Domination analysis for minimum multiprocessor scheduling

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Abstract

Let P be a combinatorial optimization problem, and let A be an approximation algorithm for P. The domination ratio $\mathrm{domr}(A, s)$ is the maximal real q such that the solution $x(I)$ obtained by A for any instance I of P of size s is not worse than at least the fraction q of the feasible solutions of I. We say that P admits an Asymptotic Domination Ratio One (ADRO) algorithm if there is a polynomial time approximation algorithm A for P such that $\lim_{s\to\infty} \mathrm{domr}(A, s) = 1$. Alon, Gutin and Krivelevich (J. Algorithms 50 (2004), 118–131) proved that the partition problem admits an ADRO algorithm. We extend their result to the minimum multiprocessor scheduling problem.

Keywords: combinatorial optimization; domination analysis; minimum multiprocessor scheduling

1 Introduction, Terminology and Notation

Let $\mathcal{P}$ be a combinatorial optimization problem, I an instance of $\mathcal{P}$, A an approximation algorithm for $\mathcal{P}$ and $x(I)$ the solution of I obtained by A. The *domination ratio* $\mathrm{domr}(A, I)$ *of A for I* is the number of solutions of I that are no better than $x(I)$ divided by the total number of feasible solutions of I. The *domination ratio* $\mathrm{domr}(A, s)$ *of A for $\mathcal{P}$* is the minimum of $\mathrm{domr}(A, I)$ taken over all instances I of $\mathcal{P}$ of size s. We say that A is an *asymptotic domination ratio one (ADRO) algorithm* for $\mathcal{P}$ if A runs in polynomial time and $\lim_{s\to\infty} \mathrm{domr}(A, s) = 1$.

Domination analysis, whose aim is to evaluate the domination ratios of various combinatorial optimization heuristics, allows one to understand the worst case behaviour of heuristics. Thus, domination analysis complements the results of the classical approximation analysis. Notice that the domination ratio avoids some drawbacks of the approximation ratio [21]. In particular, the domination ratio does not change on equivalent instances of the same problem. For example, by adding a positive constant to the weight of every arc of a weighted complete digraph, we obtain an equivalent instance of the traveling salesman problem (TSP). While the domination ratio of a TSP heuristics remains the same for both instances, the approximation ratio changes its value. For more details, see [9].

Sometimes, domination analysis provides us with a deep insight into the behaviour of heuristics. For example, it is proved in [11] that the greedy algorithm is of the minimum possible domination ratio (i.e., $1/f(s)$, where $f(s)$ is the number of feasible solutions in instances of size s) for a number of optimization problems including TSP and the assignment

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problem. In order words, the greedy algorithm may find the unique worst possible solution. (This theoretical result is in line with computational experiments with the greedy algorithm for TSP, e.g. see [14], where the authors came to the conclusion that the greedy algorithm 'might be said to self-destruct', and that it should not be used even as 'a general-purpose starting tour generator'.) Notice that this result cannot be formulated in the terms of approximation analysis (AA) since AA does not distinguish between solutions with the same objective function value.

Domination analysis was introduced by Glover and Punnen [8] and was initially used only for analysis of TSP heuristics, see, e.g., [11, 12, 18, 19]. Apart from the greedy algorithm and other constructive TSP heuristic, some authors studied local search for TSP. For the Symmetric TSP, Punnen, Margot and Kabadi [18] showed that after a polynomial number of iterations the domination number of the best improvement 2-Opt that uses small neighborhoods significantly exceeds that of the best improvement local search based on neighborhoods of much larger cardinality. Punnen, Margot and Kabadi [18] and other papers have led Gutin and Yeo [10] to the conclusion that the cardinality of the neighborhood used by a local search is not the right measure of the effectiveness of the local search. Domination ratio, along with some other parameters such as the diameter of the neighborhood digraph (see Gutin, Yeo and Zverovitch [12]), provide a much better measure.

Recently, the domination ratios of algorithms for some other combinatorial optimization problems have also been investigated [2, 5, 6, 9, 11, 15]. In [5], two heuristics for Generalized TSP have been compared. Their performances in computational experiments are very similar. Nevertheless, bounds for domination ratios show that one of the heuristics is much better than the other one in the worst case. Two greedy-type heuristics for the frequency assignment problem were compared in [15]. Again, bounds for the domination ratios allowed the authors of [15] to find out which of the two heuristics behaves better in the worst case. For more details, see a recent survey on domination analysis [10].

Let $p \geq 2$ be an integer and let S be a finite set. A *p-partition* of S is a p-tuple $(A_1, A_2, \ldots, A_p)$ of subsets of S such that $A_1 \cup A_2 \cup \ldots \cup A_p = S$ and $A_i \cap A_j = \emptyset$ for all $1 \leq i < j \leq p$.

In what follows, N always denotes the set $\{1, 2, \ldots, n\}$ and each $i \in N$ is assigned a positive integral weight $\sigma(i)$. For a subset A of N, $\sigma(A) = \sum_{i \in A} \sigma(i)$. The *minimum multiprocessor scheduling problem* (MMSP) [3] can be stated as follows. We are given a triple (N, σ, p), where p is an integer, $p \geq 2$. We are required to find a p-partition $\mathcal{C}$ of N that minimizes $\sigma(\mathcal{A}) = \max_{1 \leq i \leq p} \sigma(A_i)$ over all p-partitions $\mathcal{A} = (A_1, A_2, \ldots, A_p)$ of N.

Clearly, if $p \geq n$, then MMSP becomes trivial. Thus, in what follows, $p < n$. The size s of MMSP is $\Theta(n + \sum_{i=1}^{n} \log \sigma(i))$.

Hochbaum and Shmoys [13] proved that MMSP admits a polynomial time approximation scheme. Alon, Gutin and Krivelevich [2] proved that the partition problem, which coincides with MMSP for the special case of $p = 2$, admits an ADRO algorithm. We extend their result to MMPS with unrestricted p. While using some of the ideas from [2], our proof is based on a number of new ideas and is much more complicated.

Let $(a_1, a_2, \ldots, a_p)$ be a p-tuple of p non-negative integers such that $\sum_{i=1}^{p} a_i = n$. The number of p-partitions $(A_1, A_2, \ldots, A_p)$ of N in which $a_i = |A_i|$ equals

$$\binom{n}{a_1, a_2, \ldots, a_p} = \frac{n!}{a_1! a_2! \cdots a_p!}. \tag{1}$$

Given n, p $(p \leq n)$, let $mc(n, p)$ denote the maximum value of the multinomial coefficient

$\binom{n}{a_1,a_2,\ldots,a_p}$.

2 Preliminary Results

It is well-known that the multinomial coefficient $\binom{n}{a_1,a_2,\ldots,a_p}$ is maximal when the parameters $a_1,\ldots,a_p$ are nearly equal. It is quite possible that the following upper bound on $mc(n,p)$ is well-known. We give its proof for the sake of completeness.

Lemma 2.1 *Let $n \geq p$. Then the following holds:*

$$mc(n,p) < p^{n+1/2} \times \left(\sqrt{\frac{p}{2\pi n}}\right)^{p-1}.$$

Proof: Suppose that $(a_1,a_2,\ldots,a_p)$ is chosen in such a way as to maximize $\binom{n}{a_1,a_2,\ldots,a_p}$ for given n and p. It is not difficult to see that all $a_i \geq 1$. By (1), using the Robbins formulation of Stirling's formula [20] we get the following:

$$\begin{aligned}
\binom{n}{a_1,a_2,\ldots,a_p} &= \frac{n!}{a_1!a_2!\cdots a_p!}\\
&< \frac{\sqrt{2\pi}\ n^{n+1/2}e^{-n}e^{1/(12n)}}{\Pi_{i=1}^p\sqrt{2\pi}\ a_i^{a_i+1/2}e^{-a_i}e^{1/(12a_i+1)}}\\
&= (\sqrt{2\pi})^{1-p} \times \frac{n^{n+1/2}}{\Pi_{i=1}^p a_i^{a_i+1/2}} \times e^{1/(12n)-\sum_{i=1}^p 1/(12a_i+1)}\\
&< (\sqrt{2\pi})^{1-p} \times \frac{n^{n+1/2}}{\Pi_{i=1}^p a_i^{a_i+1/2}}.
\end{aligned}$$

By differentiating $g(x) = (x+1/2)\ln x$ twice we get $g''(x) = \frac{1}{x} - \frac{1}{2x^2}$. Since $g''(x) \geq 0$ for $x \geq 1/2$ we conclude that $g(x)$ is convex for $x \geq 1/2$. Thus, by Jensen's Inequality, $\sum_{i=1}^p g(a_i)/p \geq g(\sum_{i=1}^p a_i/p)$ as $a_1,a_2,\ldots,a_p > 1/2$. However, this is equivalent to $\Pi_{i=1}^p a_i^{a_i+1/2} \geq (n/p)^{(n/p+1/2)p}$, which together with the inequality above implies the following:

$$mc(n,p) < (\sqrt{2\pi})^{1-p} \times \frac{n^{n+1/2}}{(n/p)^{(n/p+1/2)p}} = p^{n+1/2}\left(\sqrt{\frac{p}{2\pi n}}\right)^{p-1}.$$

This completes the proof. □

Corollary 2.2 *For even n, $\frac{n!}{(n/2)!^2} < 2^n \times \sqrt{\frac{2}{\pi n}}$.*

Recall that $N = \{1,2,\ldots,n\}$. A set $\mathcal{F}$ of subsets of N is called an *antichain* if no element of $\mathcal{F}$ is contained in another element of $\mathcal{F}$. By the famous Sperner's Lemma, $|\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor}$. Consider a set $\mathcal{P}$ of p-partitions of N. We call $\mathcal{P}$ a *p-antichain* if it has no pair $(A_1,A_2,\ldots,A_p)$, $(B_1,B_2,\ldots,B_p)$ such that $A_i \subset B_i$ and $A_i \neq B_i$ for some $i \in \{1,2,\ldots,p\}$.

The following generalization of Sperner's Lemma is due to Meshalkin [17] (its further extensions are given in [4]).

Lemma 2.3 *The number of elements in a p-antichain of N is at most $mc(n,p)$.*

The next two lemmas are well known. Nevertheless, since they have short proofs, we provide such proofs.

Lemma 2.4 *The number of p-tuples $(x_1,x_2,\ldots,x_p)$ of non-negative integers satisfying $x_1 + x_2 + \cdots + x_p \leq q$ equals $\binom{q+p}{q}$.*

Proof: Consider the set $S = \{1, 2, \ldots, p+q\}$. Choose a p-element subset $T = \{t_1, t_2, \ldots, t_p\}$ of S, $t_1 < t_2 < \cdots < t_p$. Observe that every T corresponds to a p-tuple $(t_1 - t_0 - 1, t_2 - t_1 - 1, \ldots, t_p - t_{p-1} - 1)$, where $t_0 = 0$, satisfying the conditions of the lemma and vice versa. The well-known fact that there are $\binom{q+p}{p} = \binom{q+p}{q}$ p-element subsets in a $(p+q)$-element set completes the proof. □

Lemma 2.5 *For every integer $k \geq 2$, $(1 - \frac{1}{k})^{k-1} > e^{-1}$ and $(1 - \frac{1}{k})^k < e^{-1}$.*

Proof: By differentiating $\ln x$ we see that $\frac{1}{k} < \ln(k) - \ln(k-1) < \frac{1}{k-1}$ for $k \geq 2$, which implies that $\frac{-1}{k} > \ln(\frac{k-1}{k}) > \frac{-1}{k-1}$. Thus, $-1 > k \ln(1 - \frac{1}{k})$ and $(k-1)\ln(1 - \frac{1}{k}) > -1$. Exponentiating each side in the above inequalities, we obtain the desired results. □

Lemma 2.6 *Let (N, σ, p) be a triple defining an instance of MMSP ($p \geq 2$) and let $\sigma(1) \geq \sigma(2) \geq \ldots \geq \sigma(n) = 1$. Let $\tilde{\sigma} = \sum_{i=1}^{n} \sigma(i)/p$. The number g_p of p-partitions $\mathcal{A} = (A_1, A_2, \ldots, A_p)$ of N for which the objective function of MMSP satisfies*

$$\sigma(\mathcal{A}) = \max_{1 \leq i \leq p} \sigma(A_i) < \tilde{\sigma} + 1$$

is less than $p^n \left(\sqrt{\frac{8p}{\pi n}}\right)^{p-1} \sqrt{\frac{4}{\pi}}$.

Proof: Let $\mathcal{A} = (A_1, A_2, \ldots, A_p)$ be a p-partition of N such that $\sigma(\mathcal{A}) < \tilde{\sigma} + 1$. We call such a p-partition *balanced*. Then, for each $j \in \{1, 2, \ldots, p\}$, we may write

$$\sigma(A_j) = \tilde{\sigma} - i_j + \alpha_j, \tag{2}$$

where i_j is a non-negative integer and $0 \leq \alpha_j < 1$. For a p-tuple $(i_1, i_2, \ldots, i_p)$ of non-negative integers, we denote by $Q'(i_1, i_2, \ldots, i_p)$ the set of all balanced p-partitions $\mathcal{A}$ satisfying

$$0 \leq i_j - (\tilde{\sigma} - \sigma(A_j)) < 1$$

for each $j = 1, 2, \ldots, p$ (see (2)). It is not difficult to see that $Q'(i_1, i_2, \ldots, i_p)$ forms a p-antichain of S. Thus, by Lemma 2.3 and Lemma 2.1,

$$|Q'(i_1, i_2, \ldots, i_p)| \leq p^{n+1/2} \times \left(\sqrt{\frac{p}{2\pi n}}\right)^{p-1}.$$

By (2) and the definitions of $\tilde{\sigma}$, i_j and α_j, $\sum_{j=1}^{p} i_j = \sum_{j=1}^{p} \alpha_j < p$. Since $\sum_{j=1}^{p} i_j$ is integral, $\sum_{j=1}^{p} i_j \leq p - 1$ and the sum of $|Q'(i_1, i_2, \ldots, i_p)|$ over all p-tuples $(i_1, i_2, \ldots, i_p)$ of non-negative integers with $\sum_{j=1}^{p} i_j \leq p - 1$ equals the number g_p.

By the arguments above, Lemma 2.4 and Corollary 2.2, we have

$$\begin{aligned} g_p = \sum_{i_1 + i_2 + \ldots + i_p \leq p-1} |Q'(i_1, i_2, \ldots, i_p)| &\leq \binom{2p-1}{p} p^{n+1/2} \left(\sqrt{\frac{p}{2\pi n}}\right)^{p-1} \\ &= \frac{1}{2}\binom{2p}{p} p^{n+1/2} \left(\sqrt{\frac{p}{2\pi n}}\right)^{p-1} \\ &< \frac{1}{2} 2^{2p} \sqrt{\frac{2}{\pi 2p}} \times p^{n+1/2} \left(\sqrt{\frac{p}{2\pi n}}\right)^{p-1} \\ &= p^n \left(\sqrt{\frac{8p}{\pi n}}\right)^{p-1} \sqrt{\frac{4}{\pi}}. \end{aligned}$$

□

Lemma 2.7 *Let $p \geq 2$ and x be integers, and let a be a rational number such that $x > a > 0$ and ap is an integer with $ap \geq x$. If $(1 - \frac{x-1}{ap})^x > \epsilon$, then $\binom{ap}{x}(\frac{1}{p})^x(1 - \frac{1}{p})^{ap-x} > \epsilon \times \frac{a^x}{e^a x!}$.*

Proof: By Lemma 2.5 we get the following:

$$\begin{aligned}\binom{ap}{x}(\tfrac{1}{p})^x(1-\tfrac{1}{p})^{ap-x} &> \tfrac{(ap-x+1)^x}{x!}(\tfrac{1}{p})^x((1-\tfrac{1}{p})^{p-1})^a(1-\tfrac{1}{p})^{a-x}\\ &> (1-\tfrac{x-1}{ap})^x\tfrac{(ap)^x}{p^x x!}(e^{-1})^a(\tfrac{p}{p-1})^{x-a}\\ &> \epsilon\times\tfrac{a^x}{x!}\times e^{-a}.\end{aligned}$$

□

The following simple procedure allows us to obtain a *random p-partition* of a finite set S: assign each element of S independently at random to one of A_i's. (In particular for each $j \in S$ and $i \in \{1,2,\ldots,p\}$, $Prob(j \in A_i) = 1/p$.)

Lemma 2.8 *Let $p, b \geq 2$ be integers and let a be a positive rational number such that ap is an integer and $b > a$. Assume that $ap \geq b$ and $(1-\frac{b-1}{ap})^b > \frac{1}{2}$ holds. Let $\mathcal{A} = (A_1, A_2, \ldots, A_p)$ be a random p-partition of $\{1,2,\ldots,ap\}$. Then the probability that $|A_i| < b$ for all $i \in \{1,2,\ldots,p\}$ is at most $(1-\frac{a^b}{2e^a b!})^p$.*

Proof: Let B_i be the event that $|A_i| < b$. Mallows [16] proved that the probability that all B_i hold is bounded above by $\Pi_{i=1}^p Prob(B_i)$ (various more general results can be found in [7]). We will now give an upper bound for $Prob(B_i)$. Using $\epsilon = 1/2$ in Lemma 2.7, we obtain the following:

$$Prob(B_i) \quad < 1 - \binom{ap}{b}(\tfrac{1}{p})^b(1-\tfrac{1}{p})^{ap-b} \quad < 1 - \tfrac{a^b}{2e^a b!}$$

This implies that $\Pi_{i=1}^p Prob(B_i) < (1-\frac{a^b}{2e^a b!})^p$. □

Lemma 2.9 *Let (N, σ, p) be a triple defining an instance of MMSP ($p \geq 2$) and let $\sigma(1) \geq \sigma(2) \geq \ldots \geq \sigma(n) = 1$. Assume that there is a rational number q such that $0 < q < 3$ and $n = qp$. Let $\tilde{\sigma} = \sum_{i=1}^{qp}\sigma(i)/p$. Let $\mathcal{A} = (A_1, A_2, \ldots, A_p)$ be a random p-partition of $N = \{1,2,\ldots,qp\}$ and let E be the event that $\sum_{j\in A_i}\sigma(j) \leq \tilde{\sigma}+1$ for all $i \in \{1,2,\ldots,p\}$. Let $a = \lceil\frac{qp}{2q+1}\rceil/p$ and $b = \lceil 2q+1\rceil$. If $(1-\frac{b-1}{pa})^b > \frac{1}{2}$ and $ap \geq b$ then $Prob(E) \leq (1-\frac{a^b}{2e^a b!})^p$.*

Proof: If $\sigma(1) > \tilde{\sigma}+1$, then clearly $Prob(E) = 0$. Thus, we may assume that $1+\tilde{\sigma} \geq \sigma(1) \geq \sigma(2) \geq \ldots \geq \sigma(qp) = 1$. Since $\tilde{\sigma}/q \geq 1$, we have $\frac{(q+1)\tilde{\sigma}}{q} \geq 1+\tilde{\sigma}$. Thus,

$$\frac{(q+1)\tilde{\sigma}}{q} \geq 1+\tilde{\sigma} \geq \sigma(1) \geq \sigma(2) \geq \ldots \geq \sigma(qp) = 1.$$

Let m be the maximal integer for which $\sigma(m) \geq \tilde{\sigma}/(2q)$. Thus, $m\frac{(q+1)\tilde{\sigma}}{q} + (qp-m)\frac{\tilde{\sigma}}{2q} \geq p\tilde{\sigma}$. This implies that $m \geq \frac{qp}{2q+1}$. Let $S = \{1,2,\ldots,ap\}$. By Lemma 2.8, the probability that at least b elements of S are assigned to the same set A_i is at least $1-(1-\frac{a^b}{2e^a b!})^p$. We will now show that the sum of weights σ of at least b elements of S exceeds $\tilde{\sigma}+1$, which implies that $Prob(E) \leq (1-\frac{a^b}{2e^a b!})^p$.

If $\tilde{\sigma} < 2q$ then the total weight of at least b elements is at least $b \geq 2q+1 > \tilde{\sigma}+1$. And if $\tilde{\sigma} \geq 2q$ then the total weight of at least b elements is at least $b\tilde{\sigma}/(2q) \geq (2q+1)\tilde{\sigma}/(2q) = \tilde{\sigma}+\tilde{\sigma}/2q \geq \tilde{\sigma}+1$. □

3 Main Result

Recall that the size s of MMSP is $\Theta(n + \sum_{i=1}^{n} \log \sigma(i))$. Consider the following approximation algorithm H for MMSP. If $s \geq p^n$, then we simply solve the problem optimally. This takes $O(s^2)$ time, as there are at most $O(s)$ solutions, and each one can be evaluated and compared to the current best in $O(s)$ time. If $s < p^n$, then sort the elements of the sequence $\sigma(1), \sigma(2), \ldots, \sigma(n)$. For simplicity of notation, assume that $\sigma(1) \geq \sigma(2) \geq \cdots \geq \sigma(n)$. Compute $r = \lceil \log n / \log p \rceil$ and solve MMSP for $(\{1, 2, \ldots, r\}, \sigma, p)$ to optimality. Suppose we have obtained a p-partition $\mathcal{A}$ of $\{1, 2, \ldots, r\}$. Now for i from $r + 1$ to n add i to the set A_j of the current p-partition $\mathcal{A}$ with smallest $\sigma(A_j)$.

Theorem 3.1 *The algorithm H runs in time $O(s^2 \log s)$. We have* $\lim_{s \longrightarrow \infty} \mathrm{domr}(H, s) = 1$.

Proof: We may assume that every operation of addition and comparison takes $O(s)$ time (see, e.g., [1]). As we observed above, the case $s \geq p^n$ takes $O(s^2)$ time. Let $s < p^n$. The sorting part of H takes time $O(sn \log n)$. The 'optimality' part can be executed in time $O(sp^{\lceil \log n / \log p \rceil}) = O(sn)$. Using an appropriate data structure, one can find out where to add each element i for $i \geq r$ in $O(s \log p)$ time. Thus, the time complexity of H is $O(s^2 \log s)$.

In what follows, we assume that $s < p^n$. Observe that to prove that $\lim_{s \longrightarrow \infty} \mathrm{domr}(H, s) = 1$ it suffices to show that $\lim_{n \longrightarrow \infty} \mathrm{domr}(H, s) = 1$. Indeed, by $p < n$ and $s < p^n < n^n$, $\lim_{s \longrightarrow \infty} n = \infty$.

Let $\varepsilon > 0$ be arbitrary. We will show that there exists an integer n_ε such that $\mathrm{domr}(H, s) > 1 - \varepsilon$ for all $n > n_\varepsilon$. Let $n_\varepsilon = \max\{n_0, n_1, n_2, n_3\}$, where n_0, n_1, n_2 and n_3 are any integers satisfying the following inequalities for all $1/3 < a \leq 1$ and $3 < b \leq 7$.

$$(1 - \frac{b-1}{a\sqrt{\log n_0 / 3}})^b > \frac{1}{2} \qquad \varepsilon > \left(1 - \frac{a^b}{2e^a b!}\right)^{\sqrt{\log n_1 / 3}}$$

$$a\sqrt{\log n_0 / 3} \geq b$$

$$\varepsilon > \sqrt{\frac{4}{\pi}} \sqrt{\frac{8 \log \log n_2 \log \log \log n_2}{\pi \log n_2}} \qquad \varepsilon > \sqrt{\frac{4}{\pi}} \times 0.95^{\log \log n_3 - 1}$$

Let $\mathcal{I}$ be an instance (N, σ, p), where $n > n_\varepsilon$, and let $\mathcal{A} = (A_1, A_2, \ldots, A_p)$ be a p-partition of N obtained by H for $\mathcal{I}$.

Let A_j be the set of maximal weight in $\mathcal{A}$ and let m be the maximum element of A_j. Clearly, $\sigma(\mathcal{A}) = \sigma(A_j)$. Note that if $m \leq r$ or $m \leq p$, then $\mathcal{A}$ is an optimal solution. Indeed, if $m \leq r$, then H solves MMSP for $(\{1, 2, \ldots, m\}, \sigma, p)$ to optimality with value $\sigma(A_j)$. Since the value of the solution for $(\{1, 2, \ldots, n\}, \sigma, p)$ remains $\sigma(A_j)$, the solution stays optimal. If $m \leq p$, then we may assume that $r < m \leq p$. Hence, $A_j = \{m\}$ and H solves MMSP for $(\{1, 2, \ldots, m\}, \sigma, p)$ to optimality with value $\sigma(m)$. Since the value of the solution for $(\{1, 2, \ldots, n\}, \sigma, p)$ remains $\sigma(m)$, the solution stays optimal.

Thus, we may assume that

$$m > r \text{ and } m > p \tag{3}$$

Since $m > r$, m is the last element added to A_j. If we divide every $\sigma(i)$, $i = 1, 2, \ldots, n$, by $\sigma(m)$ we do not change the solution $\mathcal{A}$ of H. Thus, we may assume that $\sigma(m) = 1$.

At the time just before m was appended to A_j, $\sigma(A_i) \geq \sigma(A_j)$ for every $i \neq j$. Hence, $\sigma(A_j) \leq \sigma(m) + \sum_{i=1}^{m-1} \sigma(i)/p < \tilde{\sigma} + 1$, where $\tilde{\sigma} = \sum_{i=1}^{m} \sigma(i)/p$. Thus,

$$\sigma(\mathcal{A}) < \tilde{\sigma} + 1. \tag{4}$$

We now consider the following cases.

Case 1: $m \geq 3p$. There are p^m possible ways of putting $1, 2, \ldots, m$ into p sets of a p-partition. By (4) and Lemma 2.6, the number of p-partitions of $\{1, 2, \ldots, m\}$ that are worse than $\mathcal{A}$ is more than

$$p^m - p^m \sqrt{\frac{4}{\pi}} \left(\sqrt{\frac{8p}{\pi m}} \right)^{p-1}.$$

Clearly, no matter how we place the elements $m+1, m+2, \ldots, n$ into the sets of a p-partition $\mathcal{B}$ worse than $\mathcal{A}$ (i.e., $\sigma(\mathcal{A}) < \sigma(\mathcal{B})$), we will end up with a solution worse than $\mathcal{A}$. Thus, the number of solutions worse than $\mathcal{A}$ is more than

$$p^{n-m} \left(p^m - p^m \sqrt{\frac{4}{\pi}} \left(\sqrt{\frac{8p}{\pi m}} \right)^{p-1} \right) = p^n \left(1 - \sqrt{\frac{4}{\pi}} \left(\sqrt{\frac{8p}{\pi m}} \right)^{p-1} \right).$$

Thus,

$$\mathrm{domr}(H, \mathcal{I}) > 1 - \sqrt{\frac{4}{\pi}} \left(\sqrt{\frac{8p}{\pi m}} \right)^{p-1} \tag{5}$$

Since $m \geq 3p$, we have $\sqrt{8p/(\pi m)} < 0.95$. So if $p \geq \log \log n$, then we are done as $n \geq n_3$. If $p < \log \log n$, then by (3) and the definition of n_2 we obtain the following:

$$\mathrm{domr}(H, \mathcal{I}) > 1 - \sqrt{\frac{4}{\pi}} \left(\sqrt{\frac{8p}{\pi \log n / \log p}} \right)^{p-1} > 1 - \sqrt{\frac{4}{\pi}} \left(\sqrt{\frac{8 \log \log n \log \log \log n}{\pi \log n}} \right) > 1 - \varepsilon.$$

Case 2: $m < 3p$. We define $q = \frac{m}{p} < 3$, and note that (by (3)), $q > 1$ and $p > m/3 > \log n / (3 \log p)$. This implies that $p > \sqrt{\log n / 3}$. Let $a = \lceil \frac{qp}{2q+1} \rceil / p$ and $b = \lceil 2q + 1 \rceil$, and note that $\frac{1}{3} < a \leq 1$ and $3 < b \leq 7$ and $ap > b$ (by the definition of n_0). By the definitions of n_0, n_1 and Lemma 2.9, we have the following:

$$\mathrm{domr}(H, \mathcal{I}) > 1 - \left(1 - \frac{a^b}{2e^a b!} \right)^p > 1 - \left(1 - \frac{a^b}{2e^a b!} \right)^{\sqrt{\log n/3}} > 1 - \varepsilon. \tag{6}$$

□

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