Minimum Cost Homomorphisms to Semicomplete Bipartite Digraphs

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Abstract

For digraphs D and H, a mapping $f : V(D) \rightarrow V(H)$ is a homomorphism of D to H if $uv \in A(D)$ implies $f(u)f(v) \in A(H)$. If, moreover, each vertex $u \in V(D)$ is associated with costs $c_i(u), i \in V(H)$, then the cost of the homomorphism f is $\sum_{u \in V(D)} c_{f(u)}(u)$. For each fixed digraph H, we have the *minimum cost homomorphism problem for H*. The problem is to decide, for an input graph D with costs $c_i(u)$, $u \in V(D), i \in V(H)$, whether there exists a homomorphism of D to H and, if one exists, to find one of minimum cost. Minimum cost homomorphism problems encompass (or are related to) many well studied optimization problems. We describe a dichotomy of the minimum cost homomorphism problem for semicomplete multipartite digraphs H. This solves an open problem from an earlier paper. To obtain the dichotomy of this paper, we introduce and study a new notion, a k-Min-Max ordering of digraphs.

1 Introduction

Motivation. We consider only directed (undirected) graphs that have neither loops nor multiple arcs (edges). In this paper we solve a problem raised in [5] to find a dichotomy for the computational complexity of minimum cost homomorphism problem (MCH) for semicomplete bipartite digraphs (we define this problem below). In fact, our result leads to a complete dichotomy for the computational complexity of MCH for semicomplete k-partite digraphs ($k \geq 2$) as a (much simpler) dichotomy for the case $k \geq 3$ was obtained in [5] (see also Section 5). Our result uses and significantly extends a dichotomy for the computational complexity of MCH for bipartite undirected graphs obtained in [3].

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In our previous papers we used properties of an important notion of Min-Max ordering of digraphs. To obtain the dichotomy of this paper, we introduce and study a new notion, a k-Min-Max ordering of digraphs. We believe that properties of this notion and, in particular, Theorem 2.2 can be used to obtain further results on MCH and its special cases (see below).

The minimum cost homomorphism problem was introduced in [6], where it was motivated by a real-world problem in defence logistics. We believe it offers a practical and natural model for optimization of weighted homomorphisms. MCH's special cases include the well-known list homomorphism problem [8, 10] and the general optimum cost chromatic partition problem, which has been intensively studied [7, 11, 12], and has a number of applications, [14, 15].

Minimum cost homomorphisms. For directed or undirected graphs G and H, a mapping $f: V(G) \rightarrow V(H)$ is a *homomorphism of G to H* if $uv \in E(G)$ implies $f(u)f(v) \in E(H)$. Recent treatments of homomorphisms in directed and undirected graphs can be found in [8, 10]. Let H be a fixed directed or undirected graph. The *homomorphism problem* for H asks whether a directed or undirected input graph G admits a homomorphism to H. The *list homomorphism problem* for H asks whether a directed or undirected input graph G with lists (sets) $L_u \subseteq V(H), u \in V(G)$ admits a homomorphism f to H in which $f(u) \in L_u$ for each $u \in V(G)$.

Suppose G and H are directed (or undirected) graphs, and $c_i(u)$, $u \in V(G)$, $i \in V(H)$ are nonnegative *costs*. The *cost of a homomorphism* f of G to H is $\sum_{u \in V(G)} c_{f(u)}(u)$. If H is fixed, the *minimum cost homomorphism problem*, MinHOM(H), for H is the following optimization problem. Given an input graph G, together with costs $c_i(u)$, $u \in V(G)$, $i \in V(H)$, we wish to find a minimum cost homomorphism of G to H, or state that none exists.

A bipartite digraph is *semicomplete* if there is at least one arc between every two vertices belonging to different partite sets. In this paper, we study the *minimum cost homomorphism problem for semicomplete bipartite digraphs*, i.e., MinHOM(H) when H is a *semicomplete bipartite digraph.* Observe that MCH for semicomplete bipartite digraphs extends MCH for bipartite undirected graphs. Indeed, let B be a semicomplete bipartite digraph with partite sets U, V and arc set $A(B) = A_1 \cup A_2$, where $A_1 = \{uv: u \in U, v \in V\}$ and $A_2 \subseteq \{vu: v \in V, u \in U\}$. Let B' be a bipartite graph with partite sets U, V and edge set $E(B') = \{uv: vu \in A_2\}$. Notice that MinHOMP($B$) is equivalent to MinHOMP(B').

Min-Max ordering. Let H be a digraph. We say that an ordering $v_1, v_2, \ldots, v_p$ of $V(H)$ is a *Min-Max ordering of H* if $v_iv_r, v_jv_s \in A(H)$ implies $v_{\min\{i,j\}}v_{\min\{s,r\}} \in A(H)$ and $v_{\max\{i,j\}}v_{\max\{s,r\}} \in A(H)$. One can easily see that $v_1, v_2, \ldots, v_p$ of $V(H)$ is a Min-Max ordering of H if $i < j$, $s < r$ and $v_iv_r, v_jv_s \in A(H)$, then $v_iv_s \in A(H)$ and $v_jv_r \in A(H)$. We can define a Min-Max ordering for a bipartite undirected graph G with partite sets

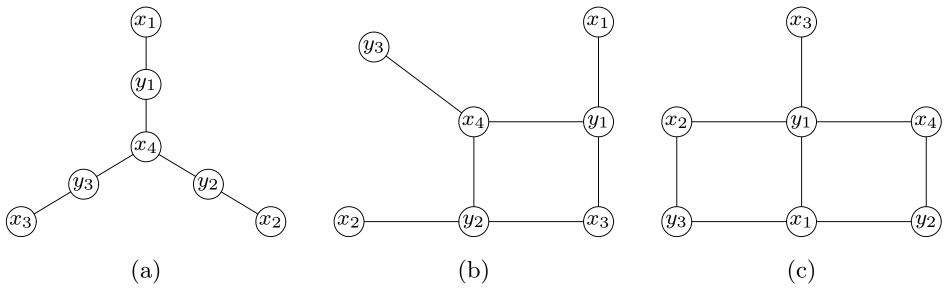

Figure 1: A bipartite claw (a), a bipartite net (b) and a bipartite tent (c).

V and U as follows: We orient all edges from V to U and apply the above definition for digraphs. Importance of Min-Max ordering for MinHOM(H) is indicated in the following two theorems.

Theorem 1.1 *[4] Let a digraph H have a Min-Max ordering. Then MinHOM(H) is polynomial-time solvable.*

A bipartite graph H with vertices $x_1, x_2, x_3, x_4, y_1, y_2, y_3$ is called

a bipartite claw if its edge set $E(H) = \{x_4y_1, y_1x_1, x_4y_2, y_2x_2, x_4y_3, y_3x_3\}$;

a bipartite net if its edge set $E(H) = \{x_1y_1, y_1x_3, y_1x_4, x_3y_2, x_4y_2, y_2x_2, y_3x_4\}$;

a bipartite tent if its edge set $E(H) = \{x_1y_1, y_1x_3, y_1x_4, x_3y_2, x_4y_2, y_2x_2, y_3x_4\}$.

See Figure 1.

Theorem 1.2 *[3] Let H be an undirected bipartite graph. If H contains a cycle C_{2k}, $k \geq 3$ or a bipartite claw or a bipartite net or a bipartite tent as an induced subgraph, then MinHOM(H) is NP-hard.*

Assume that P≠NP. Then the following three assertions are equivalent:

(i) H has a Min-Max ordering;

(ii) MinHOM(H) is polynomial time solvable;

(iii) H does not contain a cycle C_{2k}, $k \geq 3$, a bipartite claw, a bipartite net, or a bipartite tent as an induced subgraph.

Additional terminology and notation. For a graph H, $V(H)$ and $E(H)$ denote its vertex and edge sets, respectively. For a digraph H, $V(H)$ and $A(H)$ denote its vertex and arc sets, respectively. For a pair X, Y of vertex sets of a digraph H, $(X, Y)_H$ denotes the set of all arcs of the form xy, where $x \in X, y \in Y$. We omit the subscript when it is clear from the context. Also, $X \times Y = \{xy : \ x \in X, y \in Y\}$. For a set $X \subseteq V(H)$, let $N^+(X) = \{y : \ \exists x \in X \text{ with } xy \in A(H)\}$ and $N^-(X) = \{y : \ \exists x \in X \text{ with } yx \in A(H)\}$.

If xy is an arc of a digraph H, we will say that x *dominates* y, y *is dominated by* x, y is an *out-neighbor* of x, and x is an *in-neighbor* of y. We also denote it by $x \rightarrow y$. For disjoint sets $X, Y \subseteq V(H)$, $X \rightarrow Y$ means that $x \rightarrow y$ for each $x \in X$ and $y \in Y$.

An *extension* of a digraph G is a digraph D obtained from G by replacing each vertex u of G by a set of independent vertices $u_1, u_2, \ldots, u_{n(u)}$ such that for a pair u, v of vertices in G, $u_i \rightarrow v_j$ in D if and only if $u \rightarrow v$ in G.

For a bipartite digraph $H = (V, U; A)$, where V and U are its partite sets, $H^{\rightarrow}$ is the subdigraph induced by all arcs directed from V to U, $H^{\leftarrow}$ is the subdigraph induced by all arcs directed from U to V, and $H^{\leftrightarrow}$ is the subdigraph induced by all 2-cycles of H, i.e., by the set $\{xy : \ xy \in A, yx \in A\}$. The *converse* of H is the digraph obtained from H by replacing every arc xy with the arc yx.

We denote a directed cycle with p vertices by $\vec{C}_p$. For a set X of vertices of a digraph H, $D[X]$ denotes the subdigraph of H induced by X. For a digraph H, $\mathrm{UN}(H)$ denotes the *underlying graph* of H, i.e., an undirected graph obtained from H by disregarding all orientations and deleting multiple edges.

A digraph D is *strong* (or, *strongly connected*) if there is a directed path from x to y and a directed path from y to x for every pair x, y of vertices of D. A *strong component* of a digraph D is a strong maximal induced subdigraph of D.

Forbidden family. Let us introduced five special digraphs for which, as we will see later, the minimum homomorphism problem is NP-hard. The digraph C_4' has vertex set $\{x_1, x_2, y_1, y_2\}$ and arc set $\{x_1y_1, y_1x_2, x_2y_2, y_2x_1, y_1x_1\}$. The digraph C_4'' has the same vertex set, but its arc set is $A(C_4') \cup \{x_2y_1\}$. The digraph H^* has vertex set $\{x_1, x_2, y_1, y_2, y_3\}$ and arc set

$$\{x_1y_1, y_1x_2, x_2y_2, y_2x_1, x_1y_3, x_2y_3\}.$$

Let N_1 be a digraph with $V(N_1) = \{x_1, x_2, x_3, y_1, y_2, y_3\}$ and

$$A(N_1) = \{x_1y_1, y_1x_1, x_2y_2, y_2x_2, x_3y_3, y_3x_3, y_1x_2, y_1x_3, x_1y_2, x_1y_3, x_3y_2, x_2y_3\}.$$

Let N_2 be a digraph with $V(N_2) = \{x_1, x_2, x_3, y_1, y_2, y_3\}$ and

$$A(N_2) = \{x_1y_1, x_2y_2, y_2x_2, x_3y_3, y_3x_3, y_1x_2, y_1x_3, x_1y_2, x_1y_3, x_3y_2, x_2y_3\}.$$

A digraph H belongs to the family $\mathcal{HFORB}$ if H or its converse is isomorphic to one of the five digraphs above or $\mathrm{UN}(H^s)$ is isomorphic to bipartite claw, bipartite net, bipartite tent or even cycle with at least 6 vertices, where $s \in \{\rightarrow, \leftarrow, \leftrightarrow\}$.

k-Min-Max ordering. A collection $V_1, V_2, \ldots, V_k$ of subsets of a set V is called a *k-partition* of V if $V = V_1 \cup V_2 \cup \cdots \cup V_k$, $V_i \cap V_j = \emptyset$ provided $i \neq j$.

Definition 1.3 *Let $H = (V, A)$ be a digraph and let $k \geq 2$ be an integer. We say that H has a k-*Min-Max ordering *if there is a k-partition of V into subsets $V_1, V_2, \ldots, V_k$ and there is an ordering $v_1^i, v_2^i, \ldots, v_{\ell(i)}^i$ of V_i for each i such that*

(i) *Every arc of H is an (V_i, V_{i+1})-arc for some $i \in \{1, 2, \ldots, k\}$,*

(ii) $v_1^i, v_2^i, \ldots, v_{\ell(i)}^i, v_1^{i+1}, v_2^{i+1}, \ldots, v_{\ell(i+1)}^{i+1}$ *is a Min-Max ordering of the subdigraph $H[V_i \cup V_{i+1}]$ for all $i \in \{1, 2, \ldots, k\}$,*

where all indices $i + 1$ are taken modulo k. We call the ordering

$$v_1^1, v_2^1, \ldots, v_{\ell(1)}^1, v_1^2, v_2^2, \ldots, v_{\ell(2)}^2, \ldots, v_1^k, v_2^k, \ldots, v_{\ell(k)}^k$$

*a k-*Min-Max ordering *of H.*

Note that if H is a strong digraph in which the greatest common divisor of all cycle lengths is k, then $V(H)$ has a k-partition, $k \geq 2$, satisfying (i) (see Theorem 10.5.1 in [1]). A simple example of a digraph having a k-Min-Max ordering is an extension of $\vec{C}_k$.

Dichotomy and paper organization. The main result of this paper is the following:

Theorem 1.4 *Let H be an semicomplete bipartite digraph. If H contains a digraph from $\mathcal{HFORB}$ as an induced subdigraph, then MinHOM(H) is NP-hard.*

Assume that P≠NP. Then the following three assertions are equivalent:

(i) MinHOM(H) is polynomial time solvable;

(ii) H does not contain a digraph from $\mathcal{HFORB}$ as an induced subdigraph;

(iii) Each component of H has a k-Min-Max ordering for $k = 2$ or 4.

Theorem 1.4 follows from Corollaries 3.5 and 4.6.

The rest of the paper is organized as follows. In Section 2, we study properties of k-Min-Max orderings. In Section 3, we prove polynomial cases of MinHOM(H) when H is a semicomplete bipartite digraph. In Section 4, we establish NP-hard cases of the problem. In Section 5 we formulate a dichotomy for the computational complexity of MinHOM(H) when H is a semicomplete multipartite digraph. Section 6 provides a short discussion of further research.

2 Properties of k-Min-Max Orderings

Digraphs having k-Min-Max ordering have a very special structure as described in the following lemma.

Lemma 2.1 *If a digraph $H = (V, A)$ has a k-Min-Max ordering as described in Definition 1.3 and $I = \{x \in V : \ d^-(x) = 0\}$, then we can define $0 \leq L(i,j) < R(i,j) \leq \ell(j) + 1$ for all i and j such that the following holds.*

(a): *We have $N^+(v_i^j) = \{v_{L(i,j)+1}^{j+1}, v_{L(i,j)+2}^{j+1}, \ldots, v_{R(i,j)-1}^{j+1}\} \setminus I$ for each $v_i^j \in V$ with positive out-degree, where all superscripts are taken modulo k;*

(b): *For all j and $i < i'$ we have $R(i,j) \leq R(i',j)$ and $L(i,j) \leq L(i',j)$.*

Proof: If $d_H^+(v_i^j) > 0$, then let $L(i,j)$ be the maximum number such that $N^+(v_i^j) \cap \{v_1^{j+1}, v_2^{j+1}, \ldots, v_{L(i,j)}^{j+1}\} = \emptyset$ and let $R(i,j)$ be the minimum number such that $N^+(v_i^j) \cap \{v_{R(i,j)}^{j+1}, v_{R(i,j)+1}^{j+1}, \ldots, v_{l(j)}^{j+1}\} = \emptyset$. If $d_H^+(v_i^j) = 0$, then if $i = 1$ we let $L(i,j) = 1 = R(i,j) - 1$ and if $i > 1$ we let $L(i,j) = L(i-1,j)$ and $R(i,j) = R(i-1,j)$.

Suppose that $v_i^j v_{m-1}^{j+1}, v_i^j v_{m+1}^{j+1} \in A$, but $v_i^j v_m^{j+1} \notin A$ and $v_m^{j+1} \notin I$. Since $v_m^{j+1} \notin I$, there is an arc $v_t^j v_m^{j+1}$ in H. By the definition of a k-Min-Max ordering, $v_i^j v_m^{j+1} \in A$, a contradiction that proves (a).

We now prove (b). If $d^+(v_i^j) > 0$ and $d^+(v_{i'}^j) > 0$ then this follows from the k-Min-Max ordering using an analogous argument as above. If $i = 1$ and $d^+(v_i^j) = 0$, then we also see that (b) holds. We now prove the remaining cases by induction on $i + i'$. If $d^+(v_i^j) = 0$ consider $i - 1$ instead of i and if $d^+(v_{i'}^j) = 0$ then consider $i' - 1$ instead of i'. If the new i and i' are equal than (b) holds and otherwise we are done by induction. $\diamond$

The construction used in the following theorem was inspired by somewhat similar constructions in [13] and [2].

Theorem 2.2 *If a digraph H has a k-Min-Max ordering, then MinHOM(H) is polynomial-time solvable.*

Proof: Let H have a k-Min-Max ordering. Let $V_1, V_2, \ldots, V_k$ be defined as in Definition 1.3. If there is a homomorphism h of D to H, then each $h^{-1}(V_i)$, $i = 1, 2, \ldots, k$ is an independent set. Thus, we may say that the vertices of $h^{-1}(V_i)$ are of color i. We may assume that UN(D) is connected as otherwise we consider a minimum cost homomorphism of each component of D (corresponding to a component of UN(D)) separately.

Let $z \in V(D)$ be an arbitrary vertex. We will show how to decide whether there is a homomorphism f of D to H, such that $f(z) \in V_i$, for each $i \in \{1, 2, 3, \ldots, k\}$. If such a homomorphism exists for a given i, then we will furthermore show how to find a minimum cost homomorphism of D to H, such that $f(z) \in V_i$. We may assume without loss of generality that $i = 1$.

Now assign color 1 to z. Assign every out-neighbor of z color 2 and each in-neighbor of z color k. For every vertex z' with color i, we assign every out-neighbor of z' color $i+1$ modulo k and every in-neighbor of z' color $i-1$ modulo k. If some vertex of D is assigned different colors, then clearly there is no homomorphism of D to H, so assume that this is not the case.

Consider the k-partition $G_1, G_2, \ldots, G_k$ of $V(D)$ such that all vertices of G_j were assigned color j for each $j \in \{1, 2, \ldots, k\}$. Clearly, all arcs in D are (G_j, G_{j+1})-arcs, where all indices are taken modulo k and $j \in \{1, 2, 3, \ldots, k\}$. Note that if f is a homomorphism of D to H such that $f(x) \in V_1$, then for all $y_j \in G_j$ we must have $f(y_j) \in V_j$.

We will build a weighted digraph (network) $\mathcal{L}$ with vertex set $\cup_{j=1}^{k}(G_j \times V_j)$ together with two other vertices, denoted by s and t. We will also denote t by $(x, v^j_{\ell(j)+1})$ for every $j \in \{1, 2, \ldots, k\}$. The weighted arcs of $\mathcal{L}$ are as follows, where M is any constant greater than the cost of a minimum cost homomorphism of D to H.

- An arc from s to (x, v^j_1), of weight ∞, for each $x \in G_j$.
- An arc, $e^x_{i,j}$, from (x, v^j_i) to (x, v^j_{i+1}), for each $x \in G_j$ and $i \in \{1, 2, \ldots, \ell(j)\}$. Recall that when $i = \ell(j)$ the arc enters t. We define the weight of $e^x_{i,j}$ as follows. If $d^-_D(x) > 0$ and $d^-_H(v^j_i) = 0$ then the weight is ∞. If $d^+_D(x) > 0$ and $d^+_H(v^j_i) = 0$ then the weight is ∞. Otherwise the weight is $c_i(x) + M$.
- an arc from (x, v^j_i) to $(y, v^{j+1}_{L(i,j)+1})$ and an arc from $(y, v^{j+1}_{R(i,j)})$ to (x, v^j_{i+1}) for every $xy \in A(D)$ with $x \in G_j$ and every $i = 1, 2, \ldots, \ell(j)$. Furthermore the weight of these arcs are ∞.

A *cut* in $\mathcal{L}$ is a partition of the vertices into two sets S and T such that $s \in S$ and $t \in T$ and the weight of a cut is the sum of weights of all arcs going from a vertex of S to a vertex of T. We will show that a minimum weight cut in $\mathcal{L}$ has weight equal to the minimum cost homomorphism f of D to H such that $f(z) \in V_1$ plus $|V(D)|M$, if such a homomorphism exists. Otherwise, a minimum weight cut in $\mathcal{L}$ is of weight at least $|V(D)|M + M$.

Assume f is a minimum cost homomorphism of D to H such that $f(z) \in V_1$, and assume that $f(x) = v^j_{a(x)}$ for each $x \in G_j$ and for all $j = 1, 2, \ldots, k$. Define a cut in $\mathcal{L}$ as follows: $S = \{(x, v^j_i) : \ i \leq a(x),\ j = 1, 2, \ldots, k\} \cup \{s\}$ and $T = V(\mathcal{L}) - S$. Since f is a homomorphism of D to H, we note that if $d^-_D(x) > 0$ then $d^-_H(v^j_{a(x)}) > 0$ and if $d^+_D(x) > 0$ then $d^+_H(v^j_{a(x)}) > 0$. Therefore the arcs from $(x, v^j_{a(x)})$ to $(x, v^j_{a(x)+1})$ belong to the cut and contribute $c_{f(x)}(x) + M$ to the weight of the cut (and not ∞). We will now show that there are no arcs of infinite weight in the cut, which would imply that the weight of S is exactly the cost of a minimum cost homomorphism from D to H plus $|V(D)|M$.

Clearly no arc out of s belongs to the cut S. Assume for the sake of contradiction that the arc (x, v_i^j) to $(y, v_{L(i,j)+1}^{j+1})$ belongs to the cut S for some $xy \in A(D)$ with $x \in G_j$. This implies that $a(x) \geq i$ (as $(x, v_i^j) \in S$) and $a(y) < L(i,j) + 1$ (as $(y, v_{L(i,j)+1}^{j+1}) \notin S$). By Lemma 2.1 (b), this implies that $a(y) \leq L(i,j) \leq L(a(x), j)$. Thus, there is no arc from $v_{a(x)}^j$ to $v_{a(y)}^{j+1}$ in H, by the definition of $L(i,j)$. This is a contradiction to f being a homomorphism.

Now assume for the sake of contradiction that the arc $(y, v_{R(i,j)}^{j+1})$ to (x, v_{i+1}^j) belongs to the cut S for some $xy \in A(D)$ with $x \in G_j$. This implies that $a(x) < i+1$ (as $(x, v_{i+1}^j) \notin S$) and $a(y) \geq R(i,j)$ (as $(y, v_{R(i,j)}^{j+1}) \in S$). By Lemma 2.1 (b), this implies that $a(y) \geq R(i,j) \geq R(a(x), j)$. Thus, there is no arc from $v_{a(x)}^j$ to $v_{a(y)}^{j+1}$ in H, by the definition of $R(i,j)$. This is a contradiction to f being a homomorphism. We have now proved that the cut S has the stated weight when there is a homomorphism of D to H with $f(z) \in V_1$.

Assume that S' is a cut of weight less than $M + |V(D)|M$ and note that the cut S' contains exactly one arc of the form $(x, v_i^j)(x, v_{i+1}^j)$ for each $x \in V(D)$. Therefore we may define a mapping, f', from $V(D)$ to $V(H)$ by letting $f'(x) = v_i^j$ if and only if $(x, v_i^j) \in S'$ and $(x, v_{i+1}^j) \notin S'$. We will now show that f' is a homomorphism of D to H of cost equal to the weight of the cut S' minus $|V(D)|M$.

Let xy be any arc in D, and assume without loss of generality that $x \in G_j$. Let i_x and i_y be defined such that $(x, v_{i_x}^j) \in S'$ and $(x, v_{i_x+1}^j) \notin S'$ (i.e., $f'(x) = v_{i_x}^j$) and $(y, v_{i_y}^{j+1}) \in S'$ and $(y, v_{i_y+1}^{j+1}) \notin S'$ (i.e., $f'(y) = v_{i_y}^{j+1}$). As the arc $(x, v_{i_x}^j)(y, v_{L(i_x,j)+1}^{j+1})$ is not in the cut, we must have $(y, v_{L(i_x,j)+1}^{j+1}) \in S$, which implies that $i_y \geq L(i_x, j) + 1$. Furthermore as the arc $(y, v_{R(i_x,j)}^{j+1})(x, v_{i_x+1}^j)$ is not in the cut, we must have $(y, v_{R(i_x,j)}^{j+1}) \notin S$, which implies that $i_y < R(i_x, j)$. We have now shown that $L(i_x, j) + 1 \leq i_y < R(i_x, j)$. Since $xy \in A(D)$ and the arc $(x, v_{i_x}^j)(x, v_{i_x+1}^j)$ has finite weight we observe that $d_H^+(v_{i_x}^j) > 0$ and $v_{i_y}^{j+1} \notin I$, where I is defined in Definition 2.1. Definition 2.1 now implies that $v_{i_x}^j v_{i_y}^{j+1}$ is an arc in H. Therefore f' is a homomorphism. It is not difficult to see that the cost of f' is indeed equal to weight of the cut S' minus $|V(D)|M$.

It remains to recall that a minimum weight cut in a network can be found in polynomial time [1]. $\diamond$

3 Polynomial Cases

We start from a special case which is of importance when H contains no induced $\vec{C}_4$.

Lemma 3.1 *Let $H = (V, U; A)$ be a semicomplete bipartite digraph, which does not contain an induced subdigraph belonging to $\mathcal{HFORB}$ or an induced directed 4-cycle. Suppose for every $v, v' \in V$ we have $N^+(v) \subseteq N^+(v')$ or $N^+(v') \subseteq N^+(v)$. Then H has a 2-Min-Max ordering.*

Proof: We say that vertices $v_i, v_j \in V$ are *similar* if $N^+(v_i) = N^+(v_j)$. Consider similarity classes $V_1, V_2, \ldots, V_s$ of V. Moreover assume that $N^+(V_i) \subset N^+(V_j)$ for $i < j$. Set $U_1 = N^+(V_1)$ and $U_i = N^+(V_i) - \cup_{j=1}^{i-1} U_j$ for each $i > 1$. If $s = 1$ then $\mathrm{UN}(H^{\rightarrow})$ is a complete bipartite graph. By Theorem 1.2, $H^{\leftarrow}$ has a Min-Max ordering π. Observe that the orderings of vertices in V and U given by the relative order of vertices in π generate a 2-Min-Max ordering of H. Assume $s > 1$. We prove the following two claims:

(1) Let $u_i \in U_i,\ v_j \in V_j,\ j > i$, and $u_i v_j \in A$. Then $U_r \rightarrow v_j$ for each $r > i$ and $u_i \rightarrow V_t$ for each $t < j$.

Proof of (1): By the definition of U_i and V_i, if $r > j$ then $U_r \rightarrow v_j$. Now suppose that $i < r \leq j$ and $u_r v_j \notin A$ for some $u_r \in U_r$. Let $v_i \in V_i$ be arbitrary, and note that $H[\{u_i, v_i, u_r, v_j\}]$ is isomorphic to C_4' or C_4'', a contradiction. So $u_r v_j \in A$.

By the definition of U_i and V_i, if $t < i$ then $u_i \rightarrow V_t$. Now suppose that $i \leq t < j$ and that $u_i v_t \notin A$, for some $v_t \in V_t$. Let $u_j \in U_j$ be arbitrary, and note that $H[\{u_i, v_t, u_j, v_j\}]$ is isomorphic to C_4' or C_4'', a contradiction.

(2) If $u, u' \in U$ then $N^+(u) \subseteq N^+(u')$ or $N^+(u') \subseteq N^+(u)$.

Proof of (2): Assume that this is not the case, and there exist $v \in N^+(u) - N^+(u')$ and $v' \in N^+(u') - N^+(u)$. This implies that $u \neq u'$, $v \neq v'$, $uv, u'v' \in A$ and $uv', u'v \notin A$. If $u \in U_i$ and $u' \in U_j$ and $i < j$, then by (1) we note that $u'v \in A$ a contradiction. So for some i we must have $\{u, u'\} \subseteq U_i$. Analogously, if $v \in V_a$ and $v' \in V_b$ and $a < b$, then by (1) we note that $u'v \in A$, a contradiction. So for some j we must have $\{v, v'\} \subseteq V_j$. If $i > j$ then $U_i \rightarrow V_j$, which is a contradiction, so we must have $i \leq j$.

If $i < j$ then let $u'' \in U_j$ and $v'' \in V_i$ be arbitrary. Note that by (1) we must have the arcs $u''v, u''v', uv'', u'v''$ in H. By the construction of the sets U_i, U_j, V_i, V_j we now note that the underlying graph of $H[\{u, u', u'', v, v', v''\}]^{\leftrightarrow}$ is the 6-cycle $v''uvu''v'u'v''$, a contradiction.

Therefore we must have $i = j$. First assume that $i < s$. Now consider $v_s \in V_s$ and $u_s \in U_s$. By (1) there is no arc from $\{u, u'\}$ to v_s. However $H[\{u, u', v, v', v_s, u_s\}]$ is either N_1 or N_2, a contradiction.

Similarly for the case $i = s$ we derive a contradiction.

Now consider an ordering $u_1, u_2, \ldots, u_a$ of the vertices in U and an ordering $v_1, v_2, \ldots, v_b$ of the vertices in V, defined as follows. If $i < j$ then $d^+(v_i) \leq d^+(v_j)$ and if $d^+(v_i) = d^+(v_j)$ then $d^-(v_i) \geq d^-(v_j)$. Furthermore when $i < j$ then $d^+(u_i) \leq d^+(u_j)$ and if $d^+(u_i) = d^+(u_j)$ then $d^-(u_i) \geq d^-(u_j)$. Note that the ordering $v_1, v_2, \ldots, v_b$ first contains vertices from V_1 then from V_2, etc.

We will now show that for every $i \in \{1, 2, \ldots, b\}$ there exists an integer α_i such that $N^+(u_i) = \{v_1, v_2, \ldots, v_{\alpha_i}\}$. Suppose this is not the case. Then there exists an arc u_iv_j in H, such that u_iv_{j-1} is not an arc in H. Thus, both v_j and v_{j-1} belong to some V_k, as otherwise we have a contradiction to (1). This implies that $d^+(v_j) = d^+(v_{j-1})$ and $d^-(v_{j-1}) \geq d^-(v_j)$. Note that every vertex $u \in N^-(v_{j-1})$ has $N^+(u_i) \subseteq N^+(u)$, by (2). Therefore $u \rightarrow v_j$. However this implies that $d^-(v_{j-1}) < d^-(v_j)$ (as $u_i \rightarrow v_j$ but u_i does not dominate v_{j-1}), a contradiction.

Using the fact that $N^+(u_i) = \{v_1, v_2, \ldots, v_{\alpha_i}\}$ for each $i \in \{1, 2, \ldots, b\}$ and that $\alpha_i \geq \alpha_{i-1}$ for each $i \in \{2, 3, \ldots, b\}$ (as $d^+(u_i) \geq d^+(u_{i-1})$) and the similar relations for the vertices of V, we can readily conclude that H has a 2-Min-Max ordering. ◇

The *distance* $\text{dist}(x, y)$ between a pair x, y of vertices in an undirected graph G is the length of the shortest path between x and y. The *diameter* of G is the maximal distance between a pair of vertices in G.

The following theorem shows when MinHOM(H) is polynomial time solvable if H is strong and does not contain $\vec{C}_4$ as an induced subdigraph.

Theorem 3.2 *Let H be a strongly connected semicomplete bipartite digraph. Assume that H does not contain a digraph from $\mathcal{HFORB}$ or $\vec{C}_4$ as an induced subdigraph. Then H has a 2-Min-Max ordering and MinHOM(H) is polynomial time solvable.*

Furthermore, either $\text{UN}(H^{\rightarrow})$ *or* $\text{UN}(H^{\leftarrow})$ *are complete bipartite graphs, or the following holds. For every pair u, u' of distinct vertices of U we have $N^+(u) \subseteq N^+(u')$ or $N^+(u') \subseteq N^+(u)$ and for every pair v, v' of distinct vertices of V we have $N^+(v) \subseteq N^+(v')$ or $N^+(v') \subseteq N^+(v)$.*

Proof: By Theorem 2.2, to prove the first part part of this theorem (before 'Furthermore'), it suffices to show that H has a 2-Min-Max ordering. Let V and U be partite sets of H. Denote $H_1 = H^{\rightarrow}$ and $H_2 = H^{\leftarrow}$. It follows from Theorem 1.2 that $\text{UN}(H_1)$ and $\text{UN}(H_2)$ have Min-Max orderings and so do H_1 and H_2.

Let d_i be the diameter of $\text{UN}(H_i)$, $i = 1, 2$. Observe that if $\text{UN}(H_1)$ ($\text{UN}(H_2)$) is a complete bipartite graph, then, as in the proof of Lemma 3.1, a Min-Max ordering of H_2 (H_1) generates a 2-Min-Max ordering of H. Therefore, MinHOM(H) is polynomial time solvable by Theorem 2.2. Notice that $\text{UN}(H_i)$ is complete bipartite if and only if $d_i = 2$. Thus, we may assume that both $d_1 \geq 3$ and $d_2 \geq 3$.

We consider the following cases for the value of $d_1 \geq 3$.

Case 1: $d_1 > 4$.

We will show that $\text{UN}(H_2)$ is a complete bipartite graph or, equivalently, $d_2 = 2$. Assume that d_1 is odd, as the case of d_1 even can be considered similarly. Let $P = v_1u_1v_2u_2\dots v_{k-1}u_{k-1}v_ku_k$ be a shortest path of length d_1 between v_1 and u_k in $\text{UN}(H_1)$. Let $\hat{U} = \{u_1, u_2, \dots, u_k\}$ and $\hat{V} = \{v_1, v_2, \dots, v_k\}$. We will first prove that $\hat{U}\rightarrow\hat{V}$. Since P is a shortest path, we have $u_i\rightarrow v_j$ for each $u_i \in \hat{U}$ and $v_j \in \hat{V}$ provided $j \notin \{i, i+1\}$. Thus, it sufficient to prove

$$u_i\rightarrow v_i \ (i = 1, 2, \dots, k) \text{ and } u_i\rightarrow v_{i+1} \ (i = 1, 2, \dots, k-1) \tag{1}$$

Consider the subdigraph H' of H induced by four vertices $v_i, u_i, v_{i+2}, u_{i+1}$, where $i \in \{1, 2, \dots, k-2\}$. By the definition of P (including the fact that P is a shortest path), we have $v_iu_i, v_{i+2}u_{i+1}, u_{i+1}v_i, u_iv_{i+2} \in A(H)$, but $v_iu_{i+1}, v_{i+2}u_i \notin A(H)$. Since H' is not isomorphic to either $\vec{C}_4$ or C_4', we have $u_iv_i, u_{i+1}v_{i+2} \in A(H)$. This proves that $u_i\rightarrow v_i$ provided $i = 1, 2, \dots, k-2$ and $u_i\rightarrow v_{i+1}$ provided $i = 2, 3, \dots, k-1$. Thus, to prove (1) it remains to show that

$$u_{k-1}\rightarrow v_{k-1}, \ u_k\rightarrow v_k, \ u_1\rightarrow v_2 \tag{2}$$

Consider the subdigraph H'' of H induced by four vertices $v_{k-1}, u_{k-1}, v_k, u_k$. By the definition of P, we have $v_{k-1}u_{k-1}, v_ku_k, v_ku_{k-1}, u_kv_{k-1} \in A(H)$, but $v_{k-1}u_k \notin A(H)$. We have proved that $u_{k-1}\rightarrow v_k$. Since H'' is not isomorphic to C_4' or C_4'', we have $u_{k-1}v_{k-1}, u_kv_k \in A(H)$.

Consider $H[\{v_2, v_3, u_1, u_2\}]$. By the definition of P, we have $v_2u_1, v_2u_2, v_3u_2, u_1v_3 \in A(H)$, but $v_3u_1 \notin A(H)$. We have proved that $u_2\rightarrow v_2$. Since $H[\{v_2, v_3, u_1, u_2\}]$ is not isomorphic to C_4' or C_4'', we have $u_1v_2, u_2v_3 \in A(H)$. This implies that (2) and, thus, (1) has been proved.

Consider vertex $u \in U - \hat{U}$; we will show that $u \rightarrow \{v_1, v_2, \dots, v_k\}$. Suppose this is not true. Let j be the smallest index such that $uv_j \notin A(H)$. We have $v_ju \in A(H)$. Suppose $j > 1$. Since $H[\{u, v_1, u_1, v_j\}]$ is not isomorphic to C_4' or C_4'', we have $v_ju_1, v_1u \in A(H)$. Since P is a shortest path, we have $j = 2$ as otherwise $v_1uv_ju_{j+1}\dots v_ju_k$ is shorter than P. We have $v_3u \notin A(H)$ as otherwise $v_1uv_3u_3\dots v_ku_k$ is shorter than P. However, C_4' is isomorphic to $H[\{v_2, u, v_3, u_3\}]$, a contradiction.

Now assume that $j = 1$. We have $v_3u \notin A(H)$ as otherwise we have a shorter path. Since H is semicomplete bipartite, we have $uv_3 \in A(H)$. However $H[\{v_1, u, v_3, u_2\}] \cong C_4'$, a contradiction.

Analogously we can prove that $\hat{U} \rightarrow v$ for every $v \in V - \hat{V}$. Consider $u \in U - \hat{U}$, $v \in V - \hat{V}$. We show that $uv \in A(H)$. Suppose this is not true. We have $vu \in A(H)$. Since $H[\{v, u, v_1, u_1\}]$ is not isomorphic to $\vec{C}_4$, C_4' or C_4'' we have $v_1u, vu_1 \in A(H)$.

Since $H[\{v,u,v_k,u_k\}]$ is not isomorphic to C_4' or C_4'' we have $v_ku, vu_k \in A(H)$. But now $\text{dist}(v_1,u_k)=3$ in $\text{UN}(H_1)$, a contradiction.

Case 2: $d_1 = 4$.

We will show that again $\text{UN}(H_2)$ is a complete bipartite graph. Assume that $v_1u_1'v_2'u_2'v_3'$ is a shortest path between a pair $v_1 \in V$ and $v_3' \in U$ in $\text{UN}(H_1)$. Let $U_1 = N^+(v_1)$, $V_2 = N^-(U_1) - \{v_1\}$, $U_2 = N^+(V_2) - U_1$ and $V_3 = N^-(U_2) - V_2$. By the definitions, $V = \{v_1\} \cup V_2 \cup V_3$ and $U = U_1 \cup U_2$. Observe also that $(U_1, V_3) = U_1 \times V_3$, $(V_3, U_1) = \emptyset$, $(U_2, \{v_1\}) = U_2 \times \{v_1\}$ and $(\{v_1\}, U_2) = \emptyset$. Let $u_1 \in U_1$, $u_2 \in U_2$, $v_2 \in V_2$ and $v_3 \in V_3$ be arbitrary. If $u_2v_3 \notin A(H)$, then $v_3u_2 \in A(H)$, and now $H[\{u_1,v_3,u_2,v_1\}]$ is either $\vec{C}_4$ or C_4', a contradiction. Therefore, we have $u_2v_3 \in A(H)$ and consequently $(U_2,V_3) = U_2 \times V_3$. Consider v_3, u_2 where $v_3u_2 \in A(H)$. Since $H[\{u_1,v_3,u_2,v_1\}]$ is not C_4' or $\vec{C}_4$, we conclude that $u_1v_1 \in A(H)$ and consequently $(U_1,\{v_1\}) = U_1 \times \{v_1\}$.

Note that we have already proved that the underlying graph of $H_2[U_1 \cup U_2 \cup V_3 \cup \{v_1\}]$ is a complete bipartite graph. Suppose $u_1v_2 \in A(H)$. Then $u_2v_2 \in A(H)$ as otherwise $H[\{u_1,v_1,u_2,v_2\}]$ is isomorphic to C_4' or C_4'', a contradiction. Therefore, if $u_1v_2 \in A(H)$, then $(U_2,\{v_2\}) = U_2 \times \{v_2\}$. Thus, to show that $(U_2,V_2) = U_2 \times V_2$ it suffices to prove that every vertex in V_2 has an in-neighbor in U_1. Suppose this is not true, and let X_2 be the set of all vertices in V_2 that does not have an in-neighbor in U_1. Let Y_2 be the set of all vertices in U_2 that have an arc into X_2. As H is strong some vertex in H must have an arc into X_2, which implies that $Y_2 \neq \emptyset$.

If there is an arc v_3y_2 from V_3 to Y_2, then let x_2 be an out-neighbor of y_2 in X_2 and let $u_1 \in U_1$ be arbitrary. However this is a contradiction to $H[\{v_3,y_2,x_2,u_1\}]$ not being isomorphic to C_4' and C_4'', which implies that there is no arc from V_3 to Y_2.

As H is strong and there is no arc from V_3 into U_1 or Y_2, there must be an arc, say v_3u_2, from V_3 into $U_2 - Y_2$. As H is strong there must also be an arc from $V(H) - X_2 - Y_2$ into $X_2 \cup Y_2$. By the above this arc, say v_2y_2, must be from $V_2 - X_2$ to Y_2. As y_2 belongs to Y_2 there must be a vertex, say $x_2 \in X_2$, such that $y_2x_2 \in A(H)$. As $v_2 \notin X_2$ we note that there is a vertex, say $u_1 \in U_1$, such that $u_1v_2 \in A(H)$. As $u_1x_2 \notin A(H)$ and $H[\{v_2,x_2,u_1,y_2\}]$ is not isomorphic to C_4' and C_4'', we note that $x_2y_2, v_2u_1 \in A(H)$. if $u_2v_2 \notin A(H)$, then $H[\{v_1,v_2,u_1,u_2\}]$ is isomorphic to C_4' and C_4'' (as $v_1u_2 \notin A(H)$), a contradiction. As $u_2v_2 \in A(H)$ and $H[\{v_2,v_3,u_2,y_2\}]$ is not isomorphic to C_4' and C_4'' we note that $y_2v_2 \in A(H)$ (as $v_3y_2 \notin A(H)$). As $H[\{v_2,x_2,u_2,y_2\}]$ is not isomorphic to C_4' and C_4'' we note that $v_2u_2 \in A(H)$ (as $u_2x_2 \notin A(H)$). However, the underlying graph of $H[v_1,u_1,v_2,v_3,u_2'',v_2'',u_2]^{\leftrightarrow}$ is now a bipartite claw, with edges $\{v_2u_1, v_2y_2, v_2u_2, u_1v_1, y_2x_2, u_2v_3\}$, a contradiction. Therefore $U_2 \rightarrow V_2$.

We show that $u_1v_2 \in A(H)$ for every $u_1 \in U_1$ and $v_2 \in V_2$. Suppose this is not true for some $v_2 \in V_2$ and $u_1 \in U_1$. Then we have $v_2u_1 \in A(H)$. Let $v_3 \in V_3$ be arbitrary and $u_2 \in U_2 \cap N^+(v_3)$. Then $H[\{u_1,v_3,u_2,v_2\}]$ is isomorphic to C_4' or C_4''. This completes our proof that $\text{UN}(H_2)$ is a complete bipartite graph.

Case 3: $d_1 = 3$.

Consider a Min-Max ordering π for H_1. Let $\pi(x) = \min\{\pi(x') : x' \in V\}$, $\pi(t) = \max\{\pi(t') : t' \in U\}$, $\pi(y) = \max\{\pi(y') : y' \in U, x{\to}y'\}$, and $\pi(z) = \min\{\pi(z') : z' \in V, z'{\to}t\}$.

Let $T = N^-(t)$. Since H is strong, every vertex of V has an out-neighbor. Since π is a Min-Max ordering, $z'{\to}t$ for each $z' \in V$ with $\pi(z') \geq \pi(z)$. Thus, $T = \{z' \in V : \pi(z') \geq \pi(z)\}$. Let $X = N^+(x)$. Since H is strong, every vertex of U has an in-neighbor. Since π is a Min-Max ordering, $x{\to}y'$ for each $y' \in U$ with $\pi(y') \leq \pi(y)$. Thus, $X = \{y' \in U : \pi(y') \leq \pi(y)\}$.

Since $\mathrm{dist}(x, z) = 2$ in $\mathrm{UN}(H_1)$, we have $z{\to}y'$ for some $y' \in N^+(x)$. Since π is a Min-Max ordering, $z{\to}y$ (consider the arcs zy' and xy). Now for every $z'' \in V$ with $\pi(z'') \leq \pi(z)$, we have $z''{\to}y$ (as π is a Min-Max ordering). Similarly, for every $y'' \in U$ with $\pi(y'') \geq \pi(y)$, we have $z{\to}y''$.

Let $\pi(w) = \min\{\pi(w') : w' \in U\}$. Notice that $\mathrm{dist}(w, t) = 2$ in $\mathrm{UN}(H_1)$. Thus, for some $z' \in T$, we have $z'{\to}w$. Hence, $z{\to}w$ and $z{\to}X$ (as π is a Min-Max ordering). We conclude that $z{\to}U$. Similarly, we can obtain that $V{\to}y$.

Let $Y = V - T$ and $Z = U - X$. Let $x' \in X$, $y' \in Y$. We have $z{\to}x'$ and $y'{\to}y$. Hence (as π is a Min-Max ordering), $y'{\to}x'$. Thus, $Y{\to}X$. Analogously, we can prove that $T{\to}Z$.

Let $x' \in X$, $t' \in T$, $y' \in Y$ and $x'{\to}t'$. Then $t't$, ty', $y'x' \in A(H)$. Since $H[\{x', t, t', y'\}]$ is not isomorphic to $\vec{C}_4$, C'_4 or C''_4, we have $x'{\to}y'$ and $t{\to}t'$. Thus, if $x'{\to}t'$, we have $x'{\to}Y$. Analogously, if $x'{\to}t'$, we have $Z{\to}t'$. Hence,

$$x'{\to}t' \text{ implies } x'{\to}Y \text{ and } Z{\to}t' \tag{3}$$

We will now prove the following: for a pair u, u' of distinct vertices of U we have $N^+(u) \subseteq N^+(u')$ or $N^+(u') \subseteq N^+(u)$. By Lemma 3.1, this implies that H has a 2-Min-Max ordering and we are done. Suppose that we have neither $N^+(u) \subseteq N^+(u')$ nor $N^+(u') \subseteq N^+(u)$. Thus, there is a pair v, v' of vertices in V such that $u{\to}v$, $u'{\to}v'$, but uv' and $u'v$ are not arcs in H. Since $H[\{u, u', v, v'\}]$ is not isomorphic to $\vec{C}_4$, C'_4 and C''_4, we have $v{\to}u$ and $v'{\to}u'$. Now consider four cases.

Case 3.1: $v, v' \in Y$. Let $t' \in T$. By the definition of t, $t \notin \{u, u'\}$. If $u'{\to}t'$, then $H[\{t, t', v, u'\}]$ is isomorphic to $\vec{C}_4$, C'_4 or C''_4, which is impossible. Thus, $u't' \notin A(H)$ and $t'{\to}u'$. Analogously, $ut' \notin A(H)$ and $t'{\to}u$. By the fact that $t'{\to}t$ and the existence and nonexistence of previously considered arcs, we conclude that $H[\{v, v', u, u', t, t'\}]$ is isomorphic to N_1 or N_2, which is impossible.

Case 3.2: $u, u' \in Z$. We can show that this case is impossible similarly to Case 3.1 but considering x, w instead of t, t'.

Case 3.3: $v \in Y$, $v' \in T$. By (3), $u' \in Z$. By Case 3.2, we may assume that $u \in X$. Then $H[\{v, v', u', t\}]$ is isomorphic to $\vec{C}_4$, C'_4 or C''_4, which is impossible.

Case 3.4: $v, v' \in T$. By Case 3.2, we may assume that $u \in X$. By (3), $Z \rightarrow v$ and, thus, $u' \in X$. By (3), we conclude that uxu, $u'xu'$, vtv and $v'tv'$ are 2-cycles. Notice that $t \rightarrow x$, but $xt \notin A(H)$. Now it follows that $H[\{x, v, v', u, u', t\}]^{\leftrightarrow}$ is isomorphic to C_6, a contradiction.

It follows from Cases 1,2 and 3 that if neither UN(H_1) nor UN(H_2) are complete bipartite graphs, then we must have $d_1 = d_2 = 3$. In this case we have shown that for every pair u, u' of distinct vertices of U we have $N^+(u) \subseteq N^+(u')$ or $N^+(u') \subseteq N^+(u)$. However, by swapping the roles of U and V we also get that for every pair v, v' of distinct vertices of V we have $N^+(v) \subseteq N^+(v')$ or $N^+(v') \subseteq N^+(v)$. ◇

The following theorem shows when MinHOM(H) is polynomial time solvable for the case when H is not strong, and does not contain $\vec{C}_4$ as an induced subdigraph.

Theorem 3.3 *Let $H = (V, U; A)$ be a semicomplete bipartite digraph with strong components $C_1, C_2, \ldots, C_p$ $(p \geq 2)$ satisfying the following:*

- *There is no arc from C_i to C_j for $i > j$,*
- *H does not contain an induced subdigraph belonging to $\mathcal{HFORB}$ or an induced directed 4-cycle.*

Then H has a 2-Min-Max ordering and MinHOM(H) is polynomial time solvable.

Proof: Suppose there are $v, v' \in V$ and $u, u' \in U$ such that $A(H^{\rightarrow}[\{v, v', u, u'\}]) = \{vu, v'u'\}$. Note that $\{v, v', u, u'\}$ belong to a strong component of H as they are contained in a 4-cycle. Let $\{v, v', u, u'\} \subseteq V(C_t)$ for some t and let $V_1 = \{x \in V \mid x \in C_i \ , \ i < t\}$, $U_1 = \{y \in U \mid y \in C_i \ , \ i < t\}$, $V_2 = V \cap C_t$, $U_2 = U \cap C_t$, $V_3 = V - V_1 - V_2$ and $U_3 = U - U_1 - U_2$.

If there is a $w' \in U_3$ and $w \in V_3$, such that $w' \rightarrow w$, then $H[\{u, v, u', v', w, w'\}]$ is the converse of either N_1 or N_2. Therefore we must have $A(H[U_3 \cup V_3]) = V_3 \times U_3$. Analogously we must have $A(H[U_1 \cup V_1]) = V_1 \times U_1$. Consider $H' = H[U_2 \cup V_2]$ and note that H' is strong and does not contain a digraph from $\mathcal{HFORB}$ or $\vec{C}_4$ as an induced subdigraph. Therefore Theorem 3.2 implies that $U_2 \rightarrow V_2$ (as $V_2 \rightarrow U_2$ is not true), which furthermore implies that $(U_1 \cup U_2) \rightarrow (V_2 \cup V_3)$.

Let π be a Min-Max ordering of $H^{\rightarrow}$. Let $uv \in A(H)$ and $u'v' \in A(H)$ be two distinct arcs from U to V. As $d^+(u) > 0$ we note that $u \in U_1 \cup U_2$, by the above. Analogously we note that $u' \in U_1 \cup U_2$, $v \in V_2 \cup V_3$ (as $d^-(v) > 0$) and $v' \in V_2 \cup V_3$. By the above we therefore have $uv', u'v \in A(H)$. As u, u', v, v' were chosen arbitrarily, this implies that π is a 2-Min-Max ordering.

If there are no $v, v' \in V$ and $u, u' \in U$ such that $A(H^{\rightarrow}[\{v, v', u, u'\}]) = \{vu, v'u'\}$, then $H^{\rightarrow}$ satisfies the condition of Lemma 3.1. Therefore H has a 2-Min-Max ordering.$\diamond$

The following theorem shows when MinHOM(H) is polynomial time solvable for the case when H does contain $\vec{C}_4$ as an induced subdigraph.

Theorem 3.4 *Let H be a semicomplete bipartite digraph. Assume that H does not contain a digraph from $\mathcal{HFORB}$ as an induced subdigraph, but contains $\vec{C}_4$ as an induced subdigraph. Then H is an extension of a $\vec{C}_4$ and MinHOM(H) is polynomial time solvable.*

Proof: Observe that an extension L of any cycle $\vec{C}_p$, $p \geq 2$, has a p-Min-Max ordering. Thus, MinHOM(L) is polynomial time solvable by Theorem 2.2. Let $C = v_1u_1v_2u_2v_1$ be an induced 4-cycle of H. It suffices to prove that H is an extension of C.

For $i = 1, 2$, let $M^+(v_i) = \{u \in U : v_iu \in A(H),\ uv_i \notin A(H)\}$, $M^-(v_i) = \{u \in U : uv_i \in A(H),\ v_iu \notin A(H)\}$, and $M(v_i) = \{u \in U : uv_i,\ v_iu \in A(H)\}$. We have $M^+(v_1) \cap M^+(v_2) = \emptyset$ as otherwise $H[\{v_1, v_2, u_1, u_2, u_3\}] \cong H^*$, where $u_3 \in N^+(v_1) \cap N^+(v_2)$. We have $M^-(v_1) \cap M^-(v_2) = \emptyset$ as otherwise $H[\{v_1, v_2, u_1, u_2, u_3\}] \cong H^{**}$, where $u_3 \in N^-(v_1) \cap N^-(v_2)$ and H^{**} is the converse of H^*.

We have $M(v_1) \cap (M^+(v_2) \cup M^-(v_2)) \neq \emptyset$ as otherwise $H[\{v_1, u_1, v_2, u\}] \cong C_4'$, where $u \in M(v_1) \cap M^+(v_2)$ or $H[\{v_1, u_2, v_2, u\}] \cong C_4'$, where $u \in M(v_1) \cap M^-(v_2)$. Moreover, $M(v_1) \cap M(v_2) = \emptyset$ as otherwise $H[\{v_1, u_1, v_2, u\}] \cong C_4''$, where $u \in M(v_1) \cap M(v_2)$. The arguments above imply that $M(v_1) = M(v_2) = \emptyset$, and $M^+(v_1) = M^-(v_2)$ and $M^-(v_1) = M^+(v_2)$.

Similarly, we can define $M+(u_i)$, $M^-(u_i)$ and $M(u_i)$, $i = 1, 2$, and prove the relations analogous to those for $M^+(v_i)$, $M^-(v_i)$ and $M(v_i)$, $i = 1, 2$.

Let $v \in V - \{v_1, v_2\}$ and $u \in U - \{u_1, u_2\}$ be arbitrary. Without loss of generality, assume that $u \in M^+(v_1) = M^-(v_2)$ and $v \in M^+(u_2) = M^-(u_1)$ (all other cases can be treated similarly). To show that H is an extension of C, it suffices to prove that $v{\rightarrow}u$, but $uv \notin A(H)$. Suppose first that $u{\rightarrow}v$ and $v{\rightarrow}u$. Then $H[\{v, u, v_2, u_2\}] \cong C_4'$, a contradiction. Now suppose that $u{\rightarrow}v$, but $vu \notin A(H)$. Then $H[\{v, v_1, v_2, u, u_2\}] \cong H^*$, a contradiction. Thus, $v{\rightarrow}u$, but $uv \notin A(H)$ and we are done. $\diamond$

The three theorems of this section and the fact that $\vec{C}_4$ has a 4-Min-Max ordering imply the following:

Corollary 3.5 *Let H be a semicomplete bipartite digraph not containing a digraph from*

$\mathcal{HFORB}$ as an induced subdigraph. Then MinHOM(H) is polynomial time solvable and H has a k-Min-Max ordering for $k = 2$ or 4.

4 NP-hardness Cases

It is well known that the problem of finding a maximum size independent set in an undirected graph G is NP-hard. We say that a set I in a digraph D is *independent* if no vertices in I are adjacent. Clearly, the problem of finding a maximum size independent set in a digraph D (MISD) is NP-hard.

Lemma 4.1 *MinHOM(C'_4) is NP-hard.*

Proof: Let H be isomorphic to C'_4 as follows: $V(H) = \{1, 2, 3, 4\}$ and $A(H) = \{12, 21, 23, 34, 41\}$. Let D be an arbitrary digraph. We replace every arc uv of D by the digraph G_{uv} with $V(G_{uv}) = \{x, y, z, u, v\}$ and $A(G_{uv}) = \{ux, xy, yz, vz\}$. Let D' be the obtained digraph. Define the cost function as follows: $c_1(u) = 1$, $c_2(u) = 1$, $c_3(u) = 0$, $c_4(u) = 1$, $c_1(v) = 1$, $c_2(v) = 1$, $c_3(v) = 0$, $c_4(v) = 1$, $c_i(t) = 0$ when $t \notin \{u, v\}$ and $i \in \{1, 2, 3, 4\}$.

Let h be a mapping from $V(G_{uv})$ to $V(H)$, and let uv be an arc in D. If $h(u) = h(y) = h(v) = 1$ and $h(x) = h(z) = 2$, then h is a homomorphism. Thus, there is a homomorphism of D' to H, which maps all vertices of D into 1 and the vertices of D' not in D into 1 or 2.

Now let f be a homomorphism of G_{uv} to H. Observe that if $f(u) = 1$, then $f(x) = 2$, $f(y)$ is either 1 or 3, $f(z)$ is either 2 or 4, $f(v)$ is either 1 or 3. Similarly if $f(u) = 3$, then $f(v) = 1$; if $f(u) = 4$, then $f(v)$ is either 2 or 4; if $f(u) = 2$, then $f(v)$ is either 2 or 4.

Let g be a minimum cost homomorphism of D' to H and let $S = \{s \in V(D) : g(s) = 3\}$. Notice that the cost of g is $|V(D)| - |S|$ and S is an independent set by the arguments of the previous paragraph. Thus, S is an independent set of D of maximum size.

Let I be a maximum size independent set in D. The above arguments show that there is a homomorphism d of D' to H such that $d(t) = 3$ if $t \in I$ and $d(t) = 1$ if $t \in V(D) - I$. Notice that d is a minimum cost homomorphism.

Now we can conclude that MinHOM(H) is NP-hard since MISD is NP-hard. ◇

Lemma 4.2 *MinHOM(C''_4) is NP-hard.*

Proof: Let H be isomorphic to C''_4 as follows: $V(H) = \{1, 2, 3, 4\}$, $A(H) = \{12, 21, 23, 32, 34, 41\}$. Let D be an arbitrary digraph. We replace every arc uv of D by the digraph G_{uv} with $V(G_{uv}) = \{x, u, v\}$ and $A(G_{uv}) = \{ux, xv\}$. Let D' be the obtained digraph. Define the cost function as follows: $c_1(u) = 1$, $c_2(u) = 1$, $c_3(u) = 1$, $c_4(u) = 0$, $c_1(v) = 1$, $c_2(v) = 1$, $c_3(v) = 1$, $c_4(v) = 0$, $c_i(x) = 0$ for $i = 1, 2, 3, 4$.

Let h be a mapping from $V(G_{uv})$ to $V(H)$, and let uv be an arc in D. If $h(u) = h(v) = 1$ and $h(x) = 2$, then h is a homomorphism. Thus, there is a homomorphism of D' to H, which maps all vertices of D into 1 and the vertices of D' not in D into 2.

Now let f be a homomorphism of G_{uv} to H. Observe that if $f(u) = 1$, then $f(x) = 2$ and $f(v)$ is either 1 or 3. Similarly if $f(u) = 2$, then $f(v) \in \{2, 4\}$; if $f(u) = 3$, then $f(v) \in \{1, 3\}$; if $f(u) = 4$, then $f(v) = 2$.

Let g be a minimum cost homomorphism of D' to H and let $S = \{s \in V(D) : g(s) = 4\}$. Notice that the cost of g is $|V(D)| - |S|$ and S is an independent set by the arguments of the previous paragraph. Thus, S is an independent set of D of maximum size.

The rest of the proof is similar to that of Lemma 4.1. $\diamond$

The following lemma was stated in [5]. We give a proof here for the sake of completeness.

Lemma 4.3 *MinHOM(H^*) is NP-hard.*

Proof: Let H be isomorphic to H^* as follows: $V(H) = \{1, 2, 3, 4, 5\}$, $A(H) = \{12, 23, 34, 41, 15, 35\}$. We replace every arc uv of D by the digraph G_{uv} with $V(G_{uv}) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$, where $u = v_6$ and $v = v_7$, and $A(G_{uv}) = \{v_1v_2, v_2v_3, v_3v_4, v_4v_1, v_5v_6, v_5v_7, v_1v_6, v_3v_7\}$. Let D' be the obtained digraph. Define the cost function as follows: $c_1(v_6) = 1$, $c_2(v_6) = 1$, $c_3(v_6) = 1$, $c_4(v_6) = 0$, $c_5(v_6) = 1$, $c_1(v_7) = 1$, $c_2(v_7) = 1$, $c_3(v_7) = 1$, $c_4(v_7) = 0$, $c_5(v_7) = 1$, and $c_i(v_j) = 0$ for each $j \neq 6, 7$.

Let h be a mapping from $V(G_{uv})$ to $V(H)$, and let uv be an arc in D. If $h(v_i) = i$ for each $i = 1, 2, 3, 4$, $h(v_5) = 1$ and $h(u) = h(v) = 5$, then h is a homomorphism. Thus, there is a homomorphism of D' to H, which maps all vertices of D into 5.

Now let f be a homomorphism of G_{uv} to H. Observe that if $x = f(u) = f(v)$, then $x = 5$. Also, if $f(u) = 5$ then $f(v) \in \{2, 4, 5\}$ and if $f(v) = 5$ then $f(u) \in \{2, 4, 5\}$.

Let g be a minimum cost homomorphism of D' to H and let $S = \{s \in V(D) : g(s) = 4\}$. Notice that the cost of g is $|V(D)| - |S|$ and S is an independent set by the arguments of the previous paragraph. Thus, S is an independent set of D of maximum size.

The rest of the proof is similar to that of Lemma 4.1. $\diamond$

Lemma 4.4 *MinHOM(N_1) is NP-hard.*

Proof: We shall reduce the maximum independent set problem to $MinHOM(N_1)$. Let H be the following digraph isomorphic to N_1: $V(H) = \{1, 2, 3, 4, 5, 6\}$,

$$A(H) = \{12, 21, 34, 43, 56, 65, 23, 25, 14, 16, 54, 36\}.$$

Let D be an arbitrary digraph. We replace every arc uv of D with the digraph G_{uv} with $V(G_{uv}) = \{x, y, z, u, v\}$ and $A(G_{uv}) = \{ux, vz, xy, yz\}$. Consider the following cost function: $c_2(u) = c_2(v) = 1$, $c_6(u) = c_6(v) = 0$, $c_i(u) = c_i(v) = 2M + 1$ for $i \neq 2, 6$ and $c_6(y) = 2M+1$, where $M = |V(D)|$. In all remaining cases the cost is zero. Let D' be the obtained digraph, let f be a mapping from $V(D')$ to $V(H)$, and let uv be an arc in D.

Assume that $f(u) = f(v) = 2$. Then with $f(x) = f(z) = 1$ and $f(y) = 2$, we obtain a homomorphism from G_{uv} to H of cost 2. This implies there is a homomorphism of D' to H of cost $2M < 2M + 1$, and, thus, every vertex of D in D' must be colored either 2 or 6 in any minimum cost homomorphism of D' to H. Let f be a homomorphism of D' to H and let us consider the remaining options for coloring the vertices of D in D'.

Assume that $f(v) = 6$ and $f(u) = 2$. Then with $f(z) = 5$, $f(y) = 2$ and $f(x) = 1$, we obtain a homomorphism from G_{uv} to H of cost 1. Assume that $f(v) = 2$ and $f(u) = 6$. Then with $f(x) = 5$, $f(y) = 4$ and $f(z) = 3$, we obtain a homomorphism from G_{uv} to H of cost 1. Note that if $f(u) = f(v) = 6$, then $f(x) = f(z) = 5$ and $f(y) = 6$. Then the cost of f will be at least $2M + 1$ implying we cannot color both vertices u and v in color 6 in any minimum cost homomorphism of D' to H.

Now let f be a minimum cost homomorphism, let S be the vertices of D in D' colored 6 and $T = V(D) - S$. Recall that the vertices of T are colored 2. Notice that S is an independent set and the cost of f equals $|T|$.

The rest of the proof is similar to that of Lemma 4.1. $\diamond$

Lemma 4.5 *MinHOM(N_2) is NP-hard.*

We shall reduce the maximum independent set problem to MinHOM(N_2). Let H be the following digraph isomorphic to N_2 : $V(H) = \{1, 2, 3, 4, 5, 6\}$,

$$A(H) = \{12, 34, 43, 56, 65, 23, 25, 14, 16, 54, 36\}.$$

Let D be an arbitrary digraph. We replace every arc uv of D by the digraph G_{uv} with $V(G_{uv}) = \{x, y, z, u, v\}$ and $A(G_{uv}) = \{ux, vz, xy, zy\}$. We introduce the following cost function: $c_1(u) = c_1(v) = 1$, $c_5(u) = c_5(v) = 0$, $c_i(u) = c_i(v) = 2M + 1$ for $i \neq 1, 5$ and $c_4(x) = 2M + 1$ and $c_6(z) = 2M + 1$, where $M = |V(D)|$. In any other cases the cost is zero. Let D' be the obtained digraph, let f be a mapping from $V(D')$ to $V(H)$, and let uv be an arc in D.

Assume that $f(u) = f(v) = 1$. With $f(x) = f(z) = 2$ and $f(y) = 3$ with obtain a homomorphism from H' to H with cost 2. Thus, there is a homomorphism of D' to H of cost at most $2M$ (assign all vertices of D in D' color 2) and no vertex of D in D' must not be assigned any color other than 1 and 5. Let f be a homomorphism of D' to H and let us consider the remaining options for coloring the vertices of D in D'.

Assume that $f(v) = 1$ and $f(u) = 5$. With $f(z) = 2$, $f(x) = 6$ and $f(y) = 5$, we obtain a homomorphism of G_{uv} to H of cost 1. Assume that $f(v) = 5$ and $f(u) = 1$. With $f(x) = 2$, $f(z) = 4$ and $f(y) = 3$, we obtain a homomorphism of G_{uv} to H of cost 1. Note that if $f(u) = f(v) = 5$, then $f(x) \in \{4, 6\}$ and $f(z) \in \{4, 6\}$. Thus, f has cost at least $2M + 1$ implying that a minimum cost homomorphism of D' to H does not assign adjacent vertices of D color 5 (in D').

Now let f be a minimum cost homomorphism, let S be the vertices of D in D' colored 5 and $T = V(D) - S$. Recall that the vertices of T are colored 1. Notice that S is an independent set and the cost of f equals $|T|$.

The rest of the proof is similar to that of Lemma 4.1. $\diamond$

Corollary 4.6 *MinHOM(H) is NP-hard for every* $H \in \mathcal{HFORB}$.

Proof: If H is isomorphic to C'_4, C''_4, H^*, N_1 or N_2 or the converse of one of the five digraphs, then MinHOM(H) is NP-hard due to the lemmas of this section and the simple fact that if MinHOM(H) is NP-hard and H' is the converse of H then MinHOM(H') is NP-hard as well.

Let $\mathcal{B}$ be the set consisting of the following bipartite graphs: bipartite claw, bipartite net, bipartite tent and every even cycle with at least 6 vertices. If UN(H^s), where $s \in \{\rightarrow, \leftarrow\}$, is isomorphic to a graph in $\mathcal{B}$, then MinHOM(H) is NP-hard due to Theorem 1.2 and the transformation from a bipartite undirected graph to a semicomplete bipartite digraph described in the last paragraph of subsection 'Minimum Cost Homomorphisms' of Section 1. If UN($H^{\leftrightarrow}$) is isomorphic to a graph in $\mathcal{B}$, then MinHOM(H) is NP-hard as, for each bipartite undirected graph L, MinHOM(L) is equivalent to MinHOM(L^+), where L^+ is the digraph obtained from L by replacing every edge xy with two arcs xy and yx.$\diamond$

5 Dichotomy for semicomplete multipartite digraphs

A digraph D is called *semicomplete k-partite* if D can be obtained from a complete k-partite (undirected) graph G by replacing every edge xy of G by either the arc xy or the arc yx or the pair xy, yx of arcs. Let TT_p denote the acyclic tournament on $p \geq 1$ vertices. Let $p \geq 3$ and let TT_p^- be a digraph obtained from TT_p by deleting the arc from the vertex of in-degree zero to the vertex of out-degree zero. Combining the main result of this paper with the main result of [5], we obtain the following:

Theorem 5.1 *Let H be a semicomplete k-partite digraph. If $k = 2$ and H does not* *contain a digraph from $\mathcal{HFORB}$ as an induced subdigraph or if $k \geq 3$ and H is an extension of either TT_k or TT^-_{k+1} or $\vec{C}_3$, then MinHOM(H) is polynomial time solvable. Otherwise, MinHOM(H) is NP-hard.*

6 Further Research

In the case of undirected graphs H, the well-known theorem of Hell and Nešetřil [9] on the homomorphism problem implies that MinHOM(H) is NP-hard for each non-bipartite graph H. The authors of [3] obtained a complete dichotomy of the computational complexity of MinHOM(H) when H is undirected. The dichotomy obtained in this paper significantly extends the dichotomy of [3]. This indicates that the problem of obtaining a dichotomy for the computational complexity of MinHOM(H) when H is a bipartite digraph is a very difficult problem. Note that MinHOM(H) is polynomial-time solvable for some non-bipartite digraphs, for example, for acyclic tournaments [5]. Thus, a dichotomy for bipartite directed case does not coincide with a dichotomy for the general directed case. The problem of obtaining dichotomy for both cases is a very interesting open problem.

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