Fixed-Parameter Complexity of Minimum Profile Problems*

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Abstract

The profile of a graph is an integer-valued parameter defined via vertex orderings; it is known that the profile of a graph equals the smallest number of edges of an interval supergraph. Since computing the profile of a graph is an NP-hard problem, we consider parameterized versions of the problem. Namely, we study the problem of deciding whether the profile of a connected graph of order n is at most $n-1+k$, considering k as the parameter; this is a parameterization above guaranteed value, since $n-1$ is a tight lower bound for the profile. We present two fixed-parameter algorithms for this problem. The first algorithm is based on a forbidden subgraph characterization of interval graphs. The second algorithm is based on two simple kernelization rules which allow us to produce a kernel with linear number of vertices and edges. For showing the correctness of the second algorithm we need to establish structural properties of graphs with small profile which are of independent interest.

1 Introduction

The *profile* is an integer-valued graph parameter defined via vertex orderings: the profile of an ordering $\alpha : V \to \{1, \ldots, |V|\}$ of a graph $G = (V, E)$ is defined as

$$\mathrm{prf}_\alpha(G) = \sum_{v \in V} \alpha(v) - \min\{\, \alpha(w) : w \in N[v] \,\}$$

where $N[v] = \{u \in V :\ uv \in E\} \cup \{v\}$, the closed neighborhood of v; the profile of G is the smallest profile of all orderings α of G.

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Areas of application of the profile and equivalent parameters include computational biology [4, 10], archaeology [15] and clone fingerprinting [14]. Fomin and Golovach [8] established the equivalence of the profile and other parameters including one that is important in graph searching.

It is well known that computing the profile of a given graph is NP-hard [6, 17]. In fact, via the following relationship to interval graphs, the NP-hardness follows from earlier results [9]: It is known from a result of Billionnet [3] that the profile of a graph G equals the smallest number of edges of an interval supergraph of G. In view of this NP-hardness, it makes sense to study the problem in the framework of parameterized complexity. We recall some basic notions of parameterized complexity here, for a more in-depth treatment of the topic we refer the reader to [2, 7, 18].

A parameterized problem Π can be considered as a set of pairs (I, k) where I is the *problem instance* and k (usually an integer) is the *parameter*. Π is called *fixed-parameter tractable (FPT)* if membership of (I, k) in Π can be decided in time $O(f(k)|I|^c)$, where $|I|$ is the size of I, $f(k)$ is a computable function, and c is a constant independent from k and I. Let Π and Π' be parameterized problems with parameters k and k', respectively. An *fpt-reduction* R *from* Π *to* Π' is a many-to-one transformation from Π to Π', such that (i) $(I, k) \in \Pi$ if and only if $(I', k') \in \Pi'$ with $|k'| \le g(k)$ for a fixed computable function g and (ii) R is of complexity $O(f(k)|I|^c)$. A *reduction to problem kernel* (or *kernelization*) is a polynomial time fpt-reduction R from a parameterized problem Π to itself such that $|I'| \le h(k)$ for a fixed computable function h. In kernelization, an instance (I, k) is reduced to another instance (I', k'), which is called the *problem kernel*; $|I'|$ is the *size* of the kernel.

It is easy to see that a decidable parameterized problem is FPT if and only if it admits a kernelization (see, e.g., [11, 18]); however, the problem kernels obtained by this general result have impractically large size. Therefore, one tries to develop kernelizations that yield problem kernels of smaller size. The survey of Guo and Niedermeier [11] on kernelization lists some problem for which linear size kernels (the size here is the number of vertices), polynomial size kernels and exponential size kernels were obtained. For many parameterized problems, optimal size kernels have likely not been obtained yet; for example, Guo and Niedermeier [11] ask whether the feedback vertex set problem admits a linear size kernel.

1.1 New results and algorithms

What is a suitable parameter for the profile problem? If we take as parameter an upper bound on the profile, then we have a trivially fixed-parameter tractable problem. It is known that the profile of a connected graph G of order n is at least $n - 1$; i.e., $n - 1$ is a "guaranteed value" for the profile of G. Hence it makes sense to study the following parameterized problem.

Profile Above Vertex Guaranteed Value (PAVGV)
Instance: A connected graph $G = (V, E)$.

Parameter: A positive integer k.
Question: Is the profile of G at most $|V| - 1 + k$?

In Section 2, we prove that PAVGV is FPT. Our algorithm relies on the link between profile and interval graphs and a forbidden subgraph characterization for interval graphs: Lekkerkerker and Boland [16] have shown that a graph is interval if and only if it does not contain certain graphs as induced subgraphs (see Section 2 for details). There is an infinite number of possible forbidden subgraphs; therefore Cai's general result [5] for graph completion problems is not directly applicable. However, using the assumption $\mathrm{prf}(G) \le |V| - 1 + k$ we can limit the possible forbidden subgraphs to a finite number for any fixed k, and thus state a bounded search-tree algorithm that renders PAVGV fixed-parameter tractable. This algorithm is only of theoretical value because of its large branching factor; moreover, the algorithm does not imply a kernel even of moderate exponential size.

We therefore develop in Section 4 a second algorithm based on two simple kernelization rules. The first rule combines certain vertices of degree one to a single vertex. The second rule is based on the observation that if a vertex is incident with two bridges and the number of vertices on both sides of the bridges is sufficiently large, then we can suppress the vertex. The second algorithm gives us a *linear size kernel*; the algorithm is very simple and easy to implement. However, for showing its correctness we need to establish new and nontrivial structural results for graphs with small profile. These structural results, the technically most involved parts of this paper, are established in Section 3. These results are of independent interest; one such example is Theorem 3 which provides us with a tight lower bound on the profile of a 2-edge-connected graph G in terms of the order of G. Applying the first algorithm to the kernel obtained by the second algorithm, we obtain an algorithm of running time $O(n^2 + k^2 3^k (k+1)!)$.

1.2 More general parameterizations for the profile problem

From the above mentioned relationship between the profile and interval supergraphs, it follows that the profile of a graph is always at least the number of edges of the graph. Hence, one can consider the following parameterized problem:

Profile Above Guaranteed Value (PAGV)
Instance: A graph $G = (V, E)$.
Parameter: A positive integer k.
Question: Is the profile of G at most $|E| + k$?

Since $|E| \ge |V| + 1$ holds for connected graphs $G = (V, E)$, fixed-parameter tractability of PAGV implies fixed-parameter tractability of PAVGV. In fact, fixed-parameter tractability of PAGV was very recently proved by Heggernes et al. [13]. The algorithm of [13] is a bounded search-tree algorithm combined with a greedy completion algorithm. The complexity of the algorithm in [13] is $O(k^{2k}|V|^3|E|)$ and no moderate exponential size (let alone polynomial size) kernel is obtained. It would be interesting to find out whether PAGV admits a kernel of polynomial size or even a linear-size kernel.

In the final section we consider a different generalization of PAVGV.

> **Vertex Average Profile** (VAP)
> *Instance:* A graph $G = (V, E)$.
> *Parameter:* A positive integer k.
> *Question:* Is the profile of G at most $k|V|$?

This problem was introduced by Serna and Thilikos [19] who asked whether it is fixed-parameter tractable. We answer this question negatively: we show that for every constant $k \geq 2$ it is NP-complete to decide whether $\mathrm{prf}(G) \leq k|V(G)|$ for a given graph G.

2 Algorithm Based on Forbidden Subgraphs

In their seminal paper [16], Lekkerkerker and Boland proved that a graph is interval if and only if it does not contain any of the following graphs as induced subgraphs (see Figure 1 for illustrations).

1. C_i $(i > 3)$, the cycle of length i.
2. G' with $V(G') = \{x, 1, 2, 3, 4, 5, 6\}$ and $E(G') = \{x1, x2, x3, 14, 25, 36\}$.
3. G'' with $V(G'') = \{x, y, 1, 2, 3, 4, 5\}$ and $E(G'') = \{x1, x2, x3, x4, x5, y3, 12, 23, 34, 45\}$.
4. R_i $(i > 1)$, with $V(R_i) = \{x, x', y, z, 1, 2, 3, \dots, i\}$ and $E(R_i) = \{x'x, x1, x2, \dots, xi, y1, 12, 23, 34, \dots, (i-1)i, zi\}$.
5. Q_i $(i > 2)$ with $V(Q_i) = \{x, y, z, w, v, 1, 2, 3, \dots, i\}$ and $E(Q_i) = \{x1, x2, \dots, xi, y1, y2, \dots, yi, zx, zy, xy, wx, w1, 12, 23, 34, \dots, (i-1)i, vy, vi\}$.

Let G be a given graph with n vertices and m edges. It is well known that one can decide in time $O(n+m)$ whether G is an interval graph [1]. Furthermore, if G is not interval, we can find in time $O(n^2+nm)$ one of the above graphs as an induced subgraph as follows (Cai [5] describes this procedure in a more general setting): We consider $G-v$ for each $v \in V$. If $G-v$ is not interval for some v, then we consider $G-v$ instead of G (and $G-v$ must contain one of the above graphs as an induced subgraph). Otherwise, if $G-v$ is interval for all $v \in V$, then G is already one of the above graphs (and no proper induced subgraph of G contains one of the above graphs as an induced subgraph).

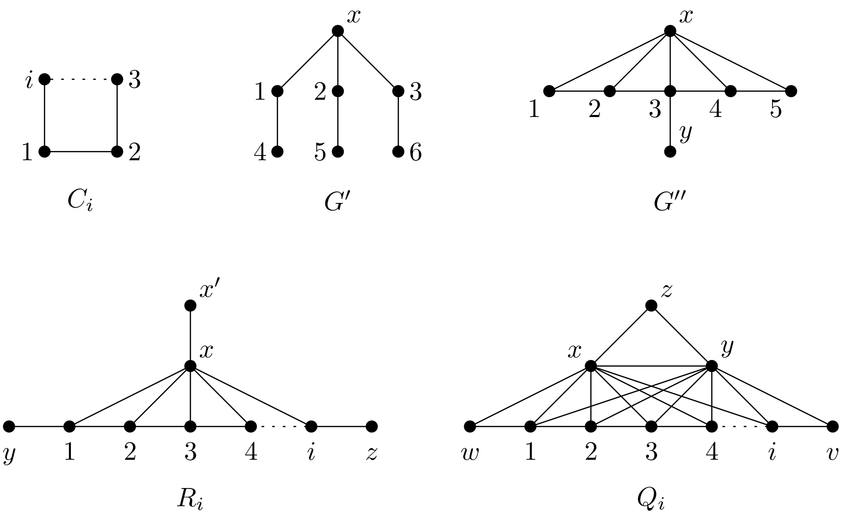

Figure 1: Forbidden induced subgraphs

Theorem 1. *Given a connected graph G with n vertices and m edges and a non-negative integer k, we can decide in time $O(nm3^k(k+1)!)$ whether $\mathrm{prf}(G) \leq n-1+k$. Hence PAVGV is fixed-parameter tractable.*

Proof. Let $G = (V, E)$ be a connected graph and let $n = |V|, m = |E|$ and $q = m - n + 2$. Clearly, $q \geq 1$. If G has an interval supergraph with at most $n - 1 + k$ edges, then $m \leq n - 1 + k$ and $q \leq k + 1$. Thus, if $q > k + 1$ we can reject the given graph G. We consider now the case where $q \leq k + 1$.

Let $\mathcal{A}$ denote the $O(n^2 + nm)$-time algorithm outlined just before Theorem 1. Notice that $O(n^2 + nm) = O(nm)$ since G is connected. Let $T(q)$ denote the running time of the following search tree algorithm. We will prove (along with describing the algorithm) by induction on q that $T(q) = O(nm \prod_{j=q}^{k+1} a(j))$, where $a(j) = 3j$ if $j > 5$ and otherwise $a(j) = 15$.

The root node of our search tree $\mathcal{T}$ is G. Consider an arbitrary node F of $\mathcal{T}$. We describe how to get all children of F. We apply the algorithm $\mathcal{A}$ either to decide that F is interval or to find one of the above graphs H as an induced subgraph. Consider the case that F is not interval and, for simplicity of notation, assume that $F = G$.

If $H = G'$ or $H = G''$, then we have to add one edge to the subgraph. There are only $21 - 6$ and $21 - 10$ possibilities, respectively. So in both cases we only have to try the at most 15 possibilities, and we add one edge. So by the induction hypothesis the bound on $T(q)$ holds.

If $H = C_i$, we add a chord of length two (i.e., a chord that lies on a 3-cycle containing two edges from the cycle). There are i ways of doing this. Now we have an induced C_{i-1}, and we again add a chord of length two (in C_{i-1}). There are $i - 1$ possibilities. Continuing this we get $i(i-1)(i-2)\cdots 7 \cdot 6 \cdot 5 \cdot 2$ possibilities of adding the $i - 3$ edges (there are only two options for a 4-cycle).

We observe that the bound on $T(q)$ still holds by the induction hypothesis.

Let $H = R_i$. Note that $|V(H)| = i + 4$ and $|E(H)| = 2i + 2$ and, thus, $|E(H)| - |V(H)| \geq i - 2$. Note that G can be built from H by adding, one by one, vertices from $V \setminus V(H)$ with edges to the already constructed induced subgraph of G. We append at least one edges for every added vertex. Thus, we have $m - n \geq i - 2$ and $q \geq i$. In order to make G an interval graph we will have to add at least one edge from $\{x', y, z\}$ to $\{1, 2, \ldots, i\}$ (but not $y1$ or zi), or from x to $\{y, z\}$. So we have $3i$ possibilities. But $a(q) \geq a(i) \geq 3i$, and hence we are done by induction.

The case $H = Q_i$ is treated similarly to the case $H = R_i$.

It remains to observe that $T(q) \leq T(1) = O(nm3^{k+1}(k + 1)!)$ for each $q \geq 1$. □

This proof implies the following simple algorithm, where the procedure $\mathcal{A}(F)$ outputs $\emptyset$ if F is an interval graph or one of the forbidden induced subgraphs of F, otherwise. For a graph H, H^* denotes the complement of H if $H = G'$ or $H = G''$. If H is a cycle $x_0x_1 \ldots x_{p-1}x_0$, then $V(H^*) = V(H)$ and $E(H^*) = \{x_ix_{i+2 \mod p} : 0 \leq i \leq p - 1\}$. If $H = R_i$, then $V(H^*) = V(H)$ and H^* only contains all the edges from $\{x', y, z\}$ to $\{1, 2, 3, \ldots, i\}$ except $y1$ and zi and H^* also contains the edges xy and xz (see the proof above). If $H = Q_i$, then $V(H^*) = V(H)$ and H^* contains all the edges from $\{z, w, v\}$ to $\{1, 2, 3, \ldots, i\}$ except $w1$ and vi and H^* also contains the edges $\{1i, xv, yw\}$.

Algorithm **add-edge**(G, k)

Input: connected graph G and integer $k \geq 0$.

Output: 'yes' if $\mathrm{prf}(G) \leq |V(G)| + k - 1$ and 'no' otherwise.

1. if $(|E(G)| - |V(G)| + 1 > k)$ output 'no';
2. $H := \mathcal{A}(G)$;
3. if $(H = \emptyset)$ output 'yes';
4. for each $e \in E(H^*)$ **add-edge**$(G + e, k - 1)$;

3 Structural Properties of Graphs with Small Profile

For this and the following sections we need additional definitions related to profiles. Let $G = (V, E)$ be a graph. An *ordering* of G is a one-to-one mapping $\alpha : V \to \{1, 2, \ldots, |V|\}$. We denote the set of orderings of G by $\mathrm{OR}(G)$. For a vertex v in G, its *neighborhood* is $N(v) = \{u \in V : uv \in E\}$ and its *closed neighborhood* is $N[v] = N(v) \cup \{v\}$. The *profile of a vertex* z of G in an ordering α of G is

$$\mathrm{prf}_\alpha(G, z) = \alpha(z) - \min\{\, \alpha(w) : w \in N[z]\,\}.$$

The *profile of a set* $Z \subseteq V$ in an ordering α of G is

$$\mathrm{prf}_\alpha(G, Z) = \sum_{z \in Z} \mathrm{prf}_\alpha(G, z).$$

The *profile of an ordering* α *of* G is $\mathrm{prf}_\alpha(G) = \mathrm{prf}_\alpha(G, V)$. An ordering α of G is *optimal* if

$$\mathrm{prf}_\alpha(G) = \min\{\, \mathrm{prf}_\beta(G) : \beta \in \mathrm{OR}(G)\,\}.$$

If α is optimal, then $\mathrm{prf}(G) = \mathrm{prf}_\alpha(G)$ is called the *profile* of G. If $X \subseteq V$ and α is an ordering of G, then let α_X denote the ordering of $G - X$ in which $\alpha_X(u) < \alpha_X(v)$ if and only if $\alpha(u) < \alpha(v)$ for all $u, v \in V(G) - X$. If $X = \{x\}$, then we simply write α_x instead of $\alpha_{\{x\}}$.

The following two lemmas will be used several times in the rest of the paper.

Lemma 1. *Let* $G = (V, E)$ *be a graph of order* n *and let* X *be a set of vertices such that* $G - X$ *is connected. If an ordering* α *has* $\{\alpha^{-1}(1), \alpha^{-1}(n)\} \subseteq V(G - X)$ *then* $\mathrm{prf}_\alpha(G, V - X) \geq \mathrm{prf}_{\alpha_X}(G - X) + |X|$.

Proof. Let $X = \{x_1, x_2, \ldots, x_r\}$ and define $X_i = \{x_1, x_2, \ldots, x_i\}$ for all $0 \leq i \leq r$. We will by induction show the following:

$$\mathrm{prf}_{\alpha_{X_i}}(G - X_i, V - X) \geq \mathrm{prf}_{\alpha_X}(G - X) + |X| - i. \tag{1}$$

The above is clearly true when $i = r$ as $X_r = X$ and $|X| = r$. If we can show that (1) is true for $i = 0$, then we are done. We will assume that (1) is true for some $i > 0$.

Since $G - X$ is connected and $\{\alpha^{-1}(1), \alpha^{-1}(n)\} \subseteq V(G - X)$, there is an edge $uv \in E(G - X)$ such that $\alpha_{X_{i-1}}(u) > \alpha_{X_{i-1}}(x_i) > \alpha_{X_{i-1}}(v)$. This implies that the profile of u is one larger in $\alpha_{X_{i-1}}$ than it is in α_{X_i}. This implies $\mathrm{prf}_{\alpha_{X_{i-1}}}(G - X_{i-1}, V - X) \geq \mathrm{prf}_{\alpha_{X_i}}(G - X_i, V - X) + 1 \geq \mathrm{prf}_{\alpha_X}(G - X) + |X| - i + 1$. We are now done by induction. □

Lemma 2 (Lin and Yuan [17])**.** *(i) If* G *is a connected graph with* n *vertices, then* $\mathrm{prf}(G) \geq n - 1$. *(ii) For a cycle* C_n *with* n *vertices we have* $\mathrm{prf}(C_n) = 2n - 3$.

For a vertex x, $d(x)$ denotes its degree, i.e., $d(x) = |N(x)|$. A slightly weaker version of the following lemma is stated in [17] without a proof.

Lemma 3. *If* G *is an arbitrary graph of order* n, $x \in V(G)$ *and* α *is an ordering of* G*, then* $\mathrm{prf}_\alpha(G) \geq \mathrm{prf}_{\alpha_x}(G - x) + d(x)$.

Proof. Let α be an ordering of G and let X be the set of vertices appearing to the left of x in α. More formally,

$$X = \{\alpha^{-1}(1), \alpha^{-1}(2), \ldots, \alpha^{-1}(\alpha(x) - 1)\}.$$

Note that for all $a \in N(x) - X$ we have $\mathrm{prf}_\alpha(G, a) \geq \mathrm{prf}_{\alpha_x}(G - x, a) + 1$. Furthermore, $\mathrm{prf}_\alpha(G, x) \geq |N(x) \cap X|$. Thus, $\mathrm{prf}_\alpha(G) - \mathrm{prf}_{\alpha_x}(G - x) \geq \mathrm{prf}_\alpha(G, x) + \sum_{a \in N(x) - X}(\mathrm{prf}_\alpha(G, a) - \mathrm{prf}_{\alpha_x}(G - x, a)) \geq |N(x) \cap X| + |N(x) - X| = d(x)$. Hence, $\mathrm{prf}_\alpha(G) \geq \mathrm{prf}_{\alpha_x}(G - x) + d(x)$. □

Theorem 3 gives a lower bound for the profile of a 2-edge-connected graph, which is important for our FPT algorithm. Lin and Yuan [17] used a concise and elegant argument to show that $\mathrm{prf}(G) \geq k(2n-k-1)/2$ for every k-connected graph G of order n. Their argument uses Menger's Theorem in a clever way, yet the argument cannot be used to prove our bound. Instead of Menger's Theorem we will apply the following well-known decomposition of 2-edge-connected graphs (see, e.g., Theorem 4.2.10 in [20]) called a *closed-ear decomposition*.

Theorem 2. *Any 2-edge-connected graph G has a partition of its edges $E_1, E_2, \ldots, E_r$, such that $G_i = G[E_1 \cup E_2 \cup \ldots \cup E_i]$ is 2-edge-connected for all $i = 1, 2, 3, \ldots, r$. Furthermore, E_j induces either a path with its endpoints in $V(G_{j-1})$ but all other vertices in $V(G_j) - V(G_{j-1})$ or a cycle with one vertex in $V(G_{j-1})$ but all other vertices in $V(G_j) - V(G_{j-1})$ for every $j = 2, 3, \ldots, r$. Moreover, G_1 is a cycle and every cycle of G can be G_1.*

Theorem 3. *If G is a 2-edge-connected graph of order n, then* $\mathrm{prf}(G) \geq \frac{3n-3}{2}$.

Proof. Let α be an optimal ordering of $V(G)$ and let y be the vertex with $\alpha(y) = n$. Since G is 2-edge-connected, y is contained in a cycle C. By Theorem 2, G has an ear-decomposition $E_1, E_2, \ldots, E_r$ such that $G[E_1] = C$. Let $G_i = G[E_1 \cup E_2 \cup \ldots \cup E_i]$, which by Theorem 2 are 2-edge-connected for all $i = 1, 2, \ldots, r$. We will prove this theorem by induction. If $r = 1$ then the theorem holds by Lemma 2(ii), as $n \geq 3$. So assume that $r \geq 2$. Let $n_i = |V(G_i)|$ for all $i = 1, 2, \ldots, r$ and note that by induction we know that $\mathrm{prf}(G_{r-1}) \geq \frac{3n_{r-1}-3}{2}$. If $n_r = n_{r-1}$ then E_r is just one edge and we are done as $\mathrm{prf}(G_r) \geq \mathrm{prf}(G_{r-1})$. So assume that $a = n_r - n_{r-1} > 0$. If $a = 1$ and $V(G_r) - V(G_{r-1}) = \{x\}$, then by Lemma 3 we obtain

$$\mathrm{prf}(G) \geq \mathrm{prf}(G_{r-1}) + d(x) \geq \frac{3n_{r-1}-3}{2} + 2 > \frac{3n-3}{2}.$$

So we may assume that $a \geq 2$. Let P be the path $G_r - V(G_{r-1})$, let x and z be the endpoints of P such that $\alpha(x) < \alpha(z)$, and let u be a neighbor of x in G_{r-1}. Let $j = \min\{\, \alpha(q) : q \in V(G_{r-1}) \,\}$, and let $Q = \{\, p \in V(P) : \alpha(p) > j \,\}$ and $M = \{\, p \in V(P) : \alpha(p) < j \,\}$, which is a partition of $V(P)$. (Note that $\alpha^{-1}(j) \in V(G_{r-1})$ and recall that $\alpha^{-1}(n) = y \in V(G_{r-1})$.) Furthermore let β denote the ordering α restricted to P (i.e., $\beta = \alpha_{V(G_{r-1})}$) and let $H = G - M$. By Lemma 1 (with $X = Q$) we obtain

$$\mathrm{prf}_{\alpha_M}(H, V(H) - Q) \geq \mathrm{prf}_{\alpha_{M \cup Q}}(H - Q) + |Q| = \mathrm{prf}_{\alpha_{V(P)}}(G_{r-1}) + |Q|.$$

We now bound $\mathrm{prf}_\beta(P)$ in the following way. Add an artificial vertex u' to the end of the ordering β and add the edges $u'x$ and $u'z$. This results in an ordering β' of $V(P) \cup \{u'\}$ where $\beta'(u') = |V(P)| + 1$. Since we have created a cycle we note that $\mathrm{prf}_{\beta'}(P \cup u') \geq 2(|V(P)| + 1) - 3$, by Lemma 2(ii). Since the profile of u' in β' is $|V(P)| + 1 - \beta(x)$ we note that the following holds.

$$\begin{array}{rcl} \mathrm{prf}_\beta(P) & \geq & 2(|V(P)|+1) - 3 - (|V(P)|+1-\beta(x)) \\ & = & |V(P)| - 2 + \beta(x) \end{array}$$

If $j = 1$ then the following holds (as $Q = V(P)$, $M = \emptyset$ and $|V(P)| - 2 + \beta(x) \geq |V(P)| - 1$):

$$\begin{array}{rcl} \mathrm{prf}_\alpha(G) & = & \mathrm{prf}_\alpha(G, V(G) - V(P)) + \mathrm{prf}_\alpha(G, V(P)) \\ & \geq & (\mathrm{prf}_{\alpha_{V(P)}}(G_{r-1}) + |Q|) + (|V(P)| - 2 + \beta(x)) \\ & = & \mathrm{prf}_{\alpha_{V(P)}}(G_{r-1}) + 2|V(P)| - 1 \end{array}$$

Now assume that $j \geq 2$. Let $R = \{\, p \in V(G_{r-1}) : \alpha(p) < \alpha(x) \,\}$ and note that $\alpha(x) = \beta(x) + |R|$. By Lemma 1 (used on the subgraph of G induced by $V(P) \cup R$ and with $X = R$) we obtain the following.

$$\mathrm{prf}_\alpha(G, V(P)) \geq \mathrm{prf}_\beta(P) + |R| \geq |V(P)| - 2 + \beta(x) + |R| = |V(P)| - 2 + \alpha(x)$$

Assume that $\alpha(x) < j$ and note that $\mathrm{prf}_\alpha(G, u) \geq \mathrm{prf}_{\alpha_M}(H, u) + j - \alpha(x)$, as $\mathrm{prf}_{\alpha_M}(H, u) \leq \alpha(u) - j$ and $\mathrm{prf}_\alpha(G, u) \geq \alpha(u) - \alpha(x)$. As $|Q| = |V(P)| - j + 1$ we obtain

$$\begin{array}{rcl} \mathrm{prf}_\alpha(G) & = & \mathrm{prf}_\alpha(G, V(H) - Q) + \mathrm{prf}_\alpha(G, V(P)) \\ & \geq & (\mathrm{prf}_{\alpha_M}(H, V(H) - Q) + j - \alpha(x)) + |V(P)| - 2 + \alpha(x) \\ & \geq & \mathrm{prf}_{\alpha_{V(P)}}(G_{r-1}) + |Q| + j + |V(P)| - 2 \\ & \geq & \mathrm{prf}_{\alpha_{V(P)}}(G_{r-1}) + 2|V(P)| - 1. \end{array}$$

Now assume that $\alpha(x) > j$. Analogously to the above we get the following:

$$\begin{array}{rcl} \mathrm{prf}_\alpha(G) & = & \mathrm{prf}_\alpha(G, V(H) - Q) + \mathrm{prf}_\alpha(G, V(P)) \\ & \geq & \mathrm{prf}_{\alpha_M}(H, V(H) - Q) + |V(P)| - 2 + \alpha(x) \\ & \geq & \mathrm{prf}_{\alpha_{V(P)}}(G_{r-1}) + |Q| + |V(P)| - 2 + \alpha(x) \\ & \geq & \mathrm{prf}_{\alpha_{V(P)}}(G_{r-1}) + 2|V(P)| - 1. \end{array}$$

So, we always have the following, which completes the proof.

$$\begin{array}{rcl} \mathrm{prf}_\alpha(G) & \geq & \mathrm{prf}_{\alpha_{V(P)}}(G_{r-1}) + 2|V(P)| - 1 \\ & \geq & \frac{3n_{r-1}-3}{2} + 2(n - n_{r-1}) - 1 \\ & = & \frac{3n-3}{2} + \frac{n-n_{r-1}}{2} - 1 \\ & \geq & \frac{3n-3}{2}. \end{array}$$

□

Let α be an optimal ordering of a connected graph G and let $V_1, V_2, \ldots, V_t$ be a partition of V. An ordering α' is obtained from α by keeping the relative order of vertices within each part V_i and laying out the parts in their order, $V_1, V_2, \ldots, V_t$. The next two results show that under certain weak conditions α' is also optimal.

Proposition 1. *Let $G = (V, E)$ be a connected graph of order n, let* $\mathrm{prf}(G) \leq n - 1 + k$ *and let α be an optimal ordering of G. Let X, Y be a partition of V such that $|X|, |Y| \geq k + 2$ and there is only one edge between $G[X]$ and $G[Y]$. Assume $\alpha^{-1}(1) \in X$ or $\alpha^{-1}(n) \in Y$. Then the ordering α' of G defined as follows is optimal: $\alpha'_{V-X} = \alpha_{V-X}$, $\alpha'_{V-Y} = \alpha_{V-Y}$, and $\alpha'(x) < \alpha'(y)$ for all $x \in X$ and $y \in Y$.*

Proof. Observe that $G[X]$ and $G[Y]$ are connected as otherwise G would be disconnected. Let xy be the single edge between $G[X]$ and $G[Y]$ and assume $x \in X$ and $y \in Y$. Let α be an optimal ordering of G with $\alpha^{-1}(n) = y' \in Y$ (the case $\alpha^{-1}(1) \in X$ is treated similarly). Let x' be the vertex with $\alpha(x') = 1$. If $x' \in Y$, then Lemma 1 implies that $\mathrm{prf}_\alpha(G, Y) \geq \mathrm{prf}_{\alpha_X}(G - X, Y) + |X|$. Since $\mathrm{prf}_\alpha(G, X) \geq \mathrm{prf}_{\alpha_Y}(G[X]) \geq |X| - 1$ and $\mathrm{prf}_{\alpha_X}(G[Y]) \geq |Y| - 1$ (both by Lemma 2(i)) and $|X| \geq k + 2$, we conclude that $\mathrm{prf}_\alpha(G) \geq |X| + |Y| + k$, a contradiction. Therefore, $x' \in X$.

Let $i = \min\{\, \alpha(y'') : y'' \in Y \,\}$ and let $j = \max\{\, \alpha(x'') : x'' \in X \,\}$. If $j < i$, we are done ($\alpha' = \alpha$), so we assume that $i < j$. Let $I = \alpha^{-1}(\{i, i+1, \ldots, j\})$. Recall that α' is defined as follows: $\alpha'_X = \alpha_X$ and $\alpha'_Y = \alpha_Y$ but $\alpha'(x'') < \alpha'(y'')$ for all $x'' \in X$ and $y'' \in Y$. We will prove that α' is optimal.

Let $H = G[X \cup (Y \cap I)]$ and let $G' = H$ if $xy \notin E(H)$ and $G' = H - xy$, otherwise. Let $\beta = \alpha_{V(G)-V(G')}$ (so β is equal to α, except we have deleted the last $n - j$ vertices in the ordering). Note that by Lemma 1 (used with the set $Y \cap I$) we get that $\mathrm{prf}_\beta(G', V(G') - (Y \cap I)) \geq \mathrm{prf}_{\beta_{Y\cap I}}(V(G') - (Y \cap I)) + |Y \cap I|$. This implies the following:

$$\mathrm{prf}_\alpha(G - xy, X) \geq \mathrm{prf}_{\alpha_Y}(G[X]) + |Y \cap I|.$$

Analogously we obtain that $\mathrm{prf}_\alpha(G - xy, Y) \geq \mathrm{prf}_{\alpha_X}(G[Y]) + |X \cap I|$, which implies

$$\mathrm{prf}_\alpha(G - xy) \geq \mathrm{prf}_{\alpha_Y}(G[X]) + \mathrm{prf}_{\alpha_X}(G[Y]) + |I| = \mathrm{prf}_{\alpha'}(G - xy) + (j - i + 1). \quad (2)$$

Suppose that $\alpha(x) > \alpha(y)$. Then (2) implies the following contradiction, as $\alpha'(y) - \alpha'(x) < j - i + 1$.

$$\mathrm{prf}_\alpha(G) \geq \mathrm{prf}_\alpha(G - xy) \geq \mathrm{prf}_{\alpha'}(G - xy) + (j - i + 1) > \mathrm{prf}_{\alpha'}(G).$$

Therefore, $\alpha(x) < \alpha(y)$. Let $l = \min\{\, \alpha(z) : z \in N[y] - \{x\} \,\}$ and let $L = \alpha^{-1}(\{\alpha(x), \alpha(x)+1, \alpha(x)+2, \ldots, l-1\})$. Note that $L = \emptyset$ if $l < \alpha(x)$. By the definition of L and the inequality in (2), we get the following:

$$\mathrm{prf}_\alpha(G) = \mathrm{prf}_\alpha(G - xy) + |L| \geq \mathrm{prf}_{\alpha'}(G - xy) + |I| + |L|.$$

When we add the edge xy to $G - xy$, we observe that, in the ordering α', the profile of y will increase by one for every vertex from Y with an α-value less then l and every vertex in X with an α-value larger than $\alpha(x)$. This is exactly the set $R_1 \cup R_2 \cup R_3 \cup R_4$, where

$$\begin{aligned} R_1 &= \{\, y'' \in Y : \alpha(x) < \alpha(y'') < l \,\}, \\ R_2 &= \{\, x'' \in X : \alpha(x) < \alpha(x'') < l \,\}, \\ R_3 &= \{\, y'' \in Y : \alpha(y'') < \min\{l, \alpha(x)\} \,\}, \\ R_4 &= \{\, x'' \in X : \max\{\alpha(x), l\} < \alpha(x'') \,\}. \end{aligned}$$

Since $R_1 \cup R_2 \subseteq L$ and $R_3 \cup R_4 \subseteq I$ (as $\alpha^{-1}(l) \in Y$), we conclude that $\mathrm{prf}_\alpha(G) \geq \mathrm{prf}_{\alpha'}(G) + |I| + |L| - |R_1| - |R_2| - |R_3| - |R_4| \geq \mathrm{prf}_{\alpha'}(G)$. □

Theorem 4. *Let $G = (V, E)$ be a connected graph of order n, let $\mathrm{prf}(G) \leq n-1+k$, and let α be an optimal ordering of G. Let $V_1, V_2, \ldots, V_t$ be a partition of V such that $|V_1|, |V_t| \geq k+2$ and there is only one edge $x_i y_i$ between $G[V_1 \cup V_2 \cup \cdots \cup V_i]$ and $G[V_{i+1} \cup V_{i+2} \cup \cdots \cup V_t]$ for each $i = 1, 2, \ldots t-1$. Let $\alpha^{-1}(1) \in V_{i_1}$, $\alpha^{-1}(n) \in V_{i_n}$ such that $1 \leq i_1 \leq i_n \leq t$.*

Then the ordering α' of G defined as follows is optimal: $\alpha'_{V-V_i} = \alpha_{V-V_i}$ for each $i = 1, 2, \ldots t$, and $\alpha'(v_i) < \alpha'(v_{i+1})$ for each $v_i \in V_i$, $v_{i+1} \in V_{i+1}$, $i = 1, 2, \ldots, t-1$.

Proof. The case $t = 2$ is covered by Proposition 1, hence assume $t \geq 3$. We distinguish the following three cases.

Case 1: $i_n = t$. Let $X = \bigcup_{i=1}^{t-1} V_i$ and $Y = V_t$. By Proposition 1 the following ordering β is optimal: $\beta_X = \alpha_X$, $\beta_Y = \alpha_Y$, and $\beta(x) < \beta(y)$ for each $x \in X, y \in Y$. Now let $X' = \bigcup_{i=1}^{t-2} V_i$, $Y' = V_{t-1} \cup V_t$. Again by Proposition 1, the following ordering β' is optimal: $\beta'_{X'} = \beta_{X'}$, $\beta'_{Y'} = \beta_{Y'}$, and $\beta'(x') < \beta'(y')$ for each $x' \in X', y \in Y'$. Combining the properties of β and β', we obtain that $\beta'_{Y'} = \alpha_{Y'}$, $\beta'_{V-V_{t-1}} = \alpha_{V-V_{t-1}}$, $\beta'_{V-V_t} = \alpha_{V-V_t}$, and $\beta'(x') < \beta'(v_{t-1}) < \beta'(v_t)$ for each $x' \in X', v_{t-1} \in V_{t-1}, v_t \in V_t$. Continuation of this argument allows us to show that α' is an optimal ordering.

Case 2: $i_1 = 1$. We argue similar as in Case 1.

Case 3: $i_n < t$. We consider the partition $V'_1, \ldots, V'_{i_n}$ where $V'_i = V_i$ for $i < i_n$ and $V'_{i_n} = \bigcup_{i=i_n}^{t} V_i$. Case 1 applies and we obtain the optimal ordering α' of G with $\alpha'_{V-V'_i} = \alpha_{V-V'_i}$ for each $i = 1, 2, \ldots, i_n$, and $\alpha'(v_i) < \alpha'(v_{i+1})$ for each $v_i \in V'_i$, $v_{i+1} \in V'_{i+1}$, $i = 1, 2, \ldots, i_n - 1$. Consider the original partition $V_1, \ldots, V_n$. Since $(\alpha')^{-1}(1) \in V_1 = V'_1$ the above Case 2 applies, and we conclude that the ordering α'' defined as follows is optimal: $\alpha''_{V-V_i} = \alpha'_{V-V_i}$ for each $i = 1, 2, \ldots, t$ and $\alpha''(v_i) < \alpha''(v_{i+1})$ for each $v_i \in V_i$, $v_{i+1} \in V_{i+1}$, $i = 1, 2, \ldots, t-1$. Consequently we have $\alpha''_{V-V_i} = \alpha_{V-V_i}$ for each $i = 1, 2, \ldots, t$, and $\alpha''(v_i) < \alpha''(v_{i+1})$ for each $v_i \in V_i$, $v_{i+1} \in V_{i+1}$, $i = 1, 2, \ldots, t-1$. □

A *bridgeless component* of a graph G is a maximal induced connected subgraph of G with no bridges. We call a connected graph G a *chain of length t* if the following holds: (a) G has bridgeless components C_i, $1 \leq i \leq t$ such that $V(G) = \bigcup_{i=1}^{t} V(C_i)$, and (b) C_i is linked to C_{i+1} by a bridge, $1 \leq i \leq t-1$. A component C_i is *nontrivial* if $|V(C_i)| > 1$, and *trivial*, otherwise. An ordering α of G is *special* if for any two vertices $x, y \in V(G)$ and $x \in V(C_i), y \in V(C_j)$, $i < j$ implies $\alpha(x) < \alpha(y)$.

The following two lemmas will be of use in the proof of Theorem 6.

Lemma 4. *Let G be a chain of order n and let η be the total number of vertices in the nontrivial bridgeless components of G. Let α be a special ordering of G with $\mathrm{prf}_\alpha(G) \leq n-1+k$. Then $\eta \leq 3k$.*

Proof. We show $\eta \leq 3k$ by induction on n. Suppose that G has a trivial component. If C_1 is trivial, then $G - C_1$ is a chain with $\mathrm{prf}_{\alpha_{V(C_1)}}(G - C_1) \leq n' - 1 + k$, where $n' = n - 1$. Thus, by the induction hypothesis, $\eta \leq 3k$. Similarly, we prove $\eta \leq 3k$ when C_t is trivial. Assume that C_i, $1 < i < t$, is trivial. Let C_i be adjacent to $x \in V(C_{i-1})$ and $y \in V(C_{i+1})$. Consider G' obtained from G by deleting C_i and appending edge xy. Observe that G' is a chain and $\mathrm{prf}_{\alpha_{V(C_i)}}(G') \leq n' - 1 + k$, where $n' = n - 1$. Thus, by induction hypothesis, $\eta \leq 3k$. So, now we may assume that $\eta = n$.

Let $C_1, \ldots, C_t$ denote the bridgeless components of G as in the definition above. Let $n_i = |V(C_i)|$. If $t = 1$, then by Theorem 3 we have $n \leq 2k + 1$ and we are done as $k \geq 1$. Now assume $t \geq 2$. Let $G' = G - V(C_t)$ and $n' = n - n_t$. Observe that G' is a chain and $\alpha_{V(C_t)}$ is a special ordering of G'. Let $k_t = \mathrm{prf}_\alpha(G, V(C_t)) - n_t + 1$ and let $k' = \mathrm{prf}_\alpha(G, V(G')) - n' + 1$. We have $k_t + k' - 1 \leq k$. Theorem 3 implies that

$$n_t - 1 + k_t = \mathrm{prf}_\alpha(G, V(C_t)) \geq \mathrm{prf}(C_t) + 1 \geq \frac{3n_t - 3}{2} + 1 = \frac{3n_t - 1}{2},$$

and thus $k_t \geq \frac{n_t+1}{2}$ and $n_t \leq 2k_t - 1$. Since $n_t \geq 3$, we have $k_t \geq 2$. By induction hypothesis, $n' \leq 3k'$. Thus $n = n' + n_t \leq 3(k - k_t + 1) + 2k_t - 1 \leq 3k$. $\square$

A connected component of a graph G is called *nontrivial* if it has more than one vertex.

Lemma 5. *Let $G = (V, E)$ be a connected graph of order n, let $X \subseteq V$ such that $G[X]$ is connected. Let $G_1, \ldots, G_r$ denote the connected components of $G - X$ and let t be the number of trivial components of $G - X$. Assume that $|V(G_i)| \leq |V(G_{i+1})|$ for $1 \leq i \leq r - 1$. If $k + n - 1 \geq \mathrm{prf}(G)$, then $k + 2 + t \geq r$ and $2k + t \geq \sum_{i=1}^{r-2} |V(G_i)|$.*

Proof. The result holds vacuously true if $r < 3$, hence assume $r \geq 3$. Let α be an optimal ordering of G. Let $I = \{\, 1 \leq i \leq r : V(G_i) \cap \{\alpha^{-1}(1), \alpha^{-1}(n)\} = \emptyset \,\}$. Clearly $|I| \geq r - 2$. Let $Y = X \cup \bigcup_{i \notin I} V(G_i)$ and $Z = V \setminus Y$. Observe that $G[Y] = G - Z$ is connected. Since $\{\alpha^{-1}(1), \alpha^{-1}(n)\} \subseteq Y$, Lemma 1 applies. Thus we see that $\mathrm{prf}(G) = \mathrm{prf}_\alpha(G)$ is at least

$$\mathrm{prf}_\alpha(G, V - Z) + \sum_{i \in I} \mathrm{prf}_\alpha(G, V(G_i)) \geq \mathrm{prf}(G - Z) + |Z| + \sum_{i \in I} \mathrm{prf}(G_i).$$

Furthermore, by Lemma 2(i),

$$\begin{aligned} k &\geq \mathrm{prf}(G) - n + 1 \geq \mathrm{prf}(G - Z) + |Z| - |Y| + \sum_{i \in I} (\mathrm{prf}(G_i) - |V(G_i)|) + 1 \\ &\geq (\mathrm{prf}(G[Y]) - |Y|) + |Z| - |I| + 1 \geq -1 + |Z| - |I| + 1 \geq |Z| - |I|. \end{aligned}$$

Thus,

$$|Z| - |I| \leq k. \tag{3}$$

Let $N = \{i \in I : \ |V(G_i)| > 1\}$. Since $|N| \leq |Z| - |I|$, we have $k \geq |N|$. Thus, $r \leq |I| + 2 = |N| + |I \setminus N| + 2 \leq k + t + 2$, and, by (3), $|Z| \leq k + |N| + |I \setminus N| \leq 2k + |I \setminus N| \leq 2k + t$. $\square$

4 Algorithm Based on Kernelization

4.1 Dealing with Vertices of Degree 1

In this section, G denotes a connected graph of order n. For an ordering α of G let $E_\alpha(G)$ denote the set of edges uv of G such that $\alpha(u) = \min_{w\in N[v]} \alpha(w)$ and $u \neq v$. The *length* $\ell_\alpha(uv)$ *of an edge* $uv \in E(G)$ *relative to* α is $|\alpha(u) - \alpha(v)|$ if $uv \in E_\alpha(G)$, and 0 if $uv \notin E_\alpha(G)$. Observe that $\mathrm{prf}_\alpha(G) = \sum_{e\in E(G)} \ell_\alpha(e)$.

Let X, Y be two disjoint sets of vertices of G and let α be an ordering of G. We say that (X, Y) is an *α-consecutive pair* if there exist integers a, b, c with $1 \leq a < b < c \leq n$ so that $X = \{\, x \in V(G) : a \leq \alpha(x) \leq b - 1 \,\}$ and $Y = \{\, y \in V(G) : b \leq \alpha(y) \leq c \,\}$. By $\mathrm{swap}_{Y,X}(\alpha)$ we denote the ordering obtained from α by swapping the α-consecutive pair (X, Y). For a set $X \subseteq V(G)$ let $E^r_\alpha(X)$ (respectively, $E^l_\alpha(X)$) denote the set of edges $uv \in E_\alpha$ with $u \in X$, $v \in V(G) \setminus X$, and $\alpha(u) < \alpha(v)$ (respectively, $\alpha(u) > \alpha(v)$).

Lemma 6. *Let α be an ordering of G and (X, Y) an α-consecutive pair such that there are no edges between X and Y. If $|E^l_\alpha(X)| \leq |E^r_\alpha(X)|$ and $|E^l_\alpha(Y)| \geq |E^r_\alpha(Y)|$, then for $\beta = \mathrm{swap}_{Y,X}(\alpha)$ we have $\mathrm{prf}_\beta(G) \leq \mathrm{prf}_\alpha(G)$.*

Proof. Observe that $E_\alpha(G) = E_\beta(G)$. Moreover, the only edges of $E_\alpha(G)$ that have different length in α and in β are the edges in $E^l_\alpha(Y) \cup E^r_\alpha(Y) \cup E^l_\alpha(X) \cup E^r_\alpha(X)$. Observe that $\ell_\beta(e) = \ell_\alpha(e) + |Y|$, $\ell_\beta(e') = \ell_\alpha(e') - |Y|$, $\ell_\beta(f) = \ell_\alpha(f) - |X|$, $\ell_\beta(f') = \ell_\alpha(f') + |X|$ for each $e \in E^l_\alpha(X)$, $e' \in E^r_\alpha(X)$, $f \in E^l_\alpha(Y)$ and $f' \in E^r_\alpha(Y)$. Using these relations and the inequalities $|E^l_\alpha(X)| \leq |E^r_\alpha(X)|$ and $|E^l_\alpha(Y)| \geq |E^r_\alpha(Y)|$, we obtain $\mathrm{prf}_\beta(G) \leq \mathrm{prf}_\alpha(G)$. □

Lemma 7. *Let α be an ordering of G and $(\{x\}, Y)$ an α-consecutive pair such that x has a neighbor z of degree 1 with $\alpha(z) > \alpha(y)$ for all $y \in Y$. If $|E^l_\alpha(Y)| \geq |E^r_\alpha(Y)|$, then for $\beta = \mathrm{swap}_{Y,\{x\}}(\alpha)$ we have $\mathrm{prf}_\beta(G) \leq \mathrm{prf}_\alpha(G)$.*

Proof. If there are no edges between x and vertices in Y then the result follows from Lemma 6 since $|E^l_\alpha(\{x\})| \leq 1 \leq |E^r_\alpha(\{x\})|$.

Now consider the case when $E^l_\alpha(\{x\}) = \{wx\}$ for a vertex w. Every edge $e \in E_\beta \setminus E_\alpha$ is of the form $e = yu$ for some vertex $y \in Y$, and xu must be in $E_\alpha \setminus E_\beta$. If $u \notin Y$ then $\ell_\beta(yu) = \ell_\alpha(yu) + 1 \leq \ell_\alpha(xu)$, and if $u \in Y$ then $\ell_\beta(yu) = \ell_\alpha(yu) \leq \ell_\alpha(xu)$. Consequently

$$\sum_{e\in E^l_\beta(Y)\cup E^r_\beta(Y)} \ell_\beta(e) \leq \sum_{e\in E^l_\alpha(Y)\cup E^r_\alpha(Y)} \ell_\alpha(e) + \sum_{e\in E_\alpha\setminus E_\beta} \ell_\alpha(e)$$

and clearly $\ell_\beta(wx)+\ell_\beta(xz) \leq \ell_\alpha(wx)+\ell_\alpha(xz)$. Hence the result also holds true in that case.

It remains to consider the case where x has neighbors in Y and $E^l_\alpha(\{x\}) = \emptyset$. Let $y' \in N(x) \cap Y$ with minimum α-value. Now $E_\beta(G) \setminus E_\alpha(G) \subseteq \{xy'\}$ and $\ell_\beta(xy') + \ell_\beta(xz) \leq \ell_\alpha(xz)$. Thus, $\mathrm{prf}_\beta(G) \leq \mathrm{prf}_\alpha(G)$. □

Lemma 8. *Let α be an ordering of G and let $(\{x\}, Y)$ be an α-consecutive pair. Let all vertices in Y be of degree 1 and adjacent with x. Then for $\beta = \mathrm{swap}_{Y,\{x\}}(\alpha)$ we have $\mathrm{prf}_\beta(G) \leq \mathrm{prf}_\alpha(G)$.*

Proof. Let y, y' denote the vertex in Y with largest $\alpha(y)$ and smallest $\alpha(y')$. Observe $\ell_\alpha(yx) = |Y|$. First assume that $E^l_\alpha(\{x\})$ contains an edge zx. We have $E_\beta(G) \subseteq E_\alpha(G) \setminus \{xy\}$, and $\ell_\beta(e) \leq \ell_\alpha(e)$ holds for all $e \in E_\beta(G) \setminus \{xz\}$. Since $\ell_\beta(zx) = \ell_\alpha(zx) + \ell_\alpha(xy)$, the result follows. Next assume that $E^l_\alpha(\{x\}) = \emptyset$. We have $E_\beta(G) \subseteq (E_\alpha(G) \setminus \{xy\}) \cup \{xy'\}$, and $\ell_\beta(e) \leq \ell_\alpha(e)$ holds for all $e \in E_\beta(G) \setminus \{xy'\}$. Since $\ell_\beta(xy') = \ell_\alpha(xy)$, the result follows. □

For $x \in V(G)$ let $N_1(x)$ denote the set of neighbors of x that have degree 1. We say that an ordering α of G is *conformal for a vertex x* of G if $\{\, \alpha(w) : w \in N_1(x) \,\}$ forms a (possibly empty) interval and $\alpha(w) < \alpha(x)$ holds for all $w \in N_1(x)$. We say that α is *conformal for a graph G* if it is conformal for all vertices of G.

Theorem 5. *For every connected graph G there exists an optimal ordering which is conformal.*

Proof. Let α be an optimal ordering of G. Let x be a vertex of G for which α is not conformal. We apply the following steps to α, until we end up with an optimal ordering which is conformal for x. In each step we transform α into an optimal ordering β in such a way that whenever α is conformal for a vertex x', so is β. Hence, we can repeat the procedure for all the vertices one after the other, and we are finally left with an optimal ordering which is conformal.

Let $w_1, w_2 \in N_1(x) \cup \{x\}$ with minimal $\alpha(w_1)$ and maximal $\alpha(w_2)$. We call a set $B \subseteq N_1(x)$ a *block* if $\{\, \alpha(b) : b \in B \,\}$ is a nonempty interval of integers. A block is *maximal* if it is not properly contained in another block.

Step 1. Assume that there exist α-consecutive pairs $(\{x\}, Y)$, (Y, Z) with the following properties: (a) Y and Z are nonempty; (b) $Y \cap N_1(x) = \emptyset$; (c) Z is a maximal block. By assumption, there is a $z \in Z$ such that $xz \in E(G)$ and $\alpha(z) > \alpha(y)$ holds for all $y \in Y$. Moreover, there are no edges between Y and Z and $E^r_\alpha(Z) = \emptyset$. If $|E^l_\alpha(Y)| \geq |E^r_\alpha(Y)|$, then we put $\beta = \mathrm{swap}_{Y,\{x\}}(\alpha)$, otherwise we put $\beta = \mathrm{swap}_{Z,Y}(\alpha)$. It follows from Lemmas 7 and 6, respectively, that β is optimal.

Step 2. Assume that there exists an α-consecutive pair $(\{x\}, Y)$ such that Y is a maximal block. We put $\beta = \mathrm{swap}_{Y,\{x\}}(\alpha)$. It follows by Lemma 8 that β is optimal.

Remark: If neither Step 1 nor Step 2 can be applied, then $\alpha(w_2) < \alpha(x)$.

Step 3. Assume that there exist α-consecutive pairs (X, Y), (Y, Z) with the following properties: (a) X and Z are maximal blocks; (b) $Y \subseteq V(G) \setminus N_1(X)$; (c) $w_1 \in X$. Note that there are no edges between X and Y and no edges between Y and Z. Furthermore, we have $E^l_\alpha(X) = \emptyset$ and $E^r_\alpha(Z) = \emptyset$ (the former follows from Property (c)). If $|E^l_\alpha(Y)| \geq |E^r_\alpha(Y)|$, then we put $\beta = \mathrm{swap}_{Y,X}(\alpha)$, otherwise we put $\beta = \mathrm{swap}_{Z,Y}(\alpha)$. In both cases it follows from Lemma 6 that β is optimal.

Remark: If none of the above Steps 1, 2, or 3, applies, then α is conformal for x. □

Note that when applying the procedure of the above proof, it is possible that we end up with exactly one maximal block X such that for a nonempty set Y the pairs (X, Y) and $(Y, \{x\})$ are α-consecutive. If $|E^l_\alpha(Y)| < |E^r_\alpha(Y)| < |E^r_\alpha(\{x\})|$, then we can neither swap X and Y nor Y and $\{x\}$ without increasing the cost of the profile.

4.2 Kernelization

For technical reasons, in this section we will deal with a special kind of weighted graphs, but they will be nothing else but compact representations of (unweighted) graphs.

We consider a *weighted graph* $G = (V, E, \rho)$ whose vertices v of degree 1 have an arbitrary positive integral weight $\rho(v)$, vertices u of degree greater than one have weight $\rho(u) = 1$. The *weight* $\rho(G)$ of $G = (V, E, \rho)$ is the sum of weights of all vertices of G. A weighted graph $G = (V, E, \rho)$ corresponds to an unweighted graph G^u, which is obtained from G by replacing each vertex v of degree 1 (v is adjacent to a vertex w) with $\rho(v)$ vertices adjacent to w. An *ordering* of a weighted graph $G = (V, E, \rho)$ is obtained as follows: take a conformal ordering α of G^u and for each vertex x with $|N_1(x)| > 1$ delete all neighbors of x degree of 1 apart from $y \in N_1(x)$ for which $\alpha(y) < \alpha(z)$ for each $z \in N_1(x) - \{y\}$. Thus, an ordering of G is an injective mapping β from $V(G)$ to $\{1, 2, \dots, \rho(G)\}$. The *profile* $\mathrm{prf}(G)$ of a weighted graph is defined exactly as the profile of an unweighted graph.

By Theorem 5 and the definitions above, $\mathrm{prf}(G) = \mathrm{prf}(G^u)$ and an optimal ordering of G can be effectively transformed into an optimal ordering of G^u. Also, $\rho(G) = |V(G^u)|$. The correspondence between G and G^u allows us to use the results given in the previous sections.

Kernelization Rule 1. *Let G be a weighted graph and x a vertex of G with $N_1(x) = \{v_1, \dots, v_r\}$, $r \geq 2$. We obtain the weighted graph $G_0 = (V_0, E_0, \rho_0)$, where $G_0 = G - \{v_2, \dots, v_r\}$ and $\rho_0(u) = \rho(u)$ for $u \in V_0 \setminus \{v_1\}$ and $\rho_0(v_1) = \sum_{i=1}^{r} \rho(v_i)$.*

The next lemma follows from Theorem 5.

Lemma 9. *Let G be a weighted connected graph and G_0 the weighted graph obtained from G by Kernelization Rule 1. Then $\mathrm{prf}(G) = \mathrm{prf}(G_0)$, and an optimal ordering α_0 of G_0 can be effectively transformed into an optimal ordering α of G.*

Let e be a bridge of a weighted connected graph G and let G_1, G_2 denote the connectivity components of $G - e$. We define the *order* of e as $\min\{\rho(G_1), \rho(G_2)\}$. Let v be a vertex of a (weighted) graph G. We say that v is *k-suppressible* if the following conditions hold: (a) v forms a trivial bridgeless component of G; (b) v is of degree 2 or 3; (c) there are exactly two bridges

e_1, e_2 of order at least $k+2$ incident with v; (d) if there is a third edge $e_3 = vw$ incident with v, then w is a vertex of degree 1.

Kernelization Rule 2 (w.r.t. parameter k). *Let v be a k-suppressible vertex of a weighted graph $G = (V, E, \rho)$ and let xv, yv be the bridges of order at least $k+2$. From G we obtain a weighted graph by removing $\{v\} \cup N_1(v)$ and adding the edge xy.*

Lemma 10. *Let $G = (V, E, \rho)$ be a weighted connected graph with $\mathrm{prf}(G) \leq \rho(G) - 1 + k$ and G' the weighted graph obtained from G by means of Kernelization Rule 2 with respect to parameter k. Then $\mathrm{prf}(G) - \rho(G) = \mathrm{prf}(G') - \rho(G')$, and an optimal ordering α' of G' can be effectively transformed into an optimal ordering α of G.*

Proof. Let v be a k-suppressible vertex of G^u and let xv, yv be the bridges of order at least $k+2$. We consider the case when $N_1(v) = \{w_1, \ldots, w_r\} \neq \emptyset$; the proof for the case when $N_1(v) = \emptyset$ is similar. Let $G^u[X]$ and $G^u[Y]$ denote the components of $G^u - v$ that contain x and y, respectively. Consider an optimal ordering α of G^u. By Theorem 5, we may assume that $\alpha(w_i) < \alpha(v)$ for every $1 \leq i \leq r$. Now by Theorem 4 ($V_1 = X$, $V_2 = \{v, w_1, \ldots, w_r\}$, $V_3 = Y$), we can find an optimal ordering α' of G^u such that $\alpha'(x') < \alpha'(w_i) < \alpha'(v) < \alpha'(y')$ for each $x' \in X$, $y' \in Y$ and $i = 1, 2, \ldots, r$.

Now it will be more convenient to argue using the weighted graphs G and G'. Using Kernelization Rule 1, we transform α' into the corresponding optimal ordering of G. For simplicity we denote the new ordering α' as well. Observe that $\mathrm{prf}_{\alpha'_{\{v,w\}}}(G', y) = \mathrm{prf}_{\alpha'}(G, y) + \mathrm{prf}_{\alpha'}(G, v) - 1 - \rho(w)$. Hence, $\mathrm{prf}(G') - \rho(G') \leq \mathrm{prf}_{\alpha'_{\{v,w\}}}(G') - \rho(G') = \mathrm{prf}(G) - \rho(G)$.

Conversely, let α' be an optimal ordering of G'. Since the bridge xy of G' is of order at least $k+2$, we may assume by Theorem 4 that for all $x' \in X$ and $y' \in Y$ we have $\alpha'(x') < \alpha'(y')$. It is straightforward to extend α' into an ordering α of G such that $\alpha_{\{v,w\}} = \alpha'$ and $\mathrm{prf}_{\alpha}(G) = \mathrm{prf}_{\alpha'}(G') + 1 + \rho(w)$. Hence $\mathrm{prf}(G) - \rho(G) \leq \mathrm{prf}_{\alpha}(G) - \rho(G) = \mathrm{prf}_{\alpha'}(G') - \rho(G')$. Thus, $\mathrm{prf}(G') - \rho(G') = \mathrm{prf}(G) - \rho(G)$. $\square$

Theorem 6. *Let $G = (V, E, \rho)$ be a weighted connected graph with $n = |V|$ and $m = |E|$. Let k be a positive integer such that $\mathrm{prf}(G) \leq \rho(G) - 1 + k$. One of the Kernelization Rules 1 and 2 can be applied with respect to parameter k, or $n \leq 12k + 6$ and $m \leq 13k + 5$.*

Proof. For a weighted graph $G = (V, E, \rho)$, let G^* be the unweighted graph with $V(G^*) = V$ and $E(G^*) = E$. Consider an optimal ordering α of G. By definition of an ordering of a weighted graph, α is obtained from a conformal ordering of G^u, so we may assume that α is conformal. Define an ordering β of G^* as follows: for each $v \in V$ we set $\beta(v) = |\{u \in V :\ \alpha(u) \leq \alpha(v)\}|$. A vertex v of G is *heavy* if $\rho(v) > 1$; let H be the set of heavy vertices of G.

Assume first that G has only one heavy vertex v and let u be the neighbor of v. Since α is conformal, v is to the left of u in α. Thus, $\mathrm{prf}_{\beta}(G^*, u) \leq$

$\mathrm{prf}_\alpha(G,u)-\rho(v)+1$. If G has more than one heavy vertex, we may transform G to G^* by setting the weights of heavy vertices, one by one, to 1. As above we get $\mathrm{prf}_\beta(G^*,U) \le \mathrm{prf}_\alpha(G,U)-\sum_{v\in H}(\rho(v)-1)$, where U is the set of the neighbors of vertices in H. Thus, $\mathrm{prf}(G^*)+\rho(G)-n \le \mathrm{prf}_\beta(G^*)+\rho(G)-n \le \mathrm{prf}(G)$, and if $\mathrm{prf}(G) \le \rho(G)-1+k$, then $\mathrm{prf}(G^*) \le n-1+k$. The last inequality will allow us to consider G^* rather than G in the rest of the proof, but for the simplicity of notation we use G instead of G^*.

Assume that none of the Kernelization Rules 1 and 2 can be applied with respect to parameter k. We will show that the claimed bounds on n and m hold. By the connection between profile and interval graphs [3] we have $m \le \mathrm{prf}(G) \le n-1+k$. Thus, $n \le 12k+6$ implies $m \le 13k+5$. Therefore, it suffices to prove that $n \le 12k+6$.

Case 1: G has no bridges of order at least $k+2$. If G is bridgeless, then by Theorem 3, we have $n-1+k \ge \mathrm{prf}(G) \ge \frac{3n-3}{2}$ and, thus, $n \le 2k+1$. Hence, we may assume that G has bridges.

For a bridge xy of G let $G[V_x^{xy}]$ and $G[V_y^{xy}]$ denote the connectivity components of $G-xy$ with $x \in V_x^{xy}$ and $y \in V_y^{xy}$. If we have $|V_x^{xy}|=|V_y^{xy}|$ for a bridge xy, then $n \le 2k+2$ since otherwise xy would have order at least $k+2$. Hence assume $|V_x^{xy}| \ne |V_y^{xy}|$ for all bridges xy of G.

Consider the oriented tree T whose vertices are the bridgeless components of G and whose arcs are the bridges xy of G, oriented from x to y if $|V_x^{xy}|>|V_y^{xy}|$. Since T is an acyclic digraph, T contains a vertex s of in-degree 0. Let S denote the bridgeless component of G corresponding to s. Let P be a connectivity component of $G-V(S)$. If P is nontrivial, P has a vertex z such that $P-z$ is connected and z is not incident to the bridge between P and S. If P is trivial, let $z=V(P)$. Let α be an optimal ordering of G. By Lemma 3, $n-1+k \ge \mathrm{prf}_\alpha(G) \ge \mathrm{prf}_{\alpha_z}(G-z)+d(z) \ge \mathrm{prf}(G-z)+1$. Thus, $\mathrm{prf}(G-z) \le |V(G-z)|-1+k$. Similarly, we can see that $\mathrm{prf}(G-V(P)) \le |V(G-V(P))|-1+k$ and, moreover,

$$\mathrm{prf}(S) \le |S|-1+k. \tag{4}$$

Therefore, as in the first paragraph of Case 1, we obtain $|V(S)| \le 2k+1$.

Let $G_1,G_2,\dots,G_r$ be the connectivity components of $G-V(S)$ and let $|V(G_i)| \le |V(G_{i+1})|$ for each $i=1,2,\dots,r-1$. Let t be the number of trivial components of $G-V(S)$. Since we have assumed that Kernelization Rule 1 cannot be applied, it follows that $t \le |V(S)| \le 2k+1$. By Lemma 5, $\sum_{i=1}^{r-2}|V(G_i)| \le 2k+t$. By the definition of S and the fact that every bridge of G is of order at most $k+1$, we have $|V(G_{r-1})| \le |V(G_r)| \le k+1$. Thus,

$$|V(G)| \le 2k+1+2k+t+2(k+1) \le 8k+4.$$

Case 2: G has some bridges of order at least $k+2$. Let C_i, $i=1,\dots,t$, denote the bridgeless components of G such that at least one vertex in C_i is incident with a bridge of order at least $k+2$. We put $X=\bigcup_{i=1}^t V(C_i)$.

Suppose that there is a component C_i incident with three or more bridges of order at least $k+2$. Then, we may assume that there are three bridges e_2,e_3,e_4 of order at least $k+2$ that connect a subgraph F_1 of G with subgraphs

F_2, F_3, F_4, respectively, and $V = \bigcup_{i=1}^{4} V(F_i)$. Let α be an optimal ordering of G. Assume without loss of generality that $\alpha^{-1}(1) \notin V(F_2)$ and $\alpha^{-1}(n) \notin V(F_2)$. Let $Q = V(F_2)$ and note that $G-Q$ is connected. Therefore Lemmas 1 and 2(i) imply $\mathrm{prf}(G) = \mathrm{prf}_\alpha(G,Q)+\mathrm{prf}_\alpha(G,V-Q) \geq |Q|-1+(|V|-|Q|-1)+|Q| \geq n+k$, a contradiction. Since G is connected, it follows that $G[X]$ is connected. Thus, $G[X]$ is a chain and we may assume that C_i and C_{i+1} are linked by a bridge b_i for each $i = 1, 2, \ldots, t-1$. Notice that each b_i is of order at least $k+2$ in G.

Let $G_1, G_2, \ldots, G_r$ be the connectivity components of $G - X$. Observe that each G_i $(1 \leq i \leq r)$ is linked with exactly one C_j $(1 \leq j \leq t)$ with a bridge e_{ij}. The bridge e_{ij} must be of order less than $k+2$, since otherwise $V(G_i) \cap X \neq \emptyset$. Hence

$$|V(G_i)| \leq k+1 \tag{5}$$

follows for all $i \in \{1, \ldots, r\}$. For each j, let $IG(j)$ be the set of indices i such that G_i is linked to C_j.

Let $N = \{\, 1 \leq i \leq t : |V(C_i)| > 1 \,\}$ and $T = \{\, 1 \leq i \leq t : |V(C_i)| = 1 \,\}$. For $i \in T$ let x_i denote the single vertex in C_i. Similarly, let $N' = \{\, 1 \leq i \leq r : |V(G_i)| > 1 \,\}$ and $T' = \{\, 1 \leq i \leq r : |V(G_i)| = 1 \,\}$. Let $H_j = G[\bigcup_{i \in IG(j)} V(G_i) \cup V(C_j)]$ for each $j = 1, 2, \ldots, t$. By Theorem 4, we may assume that there exists an optimal ordering β such that $\beta(h_i) < \beta(h_j)$ for all $i < j$, $h_i \in V(H_i)$, $h_j \in V(H_j)$. Let $\gamma = \beta_{V(G)-X}$. Clearly, γ is a special ordering of the chain $G[X]$, i.e., $\gamma(c_i) < \gamma(c_j)$ for all $i < j$, $c_i \in V(C_i)$, $c_j \in V(C_j)$.

Similarly to (4), we can prove that $\mathrm{prf}_\gamma(G[X]) \leq |X| - 1 + k$. Now by Lemma 4, $\sum_{i \in N} |V(C_i)| \leq 3k$. Lemma 5 yields that $|N'| \leq k+2$. Since none of the Kernelization Rules 1 and 2 can be applied, for each $i \in T$, x_i is linked by a bridge $x_i y_{\pi(i)}$ to at least one nontrivial $G_{\pi(i)}$, where $\pi(i) \neq \pi(i')$ whenever $i \neq i'$. Hence, $|T| \leq k+2$. Thus, $|X| = \sum_{i \in N} |V(C_i)|+|T| \leq 3k+(k+2) = 4k+2$. Using (5) and Lemma 5, we have that $\sum_{i=1}^{r} |V(G_i)| \leq 2(k+1)+2k+|T'| = 4k+2+|T'|$.

Let $Y = \bigcup_{i=1}^{r} V(G_i)$. Since Kernelization Rule 1 cannot be applied, every vertex in X is adjacent with at most one G_i with $i \in T'$. Hence $|T'| \leq |X| \leq 4k+2$. Consequently $|Y| \leq 2(4k+2) = 8k+4$. Hence $n = |X| + |Y| \leq 4k+2+8k+4 = 12k+6$ follows. $\square$

It is not too difficult to see how Theorem 1 can be extended to the following theorem. As the proof of Theorem 7 is very similar to that of Theorem 1 we will only outline the proof.

Theorem 7. *Let $G = (V, E, \rho)$ be a connected weighted graph with n vertices and m edges as considered at the beginning of this section. The cost of adding an edge $e = uv$ to G is $\rho(u) \cdot \rho(v)$. If k is a non-negative integer, then we can decide, in time $O(nm3^k(k+1)!)$, whether we can add edges of total cost (i.e. the sum of the costs of all edges added) at most $k+n-1-m$, such that the result is an interval graph.*

Proof. Note that if $\rho(u) = 1$ for all $u \in V(G)$, then this theorem is equivalent to Theorem 1. In Theorem 1 we have shown that the running time of a search tree algorithm is $O(\Pi_{j=m-n+2}^{k+1} a(j))$, where $a(j) = 3j$ if $j > 5$ and otherwise

$a(j) = 15$. This also holds for the weighted case as we can add at most as many edges as we did in the search tree in Theorem 1. So the depth of the search tree is not greater in the weighted case than in the unweighted one. □

Corollary 1. *The problem PAVGV can be solved in time* $O(|V|^2 + k^2 3^k (k+1)!)$*.*

Proof. We can apply the two kernelization rules as long as it is possible or we have concluded that $\mathrm{prf}(G) > |V| - 1 + k$. This will take time $O(|V|^2)$. Assume that $\mathrm{prf}(G) \leq |V| - 1 + k$. Then, by Theorem 6, the remaining graph H has at most $12k + 6$ vertices and $13k + 5$ edges. Applying Theorem 7 to H, we obtain the required running time. □

Remark 1. *Of the two algorithms we obtained, one of complexity* $O(nm3^k(k+1)!)$ *and the other of complexity* $O(n^2 + k^2 3^k (k+1)!)$*, the second algorithm is far more efficient.*

5 Vertex Average Profile Problem

In this final section we consider the problem Vertex Average Profile (VAP); see section 1.2. Serna and Thilikos [19] asked whether VAP is fixed-parameter tractable. The following result, announced in [12] without a proof, implies that VAP is not fixed-parameter tractable unless P = NP.

Theorem 8. *Let* $k \geq 2$ *be a fixed integer. Then it is NP-complete to decide whether* $\mathrm{prf}(H) \leq k|V(H)|$ *for a graph* H*.*

Proof. Let G be a graph and let r be an integer. We know that it is NP-complete to decide whether $\mathrm{prf}(G) \leq r$. Let $n = |V(G)|$. Let k be a fixed integer, $k \geq 2$. Define G' as follows (i and j will be chosen later): G' contains k copies of G, j isolated vertices and a clique with i vertices (all of these subgraphs of G' are vertex disjoint). We have $n' = |V(G')| = kn + i + j$. Observe that $\mathrm{prf}(K_i) = \binom{i}{2}$. By the definition of G', $k \cdot \mathrm{prf}(G) = \mathrm{prf}(G') - \mathrm{prf}(K_i) = \mathrm{prf}(G') - \binom{i}{2}$. Therefore, $\mathrm{prf}(G) \leq r$ if and only if $\mathrm{prf}(G') \leq kr + \binom{i}{2}$. If there is a positive integer i such that $kr + \binom{i}{2} = kn'$ and the number of vertices in G' is bounded from above by a polynomial in n, then G' provides a reduction from to VAP with the fixed k. Observe that $kr + \binom{i}{2} \geq k(kn + i)$ for $i = 2kn$. Thus, by setting $i = 2kn$ and $j = r + \frac{1}{k}\binom{i}{2} - kn - i$, we ensure that G' exists and the number of vertices in G' is bounded from above by a polynomial in n. □

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