1) is satisfied by the configuration C. In this case, $\mathcal{P}$ goes to a success state q_{succ} via an action *success*. Otherwise, it goes to a failure state q_{fail} via an action $fail$.

We now elaborate how to implement the second functionality, i.e., whenever $\mathcal{P}$ is in q_{check2}. We distinguish two cases. The first case is when the stack is of the form $\bot C \wr$, i.e., there is only one configuration. In this case, $\mathcal{P}$ simply goes to q_{succ} via action *success*. The second case is when the stack is of the form $\bot v \wr C' \wr C \wr$, i.e., there are at least two configurations in the stack. In this case, $\mathcal{P}$ will nondeterministically guess a position in C by popping symbols. In doing so, the counter is decremented for each symbol $\notin Q$ that is popped. Once this position is guessed, the topmost four symbols in the stack are remembered in the finite control unit. Next, we want to check these against the corresponding positions in C'; recall that such local checks suffice for Turing machines. To this end, we shall first pop the remaining symbols in C. We will then *nondeterministically* guess a position in C' by popping symbols. After this, the local check is performed (by looking at the topmost four symbols in the stack) and we make sure that the position that is guessed is correct by decrementing the counter for each remining symbol in C' below the guessed position. The PCo goes to the state q_{fail} via an action $fail$ if the former is not satisfied while the latter is satisfied. Otherwise, $\mathcal{P}$ goes to a success state via an action *succ*.

Now let $\varphi = \mathsf{EF}\langle check\rangle\neg\mathsf{EF} fail$. It is easy to check now that $\mathfrak{S}_{\mathcal{P}}, (q_0, \epsilon) \models \varphi$ iff $\mathcal{M}$ has a nonempty language.

E Proof of Theorem 5

Theorem 5. *Reachability is* NEXP*-hard for PDS with discrete clocks and one 1-reversal-bounded counter. When clock constraint constants are given in binary, only three clocks and one single-reversal counter are needed. Similarly, LTL model-checking is* coNEXP*-hard, even for a fixed formula.*

Proof. The proof follows from Lemma 8 and Lemma 9 below.

Lemma 8. *Reachability is* NEXP*-hard for pushdown systems with discrete clocks and one single-reversal counter when clock constraint constants are given in unary.*

Proof. The first case we consider is the case of unary clock constraint constants, and a non-fixed number of clocks. We reduce from the `Succinct 0-1 Knapsack` problem.

We reduce from the `Succinct 0-1 Knapsack` problem and construct a push-down system with a single-reversal counter and clocks that behaves as follows. First, it increments the counter by the value b, then it cycles through the inputs a_j, choosing non-deterministically whether the include them. If a_j is included, the counter is decremented by a_j. Finally, a unique control state from which a *success* action can be executed iff the counter is exactly zero.

The stack alphabet contains the characters $\#$, $c_1^0, c_1^1, \ldots, c_m^0, c_m^1$. The c characters will be used to count large numbers. If the ith bit of an included object's weight is 1, then we need to increment the counter 2^i times.

For example, suppose we're on the $111\ldots1$th bit of the $00\ldots0$th item. Using techniques described below, we will calculate the value of this bit. If it is 1, we have to increment $2^{(2^m-1)}$ times. To do this, we push the following sequence on the stack, again using techniques described below.

$$0c_m^0 \ldots 0c_1^0\#0c_m^1c_{m-1}^0 \ldots c_1^0\# \ldots \;\; \ldots \;\; \ldots\#0c_m^1 \ldots c_1^1\#$$

That is, an exponential number of 0s, marked — in increasing order (top of stack to bottom) — with their bit positions. From this position, the system counts from an exponential number of 0s to an exponential number of 1s. This takes $2^{(2^m-1)}$ steps. We increment the counter at each step.

To complete the proof, we need to know how to

1. enumerate all assignments to $x_1, \ldots, x_{k+m}$ and evaluate θ
2. initialise the large counter described above,
3. increment the large counter from $00\ldots0$ to $11\ldots1$.

We deal with these problems in turn.

Problem 1: We first describe how to enumerate all assignments to the inputs $x_1, \ldots, x_{k+m}$. The main trick is to use two clocks for each bit. If the bit is 1, then the two clocks are equal, otherwise they are different. It is easy to force two clocks to be equal by resetting them at the same time. To force inequality, we wait a short period of time before resetting only one of the clocks. To count through all assignments to the bits is then straightforward.

To evaluate θ we can use the guards on the transition relation of the push-down system. That is, we replace each variable with the appropriate equality check on the two clocks that represent the bits value. Since the guards allow boolean combinations of atomic clock constraints, we are done.

Problem 2: Let $y_1 \ldots y_m$ be the current bit represented by the clocks encoding $x_{k+1}, \ldots, b_{k+m}$. The first step in initialising the big counter (of $y_1 \ldots y_m$ bits) is to add $0c_1^{y_1} \ldots c_m^{y_m}\#$ to the top of the stack. This is done straightforwardly using the guards on the clock values.

We then need to write a sequence of blocks of the form $0c_1^{y_1} \ldots c_m^{y_m}\#$ for such that consecutive values of $y_1 \ldots y_m$ decrease by 1 each time, ending in $00\ldots0$.

Hence, we introduce several more clocks, over which we can assert the successor property. There will be two clocks for each bit of top two blocks on the stack. That is, we will have clocks $d_1^1, d_1^2, \ldots, d_m^1, d_m^2$ and $e_1^1, e_1^2, \ldots, e_m^1, e_m^2$. The

d clocks will encode the bits of the top block. In particular $d_i^1 = d_i^2$ iff the ith bit of the top block is 1. The e clocks will similarly encode the second to top block. We non-deterministically guess each block in turn. The first step is to "copy" the bit assignment in the d clocks to the e clocks. This can be done using the guards on the transitions. Next, we write a guessed sequence of m bits to the stack, making sure the d clocks accurately represent the guessed value. Finally, we check that this new value is the predecessor of the (now) second to top value using the boolean guard (using d_i as shorthand for $d_i^1 = d_i^2$ and similarly for e_i)

$$\bigvee_j^{n-1} \begin{bmatrix} (\neg d_j \wedge e_j) \wedge \\ \bigwedge_{i>j} (d_i \wedge \neg e_i) \wedge \\ \bigwedge_{i<j} (d_i \iff e_i) \end{bmatrix} .$$

Hence, we can initialise the large counter.

Problem 3: We will first describe a simpler version of the problem where we are counting over n bits (not 2^n). Using a pushdown system, one can count over n bits from $00\ldots0$ to $11\ldots1$ in a straightforward way. First, push $00\ldots0$ on to the stack. Thus, we hold the counter value on the stack, least significant bit first. To increment this value, first, pop from the stack until the top character is a 0, then rewrite this 0 to 1, then push enough 0s on the stack to bring the height back to n. When n is linear, this can be done using the control states of the system.

Counting from $00\ldots0$ to $11\ldots1$ over the large counter is similar, except, since n is not linear, we cannot ensure that the stack is built back up to the right height by using the control state. However, we can use the same technique as above to add the correct number of 0 bits.

Note, now, we have all the tools required to build the system we desire. The system behaves as follows. First we initialise the clocks representing the input bits $o_1 \ldots o_k$ to $00\ldots0$. Since the zeroth item holds the desired value, we increment the counter to this value. We do this by cycling through all assignments to $x_{k+1} \ldots b_{k+m}$ and calculating whether the represented bit of b is 1. If so, we write the we increment the counter as described above.

Once this is done, we cycle through the other objects. The system guesses whether to include the object. If so, it does as above to compute the value of the object, except this time it *decrements* the counter. In this way we cycle through all items until we are done, we then move to the final state, provided the counter is zero (and has not reached zero too early).

Lemma 9. *Reachability is* NEXP*-hard for pushdown systems with one single-reversal counter and three discrete clocks and atomic clock constraint constants given in binary.*

Proof. The proof is an adaptation of the proof of Lemma 8. The main changes we make are as follows:

– bit values that were previously stored as pairs of clocks are now stored together in a single clock, with bit j occupying the jth position of the binary encoding of the clock value.

– techniques due to Courcoubetis and Yannakakis [10] are used to decode and set the ith bit value.
– evaluation of the boolean guards in the proof of Lemma 8 is moved into the control state of the pushdown system, since the encoding and decoding process means that the guards of the system can no longer be used.

We first describe how to evaluate a boolean formula using the control state of the pushdown system, assuming we have a technique for evaluating the atoms of the formula. To evaluate a non-atomic formula, we store it as a tree in the control state. Evaluation uses a kind of tree automaton (the run of which is encoded into the state space). The tree automaton navigates the tree in left most, depth first order. First it moves down to the left most leaf. This will be an atomic proposition. The proposition is evaluated using the assumed technique and the value is passed up to the parent. When first returning to a parent node, it is marked as seen. If the node is a disjunction, and the value returned is 1, then the automaton returns to the parent, also carrying the 1, otherwise it moves down into the right subtree. The automaton eventually returns from this tree with a value. Since the node is marked, it detects that it has fully evaluated the disjunction and returns the value to the parent. Evaluation is analogous for conjunction. Finally, a value is returned from the root.

The evaluation above only introduces a polynomial number of control states to the pushdown system. Because the tree is navigated in left most depth first order, there are a linear number of different markings (if the right hand subtree is not visited, we can simply mark all of the nodes in this subtree without affecting the execution). Then, to keep track of the automaton, we attach the state of the automaton to the node of the tree it is at. This is only polynomial since there is only one node marked by the automaton state at a time.

Given a technique for encoding and decoding the bits of a bit vector, coupled with the evaluation of boolean formulas, we can adapt the proof of Lemma 8 in a straightforward manner. Hence, all that remains is the encoding and decoding of bits. This can be done using a technique of Courcoubetis and Yannakakis [10]. We will describe this process informally here, referring the reader to the original proof for details.

We require three clocks a, b and c. In general, the value of clocks a and b will be equal, and encode the bit vector. The value of clock c will be 0. Reading the ith bit, or setting the ith bit is a two stage process, after which the values of a, b and c will resume their general roles.

The first stage decodes the value of the ith bit, for $1 \leq i \leq n$. For intuition, consider only the most significant bit, which will be the nth bit. We can determine the value of this bit simply by checking whether the value of clock b (or, indeed, clock a) is greater than 2^{n-1} (if so, the nth bit must be 1). The idea, then, is to define an extraction process that takes a fixed amount of time. The clock a is left to advance untouched. The fact that the extraction phase takes a fixed amount of time is used to “undo” this advancement. The clock b is used to check the value of each bit in turn, from bit n down to bit i. After each bit, some manipulations involving clock c as a working clock effectively remove the most significant bit of b by setting it to zero. This allows the next bit to be checked as if it were the most significant bit.

Schematically, the bit extraction follows the diagram in Figure 1, where $R(b)$ means to reset clock b, and similarly for clock c. Note, since, in our definitions, time only progresses when a transition is fired, we assume implicit empty self-loop transitions on each state that permit time to progress.

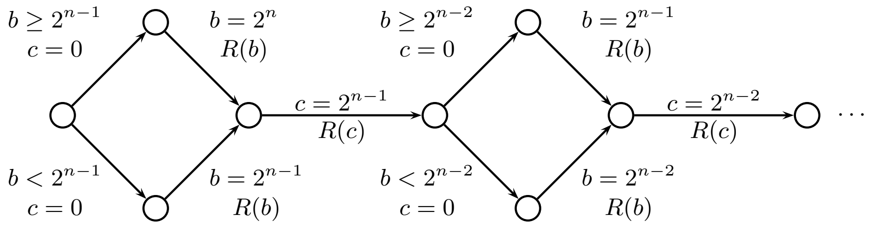

Fig. 1. Extracting a bit from a clock.

Beginning from the leftmost state, suppose we have, in binary, $b = 101$. Since the most significant bit is 1, we take the upwards branch (revealing that the most significant bit is 1) and then wait until $b = 2^3 = 1000$. This pause will let c increase to 011. That is, $1000 - 101$. We then reset b and wait for c to reach $2^2 = 100$. This requires a single step, during which b becomes 001. That is, 101 with the most significant bit set to 0. We then reset c to zero and repeat the process to find the next bit.

Observe that both the upwards and downwards branches take the same amount of time. This is by virtue of the fact that in the upper branch we know the most significant bit is 1, while in the lower branch it is zero. Hence, to extract bit i we require $2^n - 2^{i+1}$ time steps. This means clock a contains the original bit vector (call it y), plus $2^n - 2^{i+1}$. The first step of the second stage is to let 2^{i+1} time steps pass before resetting b and c, which raises a to $y + 2^n$. We let a reach 2^{n+1}, which means c contains $2^n - y$. Then, reset a and b and let c reach 2^n, before resetting it. Hence, the values of a and b are restored to y and c is set to zero, as required.

The value of the ith bit can be changed during this process by letting the first step of the second phase last 2^i steps longer or shorter as required (to switch from 0 to 1 or vice versa).

F Proof of Theorem 6

Theorem 6. *Model checking EF over PCl is* EXPSPACE*-complete. The lower bound holds for fixed formulas with two clocks with constants in the constraints given in binary, or with a non-fixed number of clocks when the constraints are in unary.*

Proof. We reduce from the tiling problem for a corridor of exponential width. Note, similar to an exponential-space Turing machine, we can assume a correct tiling, if it exists, is of finite length.

We first observe that we can use the clocks to run a PDS for an exponential number of steps. With a two clocks and binary constraints this is just a case of reseting the clocks, then making transitions at every time step (using the second clock) until the first clock is greater than 2^n (or $11\dots1$ in binary). For unary constraints, we use $2n$ clocks $c_1^1, c_1^2, \dots, c_n^1, c_n^2$. Each pair of clocks encodes a bit. The ith bit is 1 if $c_i^1 = c_i^2$ and 0 otherwise. Setting a bit value in the clocks is a matter of either reseting both clocks at the same time, or waiting for a time-step, then reseting one clock. We initialise the bits to $00\dots0$, then flip the bits to count, incrementally, from $00\dots0$ to $11\dots1$. This takes an exponential number of steps. Henceforth, we use this technique whenever we need to count an exponential number of steps.

The pushdown system has two phases. First, we guess a valid tiling of the corridor encoded as a word $t_1^1 \dots t_{2^n}^1 \# t_1^2 \dots t_{2^n}^2 \# \dots$, where $\#$ indicates the end of a row, and each t_j^i is a tile from the tile alphabet. We can make sure that the horizontal constraints on the tiling are met by restricting how the pushdown system can guess. Furthermore, we can ensure all rows are of exponential size using the techniques above.

All that remains to check is that the vertical constraints are met. To do this, the pushdown system first executes the action *guess_done*, then enters a checking phase. This phase pops characters from the stack until it is empty, except, after each pop, there is a branch that tests whether the tile at the top of the stack and the corresponding tile on the next row satisfy the vertical constraints. To do this, the system remembers the tile in the control state. Then it pops an exponential number of times, and looks at the current top of stack character. This character is the tile in the same horizontal position of the row below. If the pair satisfy the vertical relation, the system executes a *good* action, else it executes *bad*.

The EF formula is then

$$EF\,(guess\_done \wedge \neg EF bad)\ .$$

A valid tiling exists iff the pushdown system with clocks satisfies this formula.

G Prefix-Recognisable Systems

We may expand the definition of PCC by using prefix-recognisable systems as a base. That is, we generalise δ to contain rules of the form $\langle (q, w, \theta), \alpha, \beta, (q', \gamma, Y', \mathbf{w})\rangle$, where α, β, γ are regular languages over Γ. These rules transform stacks of the form uv with $u \in \alpha$ and $v \in \gamma$ to any stack $u'v$ where $u' \in \beta$. Let prPCC denote such systems.

Theorem 10. *Reachability and LTL model checking prPCC is* NEXP*-complete and* coNEXP*-complete respectively, even for prPCC with one 1-reversal-bounded counter and a fixed LTL formula.*

Proof. The upper bounds are argued in the body of the paper. For the lower bounds, we reduce from the `Succinct 0-1 Knapsack` problem. We construct a prPCC with a single-reversal counter that first increments the counter by the value b, then cycles through the inputs a_j, choosing non-deterministically whether the include them. If a_j is included, the counter is decremented by a_j. Finally, a unique final control state is reached, via the action *success*.

As a point of convenience, we will use the prefix recognisable system to simulate a pushdown system (with extra features) whose stack alphabet contains the characters $x_1^0, x_1^1, \ldots, o_{k+m}^0, x_{k+m}^1$, $\#$, $c_1^0, c_1^1, \ldots, c_m^0, c_m^1$. Intuitively, the x characters represent the current assignment to the inputs $x_1, \ldots, x_{k+m}$. These characters will appear at the bottom of the stack and represent the current bit of the current item being examined. The c characters will be used to count large numbers. If the ith bit of an included object's weight is 1, then we need increment the counter 2^i times.

For example, suppose we're on the $111\ldots1$th bit of the $00\ldots0$th item. The stack will initially contain (with the topmost character appearing leftmost)

$$x_{k+m}^1 \ldots x_{k+1}^1 x_k^0 \ldots x_1^0 \ .$$

Using techniques described below, we will calculate the value of this bit. If it is 1, we have to increment $2^{(2^m-1)}$ times. To do this, we push the following sequence on the stack, again using techniques described below.

$$0c_m^0 \ldots c_1^0 \# 0c_m^1 c_{m-1}^0 \ldots c_1^0 \# \ldots \quad \ldots \quad \ldots \# 0c_m^1 \ldots c_1^1 \#$$

That is, an exponential number of 0s, marked — in increasing order (top of stack to bottom) — with their bit positions. From this position, the system counts from an exponential number of 0s to an exponential number of 1s. This takes $2^{(2^m-1)}$ steps. We increment at each step.

To complete the proof, we need to know how to

1. enumerate all assignments to $x_1, \ldots, x_{k+m}$ and evaluate θ.
2. initialise the large counter described above,
3. increment the large counter from $00\ldots0$ to $11\ldots1$.

We deal with these problems in turn.

Problem 1: We first describe how to enumerate all assignments to the inputs of θ. Using a pushdown system, one can count over n bits from $00\ldots0$ to $11\ldots1$ in a straightforward way. First, push $00\ldots0$ on to the stack. Thus, we hold the counter value on the stack, least significant bit first. To increment this value, first, pop from the stack until the top character is a 0, then rewrite this 0 to 1, then push enough 0s on the stack to bring the height back to n. When n is linear, this can be done using the control states of the system.

We need to evaluate θ with respect to the assignments to $x_1, \ldots, x_{k+m}$ (the inputs) implied by the stack contents. We can represent θ as a circuit. We will use the stack to store working values of the outputs of the nodes of the circuit. Hence, assume our circuit has nodes $n_1, \ldots, n_l$ for some l. Moreover, we can assume we know which order to evaluate these nodes in (that is, we evaluate the inputs of a node before trying to determine its output). We use the stack characters $n_1^0, n_1^1, \ldots, n_l^0, n_l^1$ to represent whether each node outputs 1 or 0.

Input values can be checked using the rules of the prefix recognisable system: to test whether the ith bit is true for some i, we have rules of the form $(p, a, a, \Sigma^* x_i^1 \Sigma^*, (p, 1))$ and $(p, a, a, \Sigma^* x_i^0 \Sigma^*, (p, 0))$. We initialise the input nodes (in order) by pushing the appropriate $n_{l_1}^r$ on the stack. To evaluate an internal gate n_{l_1} with inputs (in order of evaluation) n_{l_2} and n_{l_3} computing the operation op, we can evaluate the inputs using rules of the form $(p, a, n_{i_1}^{r_1 op r_2} a, \Sigma^* n_{i_3}^{r_1} \Sigma^* n_{i_2}^{r_2} \Sigma^*, p')$ where p is a control state indicating that we are evaluating node n_{l_1}, p' is a control state indicating that we are to evaluate the next node (or finish) and r_1, r_2 are bit values. Once the output of the circuit has been computed, we can remember it and clear the working values (the n characters) from the stack.

Problem 2: Let $y_1 \ldots y_m$ be the current bit assignment to $x_{k+1}, \ldots, x_{k+m}$. That is $x_{k+1}^{y_1} \ldots b_{k+m}^{y_m}$ is on the top of the stack. The first step in initialising the big counter (of $y_1 \ldots y_m$ bits) is to add $0 c_1^{y_1} \ldots c_m^{y_m} \#$ to the top of the stack. This is done in two stages. First, guess the value by writing any sequence of characters of the right format (disregarding the values of $y_1, \ldots, y_m$), then check that the values match up. This can be done by evaluating the boolean formula

$$\bigvee_{1 \le i \le m} (c_i \iff x_{k+i})$$

using the technique for circuits described above. We can test the value of an atomic proposition c_i using rules of the form $(p, a, a, \Sigma^* c_i^1 \Sigma^*, (p, 1))$ and $(p, a, a, \Sigma^* c_i^0 \Sigma^*, (p, 0))$.

We then need to write a sequence of blocks of the form $0 c_1^{y_1} \ldots c_m^{y_m} \#$ such that consecutive values of $y_1 \ldots y_m$ decrease by 1 each time, ending in $00 \ldots 0$. Hence, we introduce the propositions c_i and c_i' denoting the topmost bits given by the c characters, and the second to topmost. These can be tested using rules as above whose γ components are $(\Sigma^* \setminus \#) c_i^1 \Sigma^*$ and $(\Sigma^* \setminus \#) \# (\Sigma^* \setminus \#) c_i^1 \Sigma^*$ respectively, where $(\Sigma^* \setminus \#)$ is any sequence not containing the $\#$ character.

Hence, we write a new block at a time, the check it against the formula below which asserts the successor relation.

$$\bigvee_j^{n-1} \begin{bmatrix} (\neg c_j \wedge c_j') \wedge \\ \bigwedge_{i>j} (c_i \wedge \neg c_i') \wedge \\ \bigwedge_{i<j} (c_i \iff c_i') \end{bmatrix}$$

Hence, we can initialise the large counter.

Problem 3: Counting from $00 \ldots 0$ to $11 \ldots 1$ over the large counter is similar to Problem 1, except, since n is not linear, we cannot ensure that the stack is built back up to the right height by using the control state. However, we can use the same technique as above to add the correct number of 0 bits.

Note, now, we have all the tools required to build the system we desire. The system behaves as follows. First we write $x_1^0 \dots x_k^0$ to the stack. Since the zeroth item holds b, we increment the counter to contain this value. We do this by cycling through all assignments $x_{k+1}^0 \dots b_{k+m}^0$ to $b_{k+1}^1 \dots b_{k+m}^1$ on the stack and calculating whether the represented bit of θ is 1. If so, we increment the counter as required.

Once this is done, we cycle through the other objects. The system guesses whether to include the object. If so, it does as above to *decrement* the counter by the value of the object. In this way we cycle through all items until we are done, we then move to the final state, provided the counter has returned to zero (without reaching zero too early). If we reach the final state, a solution to the knapsack problem has been found. The required LTL formula is $\mathsf{G}\neg \mathit{success}$.

H Proofs for Higher-Order PDA

Theorem 11. *The control-state reachability problem for order-2 PCo is undecidable.*

Proof. We use the undecidability of Hilbert's 10th problem [22]. We first show that we can assume we are given a conjunction of clauses of the form $x = c$, $x = y + z$ and $x = yz$ for some constant c and variables x, y, z.

Let $P(x_1, \dots, x_n)$ be a polynomial ranging over the integers. It is undecidable whether there is an assignment $\alpha_1, \dots, \alpha_n$ to $x_1, \dots, x_n$ such that $P(\alpha_1, \dots, \alpha_n) = 0$. We start with some observations appearing in Hack [15].

- The arguments can be restricted to range over the natural numbers by testing all 2^n polynomials obtained by replacing some variables with their negation.
- The range of P can be assumed positive since any root of P is also a root of P^2.
- We can separate the positive and negative coefficients in P to obtain $Q_1(x_1, \dots, x_n) \geq Q_2(x_1, \dots, x_n)$ for all assignments to $x_1, \dots, x_n$ such that all coefficients appearing in Q_1 and Q_2 are positive.

Since Q_1 and Q_2 have variables ranging over positive integers and coefficients that are all positive, we can obtain the required normal forms for Q_1 and Q_2 in a straightforward manner. Suppose we have the conjunction of these normal forms with the value of Q_1 in variable x_1 and the value of Q_2 in x_2. We assert $x_1 - x_2 = 0$ (to find a root of P) using $x_1 = x_2 + x'$ and $x' = 0$, for a fresh variable x'.

We build an order-2 PDS with single reversal counters such that the final state is reachable iff the conjunction can be satisfied. Let there be n variables $x_1, \dots, x_n$ and m clauses. The system has several sets of counters. The first set $x_1, \dots, x_n$ will hold an initial guess of the satisfying assignment to the variables. Then, for the αth clause, if it is of the form $x_i = x_j + x_k$ or $x_i = x_j x_k$ we have counters x_i^α, x_j^α and x_k^α.

The system has three stages:

1. initialise counters $x_1, \ldots, x_n$ with a guessed satisfying assignment of the variables, then
2. calculate the value of each of the clauses independently, then
3. check the independent calculations tally.

The first stage is easy to define, we describe the second and third stages in more detail.

The second stage considers each clause in turn, beginning and ending each clause with an empty stack. For clause α, there are three cases:

- Case $x_i = c_i$: nothing is to be done.
- Case $x_i = x_j + x_k$: first (re-)guess the value of x_j by simultaneously incrementing counter x^α_j and pushing a character onto the stack. Then do the same for x^α_k. Finally, increment counter x^α_i while popping from the stack. That is, copy the height of the stack into the counter x^α_i.
- Case $x_i = x_j x_k$: first (re-)guess the value of x_j by simultaneously incrementing counter x^α_j and pushing a character onto the stack. Then (re-)guess the value of x_k by simultaneously incrementing counter x^α_k and performing $push_2$ operations on the stack, giving us x_k copies of x_j on the stack. That is, a total of $x_j x_k$ stack characters. Then, increment x^α_i for every character on the stack. That is, perform pop_1 operations, incrementing x^α_i each time. When the top order-1 stack is emptied, perform a pop_2 without incrementing x^α_i. Continue until the stack is empty.

Notice that, thus far, we have only incremented the counters.

During the final stage we check that the independent guesses of x^α_i for some α and i match the initial guess of x_i. Furthermore, if we have a clause $x_i = c_i$, we check that the value of x_i is in fact c_i. To do this, we go through each i in turn, decrementing x_i and all x^α_is in tandem, then check that all are zero before carrying on to the next variable. For variables that are required to match some constant c_i, we ensure, using the control states, that we perform exactly c_i decrements.

If all tests are passed, we move to the accepting state. Such an order-2 pushdown system with single reversal counters has a non-empty language iff there is a satisfying assignment to the original formula. Hence, we obtain undecidability.