Paths, Trees and Cycles in Tournaments

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Abstract

We survey results on paths, trees and cycles in tournaments. The main subjects are hamiltonian paths and cycles, vertex and arc disjoint paths with prescribed endvertices, arc-pancyclicity, oriented paths, trees and cycles in tournaments. Several unsolved problems are included.

1 Introduction

Tournaments constitute without any doubt the most well-studied class of directed graphs. There are more than 300 research papers dealing explicitly with tournaments and many more in which tournaments are mentioned in some part of the context. One of the main reasons for this huge interest in the theory of tournaments is no doubt the excellent book of J.W. Moon [81]. In 1968, when Moon wrote his monograph, it was still possible to cover most of the results on tournaments in one short book. Now that task seems very difficult, even in a relatively long book, and it is certainly impossible to cover all major results in one survey paper. Thus we have chosen to concentrate on the theory on paths, trees and cycles in tournaments. This theory was only just starting to develop when Moon wrote his monograph. Many significant results have been proved since the late seventies. We list only some of them: A. Thomason's proof (for tournaments with a large number of vertices) of M. Rosenfeld's conjectures on the presence of every orientation of a Hamiltonian path and every non-strong orientation of a Hamiltonian cycle, respectively, in every tournament [113]; R. Häggkvist's proof (for tournaments with a large number of vertices) of Kelly's conjecture (personal communication, see also [37, 122]); C. Thomassen's proof that every 4-strong tournament is Hamiltonian-connected [115], a theorem which has led to several important results; F. Tian, Z.-C. Wu and C.-Q. Zhang's characterization of arc-pancyclic tournaments [125] and K.-M. Zhang's characterization of completely strong path-connected tournaments [137]; and N. Alon's solution of an old conjecture of T. Szele on the maximum number of Hamiltonian paths in tournaments [2].

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There have been other surveys dealing with tournaments [22, 25, 26, 33, 59, 66, 139], but none of these covers more than a small fraction of the results we are going to mention here. As the reader will see, even though we have narrowed the topic to paths, trees, and cycles, we still cannot provide a complete list of all the results in the space that a survey allows. Inevitably, our choice of material is affected by personal tastes. We hope, however, that we have succeeded in covering almost all significant progress within the area. We shall allow ourselves to step away from the main topic a few times and provide the reader with a few results on other kinds of subgraphs of tournaments. This is done to show that the results treated in the paper are closely related to other topics on tournaments.

Several of the results in this paper deal with semicomplete digraphs, a slight generalization of tournaments in which we allow arcs in both directions between some of the vertices. There are two reasons for doing this. The first is that many of the results for tournaments hold for semicomplete digraphs also, without any change in the proof. The second and more important reason is that in some proofs it has turned out to be an advantage to prove the result for semicomplete digraphs. One example of this is Thomassen's proof [115] that every 4-strong tournament is Hamiltonian-connected.

There are several other generalizations of tournaments : semicomplete multipartite digraphs [59], locally semicomplete digraphs [6, 8, 69], locally in-semicomplete digraphs [15, 17], quasi-transitive digraphs [13, 16, 60], path-mergeable digraphs [7] and several others. The main idea in considering these generalizations is to study which of the results for tournaments can be extended to the class in question. In doing so we learn something about the structure of tournaments, namely we see that some of the properties of tournaments are really not just tournament properties. We mention a few examples: every connected locally semicomplete digraph has a Hamiltonian path, every strongly connected locally semicomplete digraph has a Hamiltonian cycle [6], there are polynomial algorithms to decide if a quasi-transitive digraph has a Hamiltonian path and Hamiltonian cycle respectively [13, 60]. Recently, it was shown [14] how to use a result on semicomplete multipartite digraphs to strengthen a theorem on Hamiltonian cycles avoiding prescribed arcs in tournaments (see Section 4).

Many of the results mentioned in this survey have been extended to some of the classes mentioned above, see for example [6, 8, 12, 38, 57, 59]. We do not want to get into a discussion of the definitions or the generalizations of all of these results. Except for semicomplete multipartite digraphs, for which we refer the reader to the survey paper [59], these topics will all be covered in a forthcoming survey paper on generalizations of tournaments.

2 Terminology, notation and preliminaries

The terminology is fairly standard, generally following [27, 39]. The digraphs considered here are finite and have no loops or multiple arcs. $V(D)$ and $A(D)$ denote the vertex set and the arc set of a digraph D. The number of vertices in a digraph is its *order*. For a subset B of $V(D)$, $D\langle B\rangle$ denotes the subgraph of D induced by B.

We shall denote the arc from a vertex x to a vertex y by (x, y) or, simply, xy. If $xy \in A(D)$, we shall say that x *dominates* y and y *is dominated* by x and write $x \rightarrow y$. For disjoint sets of vertices X, Y of a digraph D we shall write $X \Rightarrow Y$ if every vertex of X dominates every vertex of Y and no vertex of Y dominates a vertex of X. An arc $xy \in A(D)$ is *single* if $yx \notin A(D)$.

A digraph T is *semicomplete* if every pair of vertices of T is joined by at least one arc. A semicomplete digraph T is a *tournament* if all arcs of T are single. A semicomplete digraph having no single arc is called *complete*. We denote by TT_n the *transitive tournament* on n vertices, i.e. the tournament with vertex set $\{1, 2, ..., n\}$ and arc set $\{(i, j) \ : \ 1 \leq i < j \leq n\}$. The *almost transitive tournament* on n vertices is obtained from TT_n by reversing the arc $(1, n)$. We define the *rotational* tournament $R(q_1, ..., q_k)$ to be the tournament with vertex set $V = \{0, 1, ..., 2k\}$ where i dominates $i + j$ (modulo $2k + 1$) if and only if $j \in \{q_1, ..., q_k\}$, where $\{q_1, ..., q_k\}$ is a k-element subset of V such that $q_i \neq 0$ and $q_i + q_j \neq 0$ (modulo $2k+1$) for all $i \neq j \in \{1, ..., k\}$.

The *out-neighbourhood* $N^+(x)$ of a vertex x in a digraph D is the set of all vertices of D dominated by x. The *in-neighbourhood* $N^-(x)$ of a vertex x is the set of all vertices of D which dominate x. The number of vertices in the out-neighbourhood (in-neighbourhood) of x is the *out-degree* $d^+(x)$ *(in-degree* $d^-(x)$) of x. A digraph D is called k*-diregular* (or, just, *diregular*) if $d^+(x) = d^-(x) = k$ for every vertex x in D. A digraph D is m*-irregular* if there exists a natural number k such that $max\{|k - d^+(x)|, |k - d^-(x)|\} \leq m$ for every vertex $x \in V(D)$. So, a diregular digraph is 0-irregular.

If the arcs of a directed walk W are distinct, we call W a *trail*. A trail is *closed* it it contains at least one arc and its first and last vertices are the same. By a *cycle (path)* we mean a directed simple cycle (path). An (x, y)*-path* is a path from x to y. The *length* of a path or cycle is the number of arcs in it. A k*-cycle* (k*-path*) is a cycle (path) of length k. A path or cycle of a digraph D is called *Hamiltonian* if it contains all the vertices of D. A digraph D is *weakly (strongly) Hamiltonian connected* if, for every ordered pair (x, y) of distinct vertices of D there is a Hamiltonian path between them (from x to y, respectively).

A digraph D is *strongly connected* (or just *strong*), if there exists an (x, y)-path and a (y, x)-path for every choice of distinct vertices x, y of D. If D is not strong, then we can order its strong components $D_1, \ldots, D_s$ so that there is no arc from D_j to D_i for $i < j$. In general this ordering is not unique but it is so for semicomplete digraphs (see Proposition 2.1). An initial (terminal) component of a digraph D is a strong component with no incoming (outgoing) arcs.

A digraph D is $k-$*strong* (or k*-strongly connected*), if for every $S \subset V(D)$ of at most $k - 1$ vertices, $D - S$ is strong. A set S of vertices of a digraph D is called an (x, y)*-separator of size* k if $D - S$ has no (x, y)-paths and $|S| = k$. S is called *trivial* if either x has out-degree zero or y has in-degree zero in $D - S$. A digraph D is k*-arc-strong* if for every subset Q of $A(D)$ of at most $k - 1$ arcs, $D - Q$ is strong. The following simple formula for finding $asc(T)$, the maximum k such that a given tournament T of order n is k-arc-strong, is obtained in [23]:

$$asc(T) = \min_{1 \leq m \leq n-1} \{s_m - m(m-1)/2\},$$

where s_m is the sum of the m smallest out-degrees in T.

An *out-branching* (*in-branching*) rooted at a vertex v in a digraph D is a spanning oriented tree T of D such that the in-degree (out-degree) of v is zero and the in-degree (out-degree) of every other vertex of T is one.

The following two easy observations will be helpful in understanding some of the remarks in later sections.

Proposition 2.1 *If a semicomplete digraph T is not strong, then there exists a unique ordering $T_1, \dots, T_k$ of its strong components such that $V(T_i) \Rightarrow V(T_j)$ whenever $i < j$. In particular, non-strong semicomplete digraphs have unique initial and terminal components.*

Proposition 2.2 *Let T be a semicomplete digraph with two distinct vertices x, y, such that there are two internally disjoint $(x, y)-$paths P_1, P_2 in T. P_1, P_2 can be merged into one $(x, y)-$path P such that $V(P) = V(P_1) \cup V(P_2)$. P can be found in linear time.*

We conclude this section with two well-known results by L. Redei [92] and P. Camion [41], respectively.

Theorem 2.3 *Every semicomplete digraph has a Hamiltonian path.*

Theorem 2.4 *A semicomplete digraph D has a Hamiltonian cycle if and only if D is strongly connected.*

The algorithmic aspects of the Hamiltonian path and cycle problems for tournaments are discussed in [59].

3 Arc-disjoint Hamiltonian paths and cycles and Kelly's conjecture

Kelly's conjecture is probably one of the best known conjectures in tournament theory.

Conjecture 3.1 *The arcs of a diregular tournament of order n can be partitioned into $(n-1)/2$ Hamiltonian cycles.*

This conjecture was verified for $n \leq 9$ by P. Seymour and C. Thomassen [103]. B. Jackson proved that every diregular tournament of order at least 5 contains a Hamiltonian cycle and a Hamiltonian path that are arc-disjoint [70]. It was proved, in [133] and [127], respectively, that there are two (three, respectively) arc-disjoint Hamiltonian cycles for $n \geq 5$ ($n \geq 15$, respectively). C. Thomassen proved the following:

Theorem 3.2 *[117] Every diregular tournament of order n has at least $\lfloor \sqrt{n/1000} \rfloor$ arc-disjoint Hamiltonian cycles.*

This result was improved by R. Häggkvist [62] to the following:

Theorem 3.3 *There is a positive constant c (in fact $c \geq 2^{-15}$) such that every diregular tournament of order n contains at least cn arc-disjoint Hamiltonian cycles.*

A better lower bound for the constant c was obtained in [65], provided n is sufficiently large. C. Thomassen [120] proved that the arcs of every diregular tournament can be covered by $12n$ Hamiltonian cycles. R. Häggkvist claims that he has proved that Kelly's conjecture is true for sufficiently large n (personal communications (1986,1994), see also [37, 122]). Häggkvist's proof which has unfortunately never been written up in full detail is based on so-called exact sorters (a concept defined by Häggkvist and which we shall not define here; some information on applications of sorters in cycle problems can be found in [64, 65]) and uses probabilistic arguments.

Let T be the tournament obtained from two diregular tournaments T_1 and T_2 each on $2m+1$ vertices, by adding all arcs from the vertices of T_1 to T_2 (i.e. $V(T_1) \Rightarrow V(T_2)$ in T). Clearly T is not strong and so has no Hamiltonian cycle. The minimum in-degree and minimum out-degree of T is m which is about $\frac{1}{4}|V(T)|$. B. Bollobás and R. Häggkvist [37] showed that if we increase the minimum in-degree and out-degree slightly, then, not only do we get many arc-disjoint Hamiltonian cycles, we also get a very structured set of such cycles. The *kth power* of a cycle C is the digraph obtained from C by adding an arc from $x \in V(C)$ to $y \in V(C)$ if and only if the length of the (x, y)-path contained in C is less than or equal to k. Hence the 1st power of C is C itself.

Theorem 3.4 *[37] For every $\epsilon > 0$ and every natural number k there is a natural number $n(\epsilon, k)$ with the following property. Let T be a tournament of order $n > n(\epsilon, k)$ such that $min\{d^+(x), d^-(x)\} \geq (\frac{1}{4} + \epsilon)n$ for every vertex $x \in V(T)$. Then T contains the kth power of a Hamiltonian cycle.*

Note that, trivially, a tournament on n vertices with minimum in-degree and out-degree at least $\frac{1}{4}n$ is strongly connected.

We now turn our attention to other results concerning arc-disjoint Hamiltonian paths and cycles in tournaments. In [117], C. Thomassen completely characterised tournaments having at least two arc-disjoint paths.

Theorem 3.5 *A tournament T fails to have two arc-disjoint Hamiltonian paths if and only if T has a strong component which is an almost transitive tournament of odd order or has two consecutive strong components of order 1.*

He also considered the existence of a Hamiltonian path P and a Hamiltonian cycle C which are arc-disjoint.

Theorem 3.6 *[117] Let T be a tournament of order at least 3 such that each arc of T is contained in a 3-cycle. Then T has a Hamiltonian path and a Hamiltonian cycle which are arc-disjoint unless T is a 3-cycle or the tournament of order 5 obtained from a 3-cycle by adding two vertices x, y and the arc xy and letting y (x) dominate (be dominated by) the vertices of the 3-cycle.*

The last theorem generalized the result of B. Jackson mentioned in the beginning of this section. However, Theorem 3.6 goes much further since almost all tournaments satisfy the assumption of the theorem (see [81]). The following conjecture, in some sense generalizing Kelly's conjecture, was proposed by C. Thomassen.

Conjecture 3.7 *[117] For every $\epsilon > 0$ almost all tournaments of order n have $\lfloor (0.5-\epsilon)n \rfloor$ arc-disjoint Hamiltonian cycles.*

P. Erdös raised the following question (see [117]).

Question 3.8 *Do almost all tournaments have $\delta(T)$ arc-disjoint Hamiltonian cycles, where $\delta(T)$ is the minimum of all in-degrees and out-degrees of T?*

C. Thomassen posed another conjecture.

Conjecture 3.9 *[117] For each integer $k \geq 2$ there exists an integer $\alpha(k)$ such that every $\alpha(k)$-strong tournament has k arc-disjoint Hamiltonian cycles.*

He conjectured that $\alpha(2) = 3$.

4 Hamiltonian cycles avoiding prescribed arcs

The following result was conjectured by C. Thomassen [117] and proved by P. Fraisse and C. Thomassen [49]:

Theorem 4.1 *If I is a set of $k-1$ arcs in a k-strong tournament T, then $T-I$ has a Hamiltonian cycle.*

This result is sharp since the deletion of a set I of k arcs from a k-strong tournament may create a vertex of indegree or outdegree 0. However, the authors of [49] realized that, for some sets I, their bound was far from being the best possible (see, e.g., Section 5 in [117]).

In [14], the following stronger result was obtained:

Theorem 4.2 *Let $T = (V, A)$ be a k-strong tournament on n vertices. Suppose that $X_1, X_2, ..., X_l$ be a partition of the vertex set V of T such that $|X_1| \leq |X_2| \leq \ldots \leq |X_l| \leq n/2$. If $k \geq k' = \sum_{i=1}^{l-1} \lfloor |X_i|/2 \rfloor + |X_l|$, then $T - \cup_{i=1}^{l} \{xy \in A : \ x, y \in X_i\}$ has a Hamiltonian cycle.*

It is easy to see that Theorem 4.1 follows from Theorem 4.2. Indeed, let I be a set of arcs in a tournament T, let G be the undirected graph obtained by ignoring all orientations of the arcs of $T\langle I \rangle$, the subgraph of T which has arc set I and no isolated vertices, and let $Y_1, ..., Y_m$ be the vertex sets of the connected components of G so that $|Y_1| \leq ... \leq |Y_m|$. By Theorem 4.2, T has a Hamiltonian cycle avoiding the arcs in I if T is k'-strong, where $k' = \sum_{i=1}^{m-1} \lfloor |Y_i|/2 \rfloor + |Y_m|$. But $k' \leq 1 + \sum_{i=1}^{m} (|Y_i| - 1) \leq 1 + \sum_{i=1}^{m} e(Y_i) \leq 1 + |I|$, where $e(Y_i)$ is the number of edges in the component of G induced by Y_i.

A simple analysis of the last calculation shows precisely when both theorems provide the same value of strong connectivity of T – namely, when I consists of one tree, plus maybe some independent arcs. In all other cases Theorem 4.2 gives a better bound. In particular, if $T\langle I\rangle$ is a union of (vertex) disjoint subtournaments of T of order $n_1, ..., n_m$ $(3 \le n_1 \le ... \le n_m)$, then, to guarantee that $T - I$ has a Hamiltonian cycle, we need T to be $(\sum_{i=1}^{m} \binom{n_i}{2} + 1)$-strong by Theorem 4.1 and to be $(n_m + \sum_{i=1}^{m-1} \lfloor n_i/2 \rfloor)$-strong by Theorem 4.2.

5 Hamiltonian paths with prescribed endvertices and Hamiltonian cycles through given arcs

In this section we consider the following three problems regarding Hamiltonian paths and cycles. Below we assume that we are given a digraph D and two distinct vertices x, y.

(1) *Weak Hamiltonian connectedness problem.* Decide if there is a Hamiltonian path with endvertices x, y (the order not specified) and find one (if it exists).

(2) *Strong Hamiltonian connectedness problem.* Decide if there is a Hamiltonian (x, y)-path and find one (if it exists).

(3) *Hamiltonian cycle through k arcs problem (the k-HCA problem).* Decide if there is a Hamiltonian cycle in D containing k specified arcs $a_1, ..., a_k$.

In the case $k = 1$ all these problems are equivalent for general digraphs, from an algorithmic point of view, and, moreover, they are NP-complete [53]. We now restrict ourselves to semicomplete digraphs, and, as we shall see, even though the first two problems are polynomial time solvable, they are not quite trivial. Moreover, the k-HCA problem is even NP-complete if k is not fixed [20].

From a theoretical point of view the complexities of the problems (as a measure of difficulty) are rather different : there is a complete mathematical characterization in the case of (1), but no such characterization has been found for (2) or (3).

The weak Hamiltonian connectedness problem for tournaments was completely solved by C. Thomassen who proved the following:

Theorem 5.1 *[115].*
Let D be a tournament and let x, y be distinct vertices of D. Then D has a Hamiltonian path connecting x and y (from x to y or from y to x), if and only if D does not satisfy any of the conditions $(1) - (4)$ *below.*

(1) D is not strong and either the initial or the terminal strong component of D (or both) contains none of x and y;

(2) D is strong, $D - x$ is not strong and either y belongs to neither the initial nor the terminal strong component of $D - x$.

(3) D is strong, $D - y$ is not strong and either x belongs to neither the initial nor the terminal strong component of $D - y$.

x

y

Figure 1: The two exceptional tournaments (the edge between x and y can be oriented arbitrarily)

(4) $D, D-x$, and $D-y$ are all strong and D is isomorphic to one of the two tournaments shown in Figure 1.

The proof of Theorem 5.1 in [115] is based on the following interesting result.

Theorem 5.2 *For every three vertices of a strong semicomplete digraph there is a Hamiltonian path connecting two of them.*

C. Thomassen pointed out in [115] that the characterization of Theorem 5.1 is valid for semicomplete digraphs except for some more digraphs with four and six vertices. It follows from Theorem 5.1 that the existence of a Hamiltonian path between specified vertices x, y can be checked in polynomial time. Moreover, the proof of Theorem 5.1 in [115] provides a polynomial algorithm for constructing a Hamiltonian path between x and y (if one exists). Theorem 5.1 immediately implies the following characterization of weakly Hamiltonian connected tournaments.

Theorem 5.3 *A tournament T with at least three vertices is weakly Hamiltonian connected, if and only if it satisfies (i),(ii), and (iii) below.*

(i) T is strong.

(ii) For each vertex x of T, $T-x$ has at most two strong components.

(iii) T is not isomorphic to either of the two tournaments shown in Fig. 1.

In [115] C. Thomassen considered not only the weak Hamiltonian connectedness problem but also the strong Hamiltonian connectedness problem (for semicomplete digraphs). This problem (for tournaments) was posed in the Russian edition (1968) of [88] and in [108]. In particular, C. Thomassen proved the following useful lemma.

Lemma 5.4 *[115] A tournament T has a Hamiltonian path from u to v if and only if T has a spanning acyclic subgraph H such that for each vertex $x \in V(T)$, H contains both (u,x)- and (x,v)-paths.*

Note that Lemma 5.4 is an easy consequence of Proposition 2.2.

C. Thomassen characterized those tournaments T which contain a Hamiltonian (x, y)-path when x and y are specified vertices of maximum and minimum degree, respectively.

Theorem 5.5 *[117] Let T be a tournament with at least two vertices and let x (respectively, y) be a vertex of T of maximum (respectively, minimum) out-degree in T. Then T has a Hamiltonian (x, y)-path unless one of the statements below holds.*

- *(i) The terminal strong component of $T - x$ consists of one vertex z dominating x, and the preceding strong component of $T - x$ consists of y only and y is dominated by x.*
- *(ii) The initial strong component of $T - y$ consists of one vertex z dominated by y, and the second strong component of $T - y$ consists of x only and x dominates y.*
- *(iii) T is isomorphic to the rotational tournament $R(1, 2)$ or to one of the two tournaments in Figure 1.*

C. Thomassen also proved the following sufficient conditions for the existence of a Hamiltonian (x, y)-path in a semicomplete digraph.

Theorem 5.6 *[115] Let D be a 2-strong semicomplete digraph and let $x, y \in V(D)$. Then D contains a Hamiltonian (x, y)-path if either (i) or (ii) below is satisfied.*

- *(i) D contains three internally disjoint (x, y)-paths each of length at least 2.*
- *(ii) D contains a vertex z which is dominated by all vertices of D except possibly x, and D contains two internally disjoint (x, y)-paths of length at least 2 and a (z, y)-path having only y in common with the above mentioned (x, y)-paths.*

Corollary 5.7 *If a semicomplete digraph T is 3-strong and y dominates x, then there is a Hamiltonian path from x to y.*

Corollary 5.8 *Every 4-strong semicomplete digraph is strongly Hamiltonian connected.*

Moreover, C. Thomassen [115] constructed infinitely many 2-strong tournaments containing an arc which is not contained in any Hamiltonian cycle and infinitely many 3-strong non-strongly Hamiltonian connected tournaments. So far no characterization of strongly Hamiltonian connected tournaments is known.

J. Bang-Jensen, Y. Manoussakis and C. Thomassen [19] considered algorithmic aspects of the strong Hamiltonian connectedness problem for semicomplete digraphs. They found a polynomial algorithm for solving the last problem based on a number of structural results and, in particular, on the following theorem which is a generalization of Theorem 5.6 (i).

Theorem 5.9 *Let T be a 2-strong semicomplete digraph on at least 10 vertices and let x, y be vertices of T such that x does not dominate y. Suppose that $T-x, T-y$ are both 2-strong. If all (x, y)-separators of size 2 (if any) are trivial, then T has a Hamiltonian (x, y)-path.*

The main result in [19] is the following:

Theorem 5.10 *There exists an $O(n^5)$ algorithm to check whether a given semicomplete digraph of order n with specified vertices x, y has a Hamiltonian (x, y)-path.*

Theorem 5.10 implies:

Corollary 5.11 *There exists an $O(n^7)$ algorithm to check whether a given semicomplete digraph of order n is strongly Hamiltonian connected.*

Corollary 5.12 *There is an $O(n^7)$ algorithm for constructing a Hamiltonian (x, y)-path (if one exists) in a semicomplete digraph of order n with two distinguished vertices x and y.*

Another interesting lemma that played an important role in the development of the algorithm in [19] is the following. If x, w, z are distinct vertices of a digraph D, then we use the notation $Q_{x,z}$, $Q_{.,w}$ to denote two disjoint paths such that the first path has initial vertex x and terminal vertex z, the second path has terminal vertex w, and $V(Q_{x,z}) \cup V(Q_{.,w}) = V(D)$. Similarly $Q_{z,x}$ and $Q_{w,.}$ denote two disjoint paths, such that the first path has initial vertex z and terminal vertex x, and the second path has initial vertex w, and $V(Q_{z,x}) \cup V(Q_{w,.}) = V(D)$.

Lemma 5.13 *Let x, w, z be distinct vertices in a semicomplete digraph T on n vertices, such that there exist internally disjoint $(x, w)-, (x, z)-$paths P_1, P_2 in T. Let $R = T - V(P_1) \cup V(P_2)$.*

1. *There are either $Q_{x,w}$, $Q_{.,z}$ or $Q_{x,z}$, $Q_{.,w}$ in T, unless $(V(P_1) \cup V(P_2) \setminus \{x\}) \Rightarrow R_t$, where R_t is the terminal component of R.*

2. *In the case when there is an arc from R_t to $V(P_1) \cup V(P_2) \setminus \{x\}$ we can find a pair of paths, such that the path with only one endvertex specified has length at least one, unless $V(T) = \{w, x, z\}$.*

3. *Moreover there is an $O(n^2)$ algorithm to find a pair of paths above if such a pair exists.*

Conjecture 5.14 *There exists a polynomial algorithm to find the length of the longest (x, y)-path in a semicomplete digraph with specified vertices x and y.*

Note that the existence of such an algorithm does not seem to follow from Theorem 5.10. The algorithm in [19] cannot be easily modified to solve this problem as well.

Now we consider known results on the k-HCA problem. A *k-path factor* in a digraph D is a collection of k vertex disjoint paths of D covering $V(D)$. C. Thomassen obtained the following theorem for tournaments with high strong connectivity (the definition of the function $f(k)$ is given in Theorem 4.1).

Theorem 5.15 *[119] If $\{x_1, y_1, ..., x_k, y_k\}$ is a set of distinct vertices in an $h(k)$-strong tournament T, where $h(k) = f(5k) + 12k + 9$, then T has a k-path factor $P_1 \cup P_2 \cup ... \cup P_k$ such that P_i is an (x_i, y_i)-path for $i = 1, ..., k$.*

The last theorem can be reformulated as follows:

Theorem 5.16 *[119] If $a_1, ..., a_k$ are pairwise independent arcs in an $h(k)$-strong tournament T, then T has a Hamiltonian cycle containing $a_1, ..., a_k$ in that cyclic order.*

Combining the ideas of avoiding and containing, C. Thomassen showed the following:

Theorem 5.17 *[119] For every set A_1 of at most k arcs in an $h(k)$-strong tournament T and for every set A_2 of at most k independent arcs of $T - A_1$, the digraph $T - A_1$ has a Hamiltonian cycle containing A_2.*

Based on the evidence from Theorem 5.10 the authors of [19] proposed the following conjecture (the truth of the case $k = 1$ follows from Theorem 5.10) :

Conjecture 5.18 *For each fixed k, the k-HCA problem is polynomially solvable for semicomplete digraphs.*

In [20] J. Bang-Jensen and C. Thomassen proved that the k-HCA problem for tournaments is NP-complete when k is not fixed. The proof of this result in [20] contains an interesting idea which we generalize below. Consider a digraph D containing a set W of k vertices such that $D - W$ is a tournament. Construct a new tournament D_W as follows. First, split every vertex $w \in W$ into two vertices w_1, w_2 such that all arcs entering w (respectively, leaving w) now enter w_1 (respectively, leave w_2). Add all possible arcs from vertices of index 1 to vertices of index 2 (whenever the arcs in the opposite direction are not already present). Add all arcs between vertices of the same index and orient them randomly. Finally, add all arcs of the kind $w_1 z$ and $z w_2$, where $w \in W$ and $z \in V(D) - W$ and the corresponding opposite arc does not exist.

It is easy to see the following:

Proposition 5.19 *Let W be a set of k vertices of a digraph D such that $D - W$ is a tournament. Then D has a cycle of length $c \geq k$ containing all vertices of W, if and only if the tournament1 D_W has a cycle of length $c + k$ through the arcs $\{w_1 w_2 : w \in W\}$.*

Proposition 5.19 allows us to study cycles through W, where $|W| = k$, in digraphs D such that $D - W$ is a tournament instead of studying cycles containing k fixed arcs in tournaments.

Note that if k is not fixed, then it is an NP-complete problem to decide the existence of a cycle through k given vertices in such a digraph. Simply take $k = |V(D)|$; then this is the Hamiltonian cycle problem for general digraphs. This proves that the k-HCA problem is NP-complete for tournaments.

Now we can formulate Conjecture 5.18 in the following way :

Conjecture 5.20 *Let k be a fixed natural number and let D denote a digraph obtained from a semicomplete digraph by adding at most k new vertices and some arcs. There exists a polynomial algorithm to decide if there is a Hamiltonian cycle in D.*

The truth of the conjecture when $k = 1$ follows from Proposition 5.19 and Theorem 5.10.

We conclude this section with a nice conjecture due to Zs. Tuza [126].

Conjecture 5.21 *[126] Let s be a positive integer and suppose that T is a semicomplete digraph such that for every $Y \subset V(T)$, $|Y| < s$, the subgraph $T - Y$ is strong and has at least one single arc. Then there exists a Hamiltonian cycle in T which has at least s single arcs.*

Zs. Tuza showed that it is enough to prove that there is a cycle of length at least $s + 1$ containing s single arcs:

Proposition 5.22 *[126] If a strong semicomplete digraph T has a cycle of length at least $s + 1$ which contains at least s single arcs, then T has a Hamiltonian cycle with at least s single arcs.*

Zs. Tuza has proved the existence of such a cycle for $s = 1, 2$ [126]. It is easy to see that $s + 1$ cannot be replaced by s in Proposition 5.22.

6 Paths with prescribed endvertices and cycles through prescribed arcs

In this section, we consider the following three problems.

1) *The k paths with fixed endvertices problem (the k-PV problem):* Given distinct vertices $u_1, ..., u_k, v_1, ..., v_k$ in a digraph D; does D have k disjoint paths $P_1, ..., P_k$ such that P_i is a (u_i, v_i)-path for $i = 1, ..., k$?

2) *(x, z)-path through k specified vertices (the k-(x, z)-PV problem):* Given distinct vertices $x, z, y_1, \ldots, y_k$ in a digraph D; does D have an (x, z)-path which contains all the vertices $y_1, \ldots, y_k$?

3) *The cycle through k fixed arcs problem (The k-CA problem):* Given a set of k arcs B in a digraph D; does D have a cycle of D containing all the arcs in B ?

For general digraphs, it is easy to see that the k-PV, $(k-1)$-(x, z)-PV and k-CA problems are polynomially equivalent. Each of the problems 2-PV, 1-(x, z)-PV and 2-CA is NP-complete for general digraphs [51].

Note also that the $(k-1)$-(x, z)-PV and the k-PV problems are even polynomially equivalent inside the class of semicomplete digraphs (the idea given in [10] in the case $k = 2$ can be easily generalized). Certainly, from a theoretical point of view, the three problems above are absolutely different.

There is a polynomial algorithm for solving the 2-PV problem for acyclic digraphs [51, 121]. J. Bang-Jensen and C. Thomassen considered this problem for semicomplete digraphs. They proved:

Theorem 6.1 *[20] There exists an $O(n^5)$ algorithm for solving the 2-PV problem (and, hence, the 2-CA problem) for a semicomplete digraph on n vertices.*

The last theorem is based on the following two structural results which are of independent interest.

Theorem 6.2 *[9] Let T be a semicomplete digraph and let x_1, x_2, y_1, y_2 be distinct vertices of T. If $T - \{x_i, y_i\}$ has three internally disjoint (x_{3-i}, y_{3-i})-paths and $T - \{x_{3-i}, y_{3-i}\}$ has two internally disjoint (x_i, y_i)-paths , for $i = 1$ and 2, then T has a pair of disjoint (x_1, y_1)- and (x_2, y_2)-paths.*

J. Bang-Jensen showed that Theorem 6.2 is the best possible in the sense that "three" cannot be replaced by "two", and "two" cannot be replaced by "one" [9].

Theorem 6.3 *[20] Let T be a semicomplete digraph and let x_1, x_2, y_1, y_2 be distinct vertices of T such that, for each $i = 1, 2$, there are two, but not three, internally disjoint (x_i, y_i)-paths in $T - \{x_{3-i}, y_{3-i}\}$. Suppose that all (x_i, y_i)-separators of size 2 in $T - \{x_{3-i}, y_{3-i}\}$ are trivial, for $i = 1, 2$. Then T has a pair of disjoint (x_1, y_1)-,(x_2, y_2)-paths.*

Because the 2-PV and 1-(x, z)-PV problems are polynomially equivalent inside the class of semicomplete digraphs, Theorem 6.1 implies the following:

Corollary 6.4 *The 1-(x, z)-PV problem is polynomially solvable for semicomplete digraphs.*

In [10] a simpler $O(n^4)$ algorithm is described for the 1-(x, z)-PV problem. This algorithm is based on the following structural characterization of those semicomplete digraphs which do not have an (x, z)-path through a specified vertex y.

Definition 6.5 *Let (T, x, y, z) denote a semicomplete digraph with special vertices x, y and z and at least 4 vertices. We say that the tuple (T, x, y, z) is of type 1 if $T - z$ has an (x, y)-path, $T - x$ contains a (y, z)-path and there exists a vertex $u \in V(D) - \{x, y, z\}$ such that there is no $(x, y)-$path in $D - \{u, z\}$ and there is no $(y, z)-$path in $D - \{u, x\}$.*

The semicomplete digraphs of type 1 constitute the simple class that forms the basis of our construction. We call them simple, because there is a very simple reason why a semicomplete digraph (T, x, y, z) of type 1 has no $(x, z)-$path through y.

Define two operations as follows:

Operation 1: Let (T, x, y, z) be given. Let T' denote a semicomplete digraph with $V(T) \cap V(T') = \emptyset$, containing a vertex x' that can reach all other vertices in T' by paths. Let (T^*, x', y, z) denote the semicomplete digraph obtained from T and T' by adding arcs as follows:

- $(V(T) \setminus \{x, z\}) \Rightarrow V(T')$
- at least one vertex in T' dominates x,

- $z \Rightarrow (V(T') \setminus x')$,
- there can be either two arcs or one arc in any direction between x' and z.

Operation 2: Let (T, x, y, z) be given. Let T'' denote a semicomplete digraph with $V(T) \cap V(T'') = \emptyset$, containing a vertex z'' that can be reached by all other vertices in T'' by directed paths. Let (T^*, x, y, z'') denote the semicomplete digraph obtained from T and T'' by adding arcs as follows:

- $V(T'') \Rightarrow (V(T) \setminus \{x, z\})$,
- at least one vertex in T'' is dominated by z,
- $(V(T'') \setminus z'') \Rightarrow x$,
- there can be either two arcs or one arc in any direction between x and z''.

Call a semicomplete digraph (T, x, y, z) $(\hat{T}, \hat{x}, \hat{y}, \hat{z})$-*obtainable* if (T, x, y, z) can be constructed from $(\hat{T}, \hat{x}, \hat{y}, \hat{z})$ by zero or more applications of operations 1 and 2 in any order.

Theorem 6.6 *[10] The semicomplete digraph (T, x, y, z) with at least 4 vertices has an $(x, z)-$path through y if and only if it is not $(\hat{T}, \hat{x}, \hat{y}, \hat{z})$-obtainable from any $(\hat{T}, \hat{x}, \hat{y}, \hat{z})$ of type 1.*

No similar structural characterization is known for the 2-PV problem for semicomplete digraphs.

In Section 5, we mentioned the result by J. Bang-Jensen and C. Thomassen [20] claiming that the k-HCA problem is NP-complete for tournaments when k is not fixed. This result follows from the original proof of the following theorem.

Theorem 6.7 *[20] The k-CA problem (and thus the k-PV problem) is NP-complete for tournaments when k is not fixed.*

We now consider structural aspects of the k-CA and k-PV problems. In [119], C. Thomassen studied the k-PV problem and proved the following:

Theorem 6.8 *For every natural number k, there exists a smallest natural number $f(k)$ such that the following holds: If $x_1, y_1, ..., x_k, y_k$ are distinct vertices in an $f(k)$-strong tournament T, then T has k disjoint paths $P_1, ..., P_k$ such that P_i is an (x_i, y_i)-path for each $i = 1, ..., k$.*

He showed that $f(k) \leq 2^{k-1}(k-1)!$ and asked if in fact $f(k) = ck$, where c is a natural number.

C.Thomassen [124] proved that the last theorem cannot be generalized to the class of all digraphs. He showed that for every natural number r there exists an r-strong digraph D with specified vertices x_1, x_2, y_1, y_2 such that D has no pair of disjoint (x_1, y_1)- and (x_2, y_2)-paths. Thus the generalization of the last theorem for general digraphs is not even true for $k = 2$. Note also the following interesting generalization of Corollary 5.8:

Theorem 6.9 *[119] Let x and y be any vertices in a $4(s+t)$-strong tournament (where s and t are non-negative integers not both zero). Then T contains a collection of $s+t$ internally disjoint paths, s of which are (x,y)-paths and t of which are (y,x)-paths and whose union contains all vertices of T.*

A digraph D is called *k-arc cyclic* if, for every set B of at most k arcs, D contains a cycle through B. Sufficient conditions for semicomplete digraphs and tournaments to be 2-arc cyclic are studied in [9] where the following theorem is proved.

Theorem 6.10 *Every 5-strong semicomplete digraph is 2-arc cyclic; every 3-strong tournament is 2-arc cyclic.*

In [9] it is noted that both results are the best possible in terms of the required connectivity.

7 Arc-disjoint paths with prescribed endvertices

The problems in Section 6 can also be formulated in terms of arc-disjoint, rather than vertex-disjoint paths. We call these analogous problems for arc-disjoint paths the k-APV, k-(x,z)-TV and k-CTA problems (T stands for trail and CT for closed trail). Here we are asking for a trail and a closed trail in the analogues of the problems (2) and (3) of Section 6.

As mentioned in Section 6, there is no degree of strong connectivity which will guarantee the existence of two disjoint paths with prescribed endvertices in a general digraph [124]. In the case of arc-disjoint paths, there is a nice and simple proof, due to Y. Shiloach [105], that k-arc-strong connectedness will guarantee the existence of k arc-disjoint paths with prescribed endvertices : Let D be a k-arc-strong digraph and $(x_1,y_1),\dots,(x_k,y_k)$ be the pairs to be connected by arc-disjoint paths. Add a new vertex z and arcs zx_i, $i=1,\dots,k$ and let all other vertices have arcs to z. This new digraph is k-arc-strong and by J. Edmonds' branching Theorem [48] it has k arc-disjoint out-branchings $F_1,\dots,F_k$ rooted at z. We may assume that F_i contains the arc zx_i. Taking a path from x_i to y_i in F_i, $i=1,\dots,k$ we obtain the desired paths.

So the arc-disjoint version seems simpler. In particular every 2-arc-strong digraph has two arc-disjoint paths with prescribed endvertices. However, this problem is still NP-complete as shown in [51].

In [11] J. Bang-Jensen considered the 1-(x,z)-TV problem for semicomplete digraphs. He proved that an obvious necessary condition is also sufficient :

Theorem 7.1 *Let T be a semicomplete digraph and x,y,z distinct vertices of T. Suppose that T has an (x,y)-path and a (y,z)-path. Then T has an (x,z)-trail through y if and only if there is no arc $e\in A(T)$ such that $T-e$ has no (x,y)-path and no (y,z)-path.*

The proof of Theorem 7.1 in [11] implies:

Corollary 7.2 *The 1-(x,z)-TV problem is solvable in time $O(n^3)$ for semicomplete digraphs. Furthermore, given three distinct vertices x, y, z, there is an $O(n^3)$ algorithm which finds an (x,z)-trail through y if one exists.*

In [11] the 2-APV problem was also solved in terms of a complete characterization and an algorithm. We need some notation before we can state the result.

Definition 7.3 *Let T be a strong semicomplete digraph and let x_1, x_2, y_1, y_2 be distinct vertices of T. The 5-tuple (T, x_1, x_2, y_1, y_2) is said to be of*

Type 1a, *if there exists a partitioning S_1, S_2 of $V(T)$ such that $x_1, x_2 \in S_2$, $y_1, y_2 \in S_1$ and there is exactly one arc from S_2 to S_1;*

Type 1b, *if it is not of Type 1a and there exists a partitioning S_1, S_2, S_3 of $V(T)$ such that the following holds for $i = 1$ or 2: $y_i \in S_1$, $x_i, y_{3-i} \in S_2$, $x_{3-i} \in S_3$, $S_1 \Rightarrow S_2 \Rightarrow S_3$ and there is exactly one arc from S_3 to S_1, namely one from the terminal component of $T\langle S_3 \rangle$ to the initial component of $T\langle S_1 \rangle$;*

Type 2r *(for some $r \geq 1$), if there exists a partitioning $S_1, S_2, \ldots, S_{2r+2}$ of $V(T)$ such that the following holds for $i = 1$ or 2 : $y_i \in S_1$, $y_{3-i} \in S_2$, $x_{3-i} \in S_{2r+1}$, $x_i \in S_{2r+2}$, $S_j \Rightarrow S_k$ for all $j < k$ with the following exceptions: there is precisely one arc from S_q to S_{q-2} for $q = 3, \ldots, 2r+2$ and it goes from the terminal component of $T\langle S_q \rangle$ to the initial component of $T\langle S_{q-2} \rangle$;*

Type 2r+1 *(for some $r \geq 1$), if there exists a partitioning $S_1, S_2, \ldots, S_{2r+3}$ of $V(T)$ such that the following holds for $i = 1$ or 2 : $y_i \in S_1$, $y_{3-i} \in S_2$, $x_i \in S_{2r+2}$, $x_{3-i} \in S_{2r+3}$, $S_j \Rightarrow S_k$ for all $j < k$ with the following exceptions: there is precisely one arc from S_q to S_{q-2} for $q = 3, \ldots, 2r+3$ and it goes from the terminal component of $T\langle S_q \rangle$ to the initial component of $T\langle S_{q-2} \rangle$.*

Theorem 7.4 *Let T be a strong semicomplete digraph and let x_1, x_2, y_1, y_2 be distinct vertices of T. Suppose that T has an (x_i, y_i)-path for $i = 1, 2$. Then T has a pair of arc-disjoint (x_1, y_1)-, (x_2, y_2)-paths if and only if T is not strong, or T is strong, but (T, x_1, x_2, y_1, y_2) is not of type 1a, 1b, 2r, or 2r+1 for any $r \geq 1$.*

The proof of Theorem 7.4 in [11] implies the following:

Corollary 7.5 *The 2-APV problem is solvable in time $O(n^4)$ for semicomplete digraphs. Furthermore, given distinct vertices x_1, x_2, y_1, y_2, there is an $O(n^4)$ algorithm to find a pair of arc-disjoint (x_1, y_1)-, (x_2, y_2)-paths, if such exist.*

8 Pancyclicity and path-connectivity

A digraph D of order n is *pancyclic* if it has k-cycles for all k such that $3 \leq k \leq n$. D is *vertex pancyclic* (*arc pancyclic*) if D has a k-cycle containing v (a) for every $v \in V(D)$ ($a \in A(D)$) and every $k \in \{3, ..., n\}$. D is called *arc-k-cyclic* if each arc of D is contained in a k-cycle. An arc of a digraph D is said to be *pancyclic* if it is contained cycles of all lengths $k = 3, ..., |V(D)|$. D is called *arc-k-anticyclic* if there is an (x, y)-path of length k in D for each arc $(x, y) \in A(D)$. A digraph D is *completely strong path-connected* if there is an $(x, y)-$ and a (y, x)-path in D, both of length k, for every pair x and y vertices of D and each $k \in \{2, ..., |V(D)| - 1\}$.

The following well known theorem is due to L. Moser.

Theorem 8.1 *[66]. Every strong tournament is pancyclic.*

J. Moon obtained a stronger result.

Theorem 8.2 *[81] Every strong tournament is vertex pancyclic.*

M. Goldberg and J.W. Moon found the following generalization of the last result.

Theorem 8.3 *[54] In every r-arc-strong tournament of order n, each vertex lies on r different cycles of length k, for $k = 3, ..., n$.*

B. Alspach [3] showed that every diregular tournament is arc pancyclic. O.S. Jacobsen [71] proved that every arc of an 1-irregular tournament on $n \geq 8$ vertices is contained in cycles of all lengths k such that $4 \leq k \leq n$. C. Thomassen generalized these results as follows:

Theorem 8.4 *[115] Let T be an m-irregular tournament on n vertices. Let k be any integer such that $4 \leq k \leq n$. If $n \geq 5m+9$, then the tournament T has an (x, y)-path with k vertices for every pair of vertices x and y. If $n \geq 5m + 3$, then every arc of T is contained in a cycle of length k.*

T_1

T_2

Figure 2: T_6-type and T_8-type tournaments. Here T_1 and T_2 are disjoint arc-3-cyclic tournaments and the edges that are not oriented can be oriented arbitrarily.

F. Tian, Z.-C. Wu and C.-Q. Zhang found a further generalization.

Theorem 8.5 *[125] Except for T_6-type tournaments and T_8-type tournaments (see Fig. 2), every arc-3-cyclic tournament is arc-pancyclic.*

In the proof of the last theorem, the authors of [125] extensively used Lemma 5.4. Theorem 8.5 implies the following results:

Corollary 8.6 *[125] At most one arc of an arc-3-cyclic tournament is not pancyclic.*

Corollary 8.7 *[130] If T is a tournament of order n which is arc-k-pancyclic for $k = 3$ and $k = n$ then T is arc pancyclic.*

Corollary 8.8 *[132] Let T be an arc-3-cyclic tournament of order n such that the in-degree and the out-degree of each vertex are at least k, where $n \leq 4k - 3$; then T is arc-pancyclic.*

Corollary 8.9 *[115] There are infinitely many 2-strong tournaments containing an arc which is not in any Hamiltonian cycle.*

To see the last result, note that each T_6-type digraph is 2-strong, arc-3-cyclic and contains an arc which is not in any Hamiltonian cycle.

M. Darrah, Y.-P. Liu and C.-Q. Zhang recently found a generalization of Corollary 8.7:

Theorem 8.10 *[44] Let D be an arc-3-cyclic semicomplete digraph and let $a = uv \in A(D)$. If the arc a is contained in a cycle of length r and $vu \notin A(D)$, then a is contained in cycles of lengths h, for all $h = 3, 4, ..., r$.*

We formulate another result from [44].

Theorem 8.11 *Let D be an arc-3-cyclic semicomplete digraph and let $a = uv \in A(D)$. If the arc a is contained in a cycle of length r and the minimum in-degree and out-degree of the vertices u and v is at least three, then a is contained in cycles of lengths h, for all $h = 6, 7, ..., r$.*

Y. Zhu and F. Tian considered sufficient conditions for arc-3-cyclicity of a tournament.

Theorem 8.12 *[140] If T is a tournament of order n such that $d^+(v)+d^-(u) \geq n-2$ for every arc uv, then T is arc-pancyclic.*

Note that if we replace $n - 2$ by $n - 1$ above then this is trivial.

K.-M. Zhang characterized completely strong path-connected tournaments using Theorem 8.5.

Theorem 8.13 *[137] Let T be a tournament. The following statements are equivalent:*

(1) T is completely strong path-connected;

(2) T is arc-3-cyclic and arc-3-anticyclic, and T is not isomorphic to a T_0-type tournament (i.e. a tournament with partition $V(T) = X \cup Y \cup \{z\}$ of the vertex set so that $T\langle X\rangle \Rightarrow T\langle Y\rangle \Rightarrow z \Rightarrow T\langle X\rangle$);

(3) T is 2-strong and T is arc-3-cyclic and arc-3-anticyclic;

(4) T is arc-3-cyclic and arc-3-anticyclic, and for every arc $(v, w) \in A(T)$ there is a (v, w)-path of length at least $n/2$.

Theorem 8.13 immediately implies the following result conjectured in [138]:

Corollary 8.14 *A tournament T is completely strong path-connected if and only if T is arc-3-cyclic, arc-3-anticyclic and arc-$(n-1)$-anticyclic.*

K.-M. Zhang and Z.-S. Wu [138] conjectured that the following weaker form of the last result is true:

A tournament T is completely strong path-connected if and only if T is arc-3-anticyclic and arc-$(n-1)$-anticyclic.

In 1987, C.-G. Yang [131] constructed counterexamples to this conjecture (the minimum number of vertices is 91). Recently, Y. Guo [56] managed to obtain smaller counterexamples of order 23.

Some further results on the topic can be found in the recent papers [58, 87].

9 Complementary cycles

B. Bollobás (see [95]) posed the following problem.

Problem 9.1 *Let k be a positive integer. What is the least integer $g(k)$ so that all but a finite number of $g(k)$-strong tournaments contain a spanning subgraph consisting of k vertex-disjoint cycles?*

Clearly, $g(1) = 1$. K.B. Reid [94] proved the following:

Theorem 9.2 *Let T be a 2-strong tournament of order $n \geq 6$. Then T contains two vertex-disjoint cycles of lengths 3 and $n-3$, unless T is isomorphic to the rotational tournament $R(1, 2, 4)$.*

Therefore, $g(2) = 2$. For $k \geq 3$, the problem remains open. K.B. Reid [95] proved that $k \leq g(k) \leq 3k - 4$. The last theorem also provides a partial solution to the following problem by C. Thomassen (see [94]).

Problem 9.3 *Let r and q be two positive integers. Does there exist a positive integer $s = s(r, q)$ so that all but a finite number of s-strong tournaments can be partitioned into an r-strong and a q-strong subtournament?*

From Theorem 9.2 we have $s(1, 1) = 2$. For all other r and q, the problem is still open. Z.M. Song raised a further problem.

Problem 9.4 *[107] Let k be a positive integer. What is the least integer $f(k)$ so that, for every sequence of k integers $n_1, ..., n_k$, satisfying the following conditions*

$$n_i \geq 3,\ i = 1, ..., k, \textit{ and } \sum_{i=1}^{k} n_i = n,$$

all but a finite number of $f(k)$-strong tournaments contain k vertex-disjoint cycles of lengths $n_1, ..., n_k$?

The main result of [107] gives a partial solution to the last problem.

Theorem 9.5 *[107] Let T be a 2-strong tournament of order $n \geq 6$. Suppose that T is not isomorphic to the rotational tournament $R(1, 2, 4)$. Then, for all m such that $m = 3, 4, ..., n-3$, T contains an m-cycle and an $(n-m)$-cycle which are vertex-disjoint.*

Hence, we obtain that $f(2) = g(2) = 2$. The following conjecture was made by Z.M. Song.

Conjecture 9.6 *[107] For all $k \geq 2$, $f(k) = g(k)$.*

10 Oriented paths and cycles

An *oriented path* in a tournament T is a sequence of distinct vertices $Q = v_0, v_1 ..., v_k$. The *type* of Q is $e^k = e_1 e_2 ... e_k$, where $e_i = +1$ if $v_{i-1} \rightarrow v_i$ and $e_i = -1$ if $v_i \rightarrow v_{i-1}$ (see [97]). A type e^k $(k < n)$ *is realizable* in a tournament T of order n if T has an oriented path of type e^k. An *oriented cycle* and its type are defined similarly. Clearly the type of a spanning oriented cycle contains n elements +1 or -1. A type e^n *is realizable* in a tournament T if T contains a spanning oriented cycle of type e^n. An oriented path (or cycle) is called *antidirected* if, for each i, $e_i = -e_{i+1}$. Note that an antidirected cycle always has even number of vertices. The notion of an antidirected path was introduced by B. Grünbaum in [55] who characterized tournaments containing a spanning antidirected path.

Theorem 10.1 *Every tournament has a spanning antidirected path except for the three rotational tournaments $R(1), R(1, 2)$ and $R(1, 2, 4)$.*

Grünbaum conjectured that all tournaments of even order n $(n \geq 10)$ have an antidirected cycle. M. Rosenfeld [97] gave another proof of the previous theorem and conjectured the following:

Conjecture 10.2 *(Rosenfeld's path conjecture) There exists an integer n_0 such that every type e^{n-1} is realizable in every tournament T of order n provided $n \geq n_0$.*

C. Thomassen [114] established Grünbaum's conjecture for $n \geq 50$. This bound was later improved to 28 by M. Rosenfeld [98].

Attacking Rosenfeld's path conjecture, B. Alspach and M. Rosenfeld [4] and H.J. Straight [109] established the conjecture for oriented paths with two blocks (a block being a maximal directed subpath). B. Alspach and M. Rosenfeld [4] also considered the case when the i'th block has length at least $i+1$. R. Forcade [50] proved that Rosenfeld's path conjecture is true if n is a power of 2. K.B. Reid and N.C. Wormald [96] showed that every oriented path of order p is contained in every tournament of order $3p/2$. Improving this bound, C.-Q. Zhang [135] proved the following weakening of Rosenfeld's path conjecture:

Theorem 10.3 *Every type e^{n-2} is realizable in every tournament T of order n.*

Since only strong tournaments have Hamiltonian cycles, the types $e^n = 11...1$ and $e^n = -1-1...-1$ when $n \geq 3$ are not realizable in non-strong tournaments. The next conjecture is also due to M. Rosenfeld [98].

Conjecture 10.4 *(Rosenfeld's cycle conjecture) There is an integer n_0 such that every type e^n containing at least one 1 and at least one -1 is realizable in each tournament T of order $n \geq n_0$.*

B. Grünbaum (see [98]) verified the last conjecture for types $11...1-1$ and $-1-1...-1+1$. C. Thomassen [114] and M. Rosenfeld [98] established the conjecture for antidirected cycles, A. Benhocine and A.P. Wojda [29] showed that the conjecture is true for oriented cycles with just two blocks. Finally, A. Thomason [113] proved both conjectures of M. Rosenfeld for tournaments where the number of vertices is sufficiently high. The main result of [113] is the following:

Theorem 10.5 *Let C be a non-strongly oriented cycle of order n and let T be a tournament of order n. If T contains a transitive subtournament of order 129 (in particular if $n \geq 2^{128}$), then T contains C.*

Hence, A. Thomason obtained $n_0 = 2^{128}$. However, he conjectured [113] that $n_0 = 8$ for Rosenfeld's path conjecture and $n_0 = 9$ for Rosenfeld's cycle conjecture. Proving Theorem 10.5, A. Thomason also established the following theorems of independent interest concerning the existence of oriented paths with specified end vertices.

Theorem 10.6 *Let P be an oriented path of order $n-1$ and first block length k. Let T be a tournament of order $n \geq 2$ and let K be a set of at least $k+1$ vertices of T. Then there is a copy of P in T with initial vertex in K.*

Theorem 10.7 *Let P be an oriented path of order $n-2$ with first block length k and last block length q, where $k+q \leq n-5$. Let T be a tournament of order $n \geq 3$ and let K, Q be disjoint subsets of T of orders at least $k+1$ and $q+1$ respectively. Then there is a copy of P in T with initial vertex in K and end vertex in Q.*

The last two theorems are true for all $n \geq 3$ and best possible in the sense that the values for $|K|$ and $|Q|$ can not be reduced.

11 Claws and Hamiltonian paths with additional arcs

In our paper, a *claw* is an out-branching B such that each vertex of B except the root has out-degree 0 or 1. The *degree* of a claw B is the out-degree of the root of B. Let $C(n,d)$ denote the set of all claws of order n and degree d. Note that Redei's Theorem implies that every tournament on n vertices contains the unique element of $C(n,1)$. M. Saks and V. Sós [101] conjectured that a tournament of order n contains all elements of $C(n,d)$ for $1 \leq d \leq n/2$. X. Lu [76] showed that the assertion of the conjecture is not true. He also proved that every tournament of order n has all claws from $C(n,d)$ for $d \leq n/4$. X. Lu strengthened the last result in [77] proving the following:

Theorem 11.1 *Every tournament of order n contains all claws from $C(n,d)$ for $1 \leq d \leq 3n/8$.*

X. Lu raises the following problem:

Problem 11.2 *[77] Find*

$$c_0 = \sup\{c : \textit{every tournament of order } n \textit{ has all claws from } C(n,cn)\}.$$

Lu [78] proved that $3/8 \leq c_0 \leq 25/52$.

Let $H(n,i)$ denote the digraph on n vertices $v_1,...,v_n$ with the arc set $\{(v_k,v_{k+1}) : 1 \leq k \leq n-1\} \cup \{(v_1,v_i)\}$ and let T^* denote the tournament on five vertices 1,2,3,4,5 and ten arcs 12,14,31,51,23,42,52,34,35,45. V. Sós conjectured (see [90]) that every tournament on n vertices contains $H(n,i)$ for each n and i, $4 \leq i \leq n-1$. V. Petrović [90] has verified this conjecture. He obtained the following result.

Theorem 11.3 *[90] Every tournament T on n vertices contains $H(n,i)$ for each n and i, $3 \leq i \leq n$ unless either*

(a) $i = 3$ and $T = R(1)$ or any non-strong tournament with initial strong component $R(1)$, or

(b) $i = 5$ and T is the strong tournament T^* on 5 vertices or any non-strong tournament with initial strong component T^*.

12 Oriented trees

We first consider arc-disjoint branchings and then a conjecture by D. Sumner and related results.

By J. Edmonds' well known branching theorem [48], a digraph D has k arc-disjoint out-branchings rooted at some fixed vertex s if and only if there are k arc-disjoint paths from s to every other vertex of D. By applying the same method as in Shiloach's proof, mentioned in Section 7, this characterization can easily be extended to cover the case when the out-branchings have specified roots $s_1,\dots,s_k$, some of which may be identical. Hence that problem is polynomially solvable for general digraphs.

In contrast, if we ask for an out-branching F_u^+ and and in-branching F_v^- which are arc-disjoint and have roots u and v, respectively, then the problem is NP-complete, even in the case when $u = v$. The proof, due to C. Thomassen, is given in [11]. C. Thomassen [118] (see also [11]) made the following conjecture :

Conjecture 12.1 *There exists a natural number k such that every k-arc-strong digraph D with specified vertices u and v contains arc-disjoint branchings F_u^+, F_v^-, where F_u^+ is an out-branching rooted at u and F_v^- is an in-branching rooted at v.*

In [11] J. Bang-Jensen proved this conjecture for tournaments. As we shall see, the case when $u = v$ is significantly easier than the case when $u \neq v$.

We first look at the case when $u = v$. At first glance, one might think that the following, obviously necessary, condition is also sufficient: for every $x \in V(T) - v$ there exists a closed trail through x and v. A 3-cycle shows that this is not sufficient. Examples in [11] show that when $n > 3$ this condition is not sufficient either.

Theorem 12.2 *Let T be a strong semicomplete digraph and v an arbitrary vertex of $V(T)$. There exist arc-disjoint branchings F_v^-, F_v^+ rooted at v if and only if there is no arc e such that $T - e$ has neither an in-branching nor an out-branching rooted at v.*

Theorem 12.2 and its proof in [11] implies the following:

Corollary 12.3 *There exists an $O(n^3)$ algorithm to decide if a given semicomplete digraph T with a specified vertex v has a pair of arc-disjoint branchings F_v^-, F_v^+ rooted at v. Furthermore we can find such branchings in time $O(n^3)$ if they exist.*

Theorem 12.2 and Corollary 12.3 have been generalized in [16] :

Theorem 12.4 *Let D be a strong digraph and v a vertex of D such that $V(D) = \{v\} \cup N^-(v) \cup N^+(v)$. Let $A = \{A_1, A_2, \ldots, A_k\}$ ($B = \{B_1, B_2, \ldots, B_r\}$) denote the set of terminal (initial) strong components in $D\langle N^+(v)\rangle$ ($D\langle N^-(v)\rangle$). Then D contains a pair of arc-disjoint branchings F_v^+, F_v^- such that F_v^+ is an out-branching rooted at v and F_v^- is an in-branching rooted at v if and only if there exist two disjoint arc sets $E_A, E_B \subset A(D)$ such that all arcs in $E_A \cup E_B$ go from $N^+(v)$ to $N^-(v)$ and every strong component $A_i \in A$ ($B_j \in B$) is incident with an arc from E_A (E_B).*

Corollary 12.5 *[16] Let D be a strong digraph and v a vertex of D such that $V(D) = \{v\} \cup N^-(v) \cup N^+(v)$. There exists an $O(n^3)$ algorithm to decide if D has arc-disjoint branchings F_v^-, F_v^+ rooted at v.*

These results show that it is enough to insist that the special vertex is adjacent to all other vertices to guarantee that the problem is polynomially solvable.

Now we turn our attention to the case $u \neq v$.

Theorem 12.6 *[11] Let T be a tournament and u, v distinct vertices of $V(T)$. Let*

$$A = \{x \in V(T)|ux, vx \in A(T)\}, \qquad B = \{x \in V(T)|ux, xv \in A(T)\}$$

$$C = \{x \in V(T)|xu, vx \in A(T)\}, \qquad D = \{x \in V(T)|xu, xv \in A(T)\}$$

There exist arc-disjoint branchings F_u^+, F_v^- if and only if T satisfies none of the following six conditions:

1. *$|V(T)| \leq 3$ or ($|V(T)| = 4$ and v dominates u);*

2. *T is not strong and either u is not in the initial component, or v is not in the terminal component of T;*

3. *T is strong and there exists and arc $e \in A(T)$ such that $T - e$ has no out-branching from u and no in-branching at v;*

4. *T is strong, u dominates v, $B = \emptyset$, $A, D \neq \emptyset$ and*

$$(*) \begin{cases} \text{there is exactly one arc } e \text{ going out of } A_l, \\ \text{the terminal component of } T\langle A\rangle, \text{ and there is exactly one} \\ \text{arc } e' \text{ entering } D_1, \text{ the initial component of } T\langle D\rangle, \text{ and } e \neq e' \end{cases}$$

   and finally every path from a vertex of A to a vertex of D in $T - \{u, v\}$ contains e and e';

5. *T is strong, v dominates u, $B = \{b\}$, $A, D \neq \emptyset$, (T, u, v) satisfies (*), there is no trail from A to D in $T - \{u, v\}$ which contains b and every path from A to D in $T - \{u, v\}$ contains e and e';*

6. *T is strong, v dominates u, $B = \emptyset$, $A, D \neq \emptyset$, (T, u, v) satisfies (*), there exist arc-disjoint (u, v)-paths P, P' and for every pair of such paths, either $e, e' \in A(P)$ or $e, e' \in A(P')$.*

It is not obvious from this characterization that there exists a polynomial algorithm to check the existence of the desired branchings. The proof in [11] that such an algorithm exist is based on Corollaries 7.2,7.5:

Corollary 12.7 *[11] There exists an $O(n^4)$ algorithm which given a tournament T and distinct vertices u and v decides whether T has a pair of arc-disjoint branchings F_u^+, F_v^- and finds such a pair if one exists.*

The last part of the corollary is not explicitly stated in [11], but the details are implicit in the proofs.

Theorems 12.2 and 12.6 imply the truth of Conjecture 12.1 for tournaments.

Corollary 12.8 *Every 2-arc-connected tournament T contains arc-disjoint branchings F_u^+, F_v^- for any choice of vertices u and v in $V(T)$.*

Based on the evidence from Corollary 12.5, we make the following conjecture:

Conjecture 12.9 *There exists a polynomial algorithm which given any digraph D and distinct vertices u and v of $V(D)$ such that both u and v are adjacent to all other vertices of D, decides whether D has a pair of arc-disjoint branchings F_u^+, F_v^-.*

A *caterpillar* is an undirected tree such that when its vertices of degree 1 are removed the graph remaining is an undirected path, called the *spine*. Consider the next conjecture due to D. Sumner (see [96].

Conjecture 12.10 *Let $t = t(n)$ denote the the least number such that every tournament on t vertices contains every oriented tree of order n. For every n, $t(n) = 2n-2$.*

In support of that conjecture, K.B. Reid and N.C. Wormald [96] proved that every tournament on $2n-2$ vertices contains each orientation of a caterpillar of order n with diameter at most 4 as well as each orientation of a caterpillar of order n whose spine is a (directed) path.

N.C. Wormald [128] showed that $t(n) \leq (1+o(1))n \log_2 n$. R. Häggkvist and A. Thomason [63] proved the following:

Theorem 12.11 *Every oriented tree of order n is contained in every tournament of order $12n$.*

Moreover, for large n they improved this result and showed that $t(n) \leq (4+o(1))n$ [63].

N.C. Wormald [129] raises the following question: for n large enough, what oriented trees are contained in all tournaments on n vertices. In [28] E.A. Bender and N.C. Wormald proved the following:

Theorem 12.12 *[28] There is a set τ^* containing almost all oriented labeled trees, such that a random labeled tournament of order n almost surely contains at least one isomorphic copy of all oriented trees of order n in τ^*.*

They ask the following:

Question 12.13 *[28] Is there a set τ^* containing almost all oriented labeled trees, such that every tournament on n vertices contains at least one isomorphic copy of all oriented trees of order n in τ^*?*

13 Monochromatic paths in arc-coloured tournaments

We call the tournament T *r-arc-coloured* if the arcs of T are coloured with r colours $1, ..., r$. Let T_i denote a spanning subgraph of T formed by all arcs of colour i, $i = 1, ..., r$. We say that a tournament T has *finite monochromatic radius* if T contains a vertex v such that, for every other vertex w in T, there is a (v, w)-path in at least one of the digraphs $T_1, T_2, ..., T_r$. Let TR_3 and ST_3, respectively, denote the transitive tournament of order 3 and the strong tournament of order 3, both of whose arcs are coloured with three distinct colours.

In [102] B. Sands, N. Sauer and R. Woodrow studied monochromatic paths in r-arc-coloured digraphs. They proved the following (for a direct proof, see [93]):

Theorem 13.1 *Every 2-arc-coloured tournament has finite monochromatic radius.*

As pointed out by B. Sands, N. Sauer and R. Woodrow [102], the monochromatic radius of a 3-arc-coloured tournament might be infinite. Moreover, they stated the following conjecture (according to [102], P. Erdös also conjectured this).

Conjecture 13.2 *[102] For each r, there is a least positive integer $f(r)$ such that every r-arc-coloured tournament contains a set S of $f(r)$ vertices with the property that for every vertex v not in S there is a path from a vertex in S to v in at least one of the digraphs $T_1, T_2, ..., T_r$. In particular, does $f(3)$ equal 3 ?*

A. Bialostocki [34] proved that if there is a counterexample to Conjecture 13.2, then the tournament has to have at least 19 vertices. A. Bialostocki and N. Sauer [36] introduced the following interesting notion. Let T be a tournament and $c : A(T)\rightarrow\{1, ..., r\}$ be an r-colouring of the arcs of T. The *reachability graph* is a (semicomplete) digraph $R = R(T, c)$ such that $V(R) = V(T)$ and $v\rightarrow u$ in R if and only if there is a monochromatic path from v to u in T. The set $\mathcal{R}_r(s)$ is defined as the set of all tournaments having the property that for every r-coloring c of $A(T)$, the reachability graph $R(T, c)$ contains K_s^*, the complete digraph of order s, as a subgraph. A tournament T is said to satisfy property $M(k)$ if, for every transitive subtournament T_1 of order less than k, there exists a vertex $x \in V(T)$ which is dominated by every vertex of T_1.

Theorem 13.3 *[36] If a tournament T satisfies $M(k)$, then $T \in \mathcal{R}_2(k)$.*

The authors of [36] also introduced a so-called continuous colouring of the arcs of a tournament, settled some its properties and raised several problems.

B. Sands, N. Sauer and R. Woodrow raised the following problem:

Problem 13.4 *[102] Let T be a 3-arc-coloured tournament which does not contain TR_3. Must T have finite monochromatic radius?*

In support to the positive answer to this question, M.G. Shen proved:

Theorem 13.5 *[104] Every r-arc-coloured tournament which does not contain TR_3 or SR_3 has finite monochromatic radius.*

Obviously, Theorem 13.5 implies Theorem 13.1 as well as the following result.

Theorem 13.6 *[104] Let T be an r-arc-coloured tournament whose vertices can be partitioned into disjoint blocks $B_1, ..., B_h$, $h \geq 1$, such that*

(i) no block contains a 3-arc-coloured triangle (i.e. TR_3 or ST_3);

(ii) for every pair $1 \leq i < j \leq h$ all arcs between B_i and B_j have the same colour c_{ij};

(iii) the transitive tournament TT_h whose arc (i,j) has the colour c_{ij} does not contain TR_3.

Then T has finite monochromatic radius.

Ramsey and anti-Ramsey type problems for tournaments are considered in [21, 32, 35, 42, 61, 79].

14 The number of paths and cycles in tournaments

We first consider results on the numbers of Hamiltonian paths and cycles in tournaments.

Solving a conjecture of T. Szele [110], N. Alon [2] showed:

Theorem 14.1 *The maximum number, $P(n)$, of Hamiltonian paths in a tournament on n vertices satisfies*

$$P(n) \le cn^{1.5} n!/2^{n-1}$$

where c is independent of n.

Szele [110] proved that

$$P(n) \ge n!/2^{n-1}$$

and hence the gap between the upper and lower bounds for $P(n)$ is only $O(n^{1.5})$. It would be interesting to close this gap and determine $P(n)$ up to a constant factor.

J.W. Moon [83] dealt with the minimum number of Hamiltonian paths in strong tournaments of order n.

Theorem 14.2 *If $p(n)$ denotes the minimum number of Hamiltonian paths in a strong tournament of order n, then*

$$6^{(n-1)/4} \le p(n) \le \begin{cases} 3.5^{(n-3)/3}, & \textit{if } \ n = 0 \ (\textit{mod } \ 3), \\ 5^{(n-1)/3}, & \textit{if } \ n = 1 \ (\textit{mod } \ 3), \\ 9.5^{(n-5)/3}, & \textit{if } \ n = 2 \ (\textit{mod } \ 3). \end{cases}$$

C. Thomassen [116] considered the problem of finding the minimum number of Hamiltonian cycles in sets of tournaments with some restrictions. He showed that there is a constant $\alpha > 1$ such that each 2-strong tournament of order n has at least α^n Hamiltonian cycles. This result is best possible in the sense that we must have $\alpha < 2$ [116]. Using Corollary 5.8 C. Thomassen [120] proved the following:

Theorem 14.3 *Every strong tournament with minimum out-degree at least $3k+3$ has at least $4^k k!$ Hamiltonian cycles.*

A tournament T is called *domination orientable* if there is a labeling of $V(T)$, $v_1, ..., v_n$, so that v_i dominates v_{i+j}, $j = 1, 2, ..., d^+(v_i)$, for every $i = 1, 2, ..., n$. C.-Q. Zhang [136] considered the number of Hamiltonian cycles in domination orientable tournaments. In particular, he proved the following:

Theorem 14.4 *Let $\delta(T)$ be the minimum of both the minimum out-degree and in-degree of T. The number of Hamiltonian cycles in a domination orientable tournament T of order n is at least $3 \times 2^{n-6}$ when $\delta(T) \geq 2$ and $n \geq 6$, $17 \times 2^{n-8} - 5$ when $\delta(T) \geq 3$ and $n \geq 8$, and $140 \times 2^{n-11} - 9$ when $\delta \geq 4$ and $n \geq 11$.*

Let $c(T, k)$ denote the number of k-cycles in a tournament T. J.W. Moon [80] showed that if T_n is a non-trivial strong tournament of order n, then $c(T_n, k) \geq n - k + 1$. M. Burzio and D.C. Demaria [40] characterized all tournaments of order n containing exactly $n - 2$ 3-cycles. M. Burzio and D.C. Demaria [40] used the last result to show that there are 2^{n-4} strong tournaments of order n with only $n - 2$ 3-cycles. Another proof of this result is obtain in [85]. M. Las Vergnas [75] showed A_n (obtained from TT_n by reversing the arcs of the unique Hamiltonian path) is the only strong tournament with exactly $n - k + 1$ k-cycles, for each k, $4 \leq k \leq n - 1$.

The problem of finding the number of tournaments having a unique Hamiltonian cycle has attracted several researches. R. J. Douglas [46] gave a structural characterization of tournaments having a unique Hamiltonian cycle. This result implies a formula for the number s_n of non-isomorphic tournaments of order n with a unique Hamiltonian cycle. This characterization as well as the formula are rather complicated. G.P. Egorychev [47] obtained a simplification by considering a generating function for the number of tournaments with unique Hamiltonian cycle. M. R. Garey [52] showed that s_n could be expressed as a Fibonacci number ($s_n = f_{2n-6}$); his derivation was based on Douglas's characterization. J. W. Moon [84] obtained a direct proof of Garey's formula that is essentially independent of Douglas's characterization. Another proof of Garey's formula is provided in [45].

Simple formulae for the number $c(T, 3)$ and the maximum numbers of 3-cycles and 4-cycles in a tournament of order n are well-known (cf. [24, 43, 72, 81]). J.W. Moon [86] introduced $u(T_n)$, the number of vertices x in a tournament T_n of order n such that each arc oriented towards x belongs to at least one 3-cycle. He determines the minimum value of $c(T_n, 3)$ when $u(T_n) = n$ and the maximum value of $c(T_n, 3)$ when $u(T_n) = 3$. He discusses some applications of these results.

In [73], A. Kotzig determined a formula for the number of 4-cycles in a tournament T in terms of the out-degrees of T and the cardinalities of the pairwise intersections of distinct out-neighbourhoods (see [25]). B. Alspach and C. Tabib [5] found a lower and an upper bound for the number $c(T, 4)$ in terms of the out-degrees of T only. They also proved the following:

Theorem 14.5 *Let $d_1^+, ..., d_n^+$ be the out-degrees of the vertices in a domination orientable tournament T. The maximum number of 4-cycles contained in any tournament with the same list of the out-degrees is*

$$\binom{n}{4} - \sum_{i=1}^{n} \binom{d_i^+}{3}$$

and this bound can be attained.

A *k-homogeneous tournament* T is a tournament with at least one k-cycle such that every arc lies on the same number t of k-cycles. J. Plesnik [91] proved that every arc of a 3-homogeneous tournament of order $4k-1$ lies on exactly $2k(k-1)$ 4-cycles. A tournament of order $4k+1$ ($k \geq 1$) is called *3-near-homogeneous* if every arc lies on k or $k+1$ 3-cycles [111].

Theorem 14.6 *[5, 111] Let T be a diregular tournament on $2n+1$ vertices. Then*

$$c(T,4) \geq \begin{cases} (2n+1)(n^3-n)/8, & \text{if } n \text{ is odd,} \\ (2n+1)n^3/8, & \text{if } n \text{ is even.} \end{cases}$$

Moreover, these bounds are attained if and only if T is 3-homogeneous when n is odd and T is 3-near-homogeneous when n is even.

D.M. Berman [30] obtained an upper bound for the number of 5-cycles in a domination orientable tournament of order n. A tournament T is called *doubly diregular* if, for every $v \in V(T)$, $T\langle N^+(v)\rangle$ is diregular. It is easy to check that every doubly diregular tournament is diregular. The connection between doubly diregular and k-homogeneous tournaments is considered in the following two theorems.

Theorem 14.7 *[74] A non-trivial tournament is doubly diregular if and only if it is 3-homogeneous.*

Theorem 14.8 *[100] A non-trivial tournament is doubly diregular if and only if it is 4- and 5-homogeneous.*

The method of the proof of the last theorem in [100] is algebraic and serves also to show that a diregular tournament is doubly diregular if and only if the minimal polynomial of its adjacency matrix has degree 3.

We conclude by the restriction to tournaments of the well-known conjecture of A. Adám [1] on cycles in digraphs.

Conjecture 14.9 *Every non-transitive tournament contains an arc whose reversal reduces the total number of cycles.*

C. Thomassen [123] disproved this conjecture in its most general setting, i.e. for general directed multigraphs. Conjecture 14.9 was first stated in [25].

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