Almost minimum diameter orientations of semicomplete multipartite and extended digraphs

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Abstract

An orientation of a digraph D is a spanning subdigraph of D obtained from D by deleting exactly one arc between x and y for every pair $x \neq y$ of vertices such that both xy and yx are in D. Almost minimum diameter orientations of certain semicomplete multipartite and extended digraphs are considered, several generalizations of results on orientations of undirected graphs are obtained, some conjectures are posed.

1 Introduction, terminology and notation

An *orientation* of a digraph D is a spanning subdigraph of D obtained from D by deleting exactly one arc between x and y for *every* pair $x \neq y$ of vertices such that both xy and yx are in D. In this paper, we consider almost minimum diameter orientations of certain semicomplete multipartite and extended digraphs. We prove several generalizations of results on orientations of undirected graphs obtained in [5, 12, 14], remark that some others can be shown analogously, and pose two conjectures on the topic.

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While there is a large number of papers considering minimum diameter orientations of undirected graphs, see e.g. [3, 6, 8, 9, 10, 11, 13, 14], there is only one recent work [7] on minimum diameter orientations of digraphs, where the authors provide a motivation for studying the topic. It is worth noting that there are a few papers [2, 4, 21] considering finite diameter orientations of mixed graphs (or, equivalently, of directed graphs), but none of these papers has dealt with minimizing the diameter of an orientation of a given digraph. We restrict our attention to special classes of digraphs since even the problem to check whether a given undirected graph has an orientation of diameter 2 is proved to be $\mathcal{NP}$-complete by Chvátal and Thomassen [3] and the upper bound on the diameter of an orientation of an undirected graph obtained in [3] is far from best possible for many classes of undirected graphs.

We use the standard terminology and notation on digraphs as given in [1]. We still provide most of the necessary definitions for the convenience of the reader.

A *biorientation* of a digraph D is a spanning subdigraph of D obtained from D by deleting exactly one arc between x and y for *some* pairs $x \neq y$ of vertices such that both xy and yx are in D. Clearly, an orientation of a digraph is a special case of biorientation. A digraph D is *symmetric* if for every pair $x \neq y$ of vertices in D either there is no arc between x and y or both xy and yx are in D. Symmetric digraphs are in natural correspondence to undirected graphs: for an undirected graph G, the symmetric digraph $\overleftrightarrow{G}$ is obtained from G by replacing every edge xy with the pair xy, yx of arcs. Let $D = (V, A)$ be a digraph and let x, y be a pair of vertices in D. If $xy \in A$, we say x *dominates* y, and y *is dominated by* x, and denote it by $x \rightarrow y$. If $X, Y \subset V$, then $X \rightarrow Y$ means that every vertex of X dominates every vertex of Y. Notice that $X \rightarrow Y$ does not mean that there is no arc from Y to X. The *converse* of a digraph D is the digraph obtained from D by replacing every arc xy of D by the arc yx.

All paths and cycles we consider in this paper are directed and simple. A path from x to y is an (x, y)-*path*. The *distance*, $\text{dist}_D(x, y)$, *from* x *to* y in D is the least length of an (x, y)-path if y is reachable from x, and is equal to ∞, otherwise. We assume that $\text{dist}_D(x, x) = 0$ for every vertex $x \in V$. The *diameter* $\text{diam}(D)$ of D is the maximum distance from a vertex to another vertex of D. Clearly, D is of finite diameter if and only if D is strong. The minimum diameter of an orientation of a digraph D will be denoted by $\text{diam}_{\min}(D)$.

A digraph D is *semicomplete* if there is at least one arc between any pair of distinct vertices of D. The $(s_1, s_2, ..., s_n)$-*extension* (or just *extension*) $D(s_1, s_2, ..., s_n)$ of a digraph D with vertices labelled, say, $1, 2, ..., n$ is obtained from D by replacing every vertex i by a set of s_i independent (i.e. with no arc between them) vertices; more formally,

$$V(D(s_1, s_2, ..., s_n)) = \{(p_i, i) : \ 1 \leq p_i \leq s_i, \ 1 \leq i \leq n\}$$

and $(p, i) \rightarrow (q, j)$ in $D(s_1, s_2, ..., s_n)$ if and only if $i \rightarrow j$ in D. A digraph D is semicomplete k-partite, $k \geq 2$, if the vertices of D can be partitioned into k *partite sets* $V_1, V_2, ..., V_k$

such that every partite set is an independent set, but for every pair x, y of vertices from distinct partite sets, xy or yx (or both) is in D. When $k = 2$ we speak of *semicomplete bipartite digraphs*; when $k \geq 2$ we speak of *semicomplete multipartite digraphs*. Clearly, every semicomplete digraph with n vertices is a semicomplete n-partite digraph, and an extension of a semicomplete digraph with n vertices is a semicomplete n-partite digraph.

Semicomplete digraphs, semicomplete multipartite digraphs and some families of extended digraphs have been extensively studied in the literature (cf. [1] and the bibliography therein).

2   Semicomplete multipartite digraphs

Almost minimum diameter orientations of semicomplete digraphs and semicomplete bipartite digraphs have been studied in [7], where the following two theorems were proved. Note that Theorem A was proved for a larger family of digraphs, quasi-transitive digraphs; the digraph $\overleftrightarrow{K}_{1,n-1}$ clearly has no strongly connected orientation.

Theorem A *If D is a strong semicomplete digraph, then*

$$\mathrm{diam_{min}}(D) \leq max\{3, \mathrm{diam}(D)\},$$

and the bound is sharp.

Theorem B *If D is a strong semicomplete bipartite digraph and $D \neq \overleftrightarrow{K}_{1,n-1}$, then*

$$\mathrm{diam_{min}}(D) \leq max\{5, \mathrm{diam}(D)\},$$

and the bound is sharp.

Based on Theorems A and B as well as on Theorem 2.3 stated and proved below, one may guess that there is an absolute constant c such that for every strong semicomplete multipartite digraph D, we have $\mathrm{diam_{min}}(D) \leq \max\{c, \mathrm{diam}(D)\}$. However, the following example shows that this guess is wrong.

To simplify the following discussion, let us first consider the definitions of forward and backward arcs. For an ordering $\alpha = v_1, v_2, ..., v_n$ of the vertices of a digraph D, an arc $v_i v_j \in A(D)$ is *forward* (*backward*) if $i < j$ ($i > j$).

For every integer $l \geq 1$, let D_l be the semicomplete multipartite digraph with partite sets $V' = \{x_0\}$, $V'' = \{y_0\}$, $V_1 = \{x_1, y_1\}$, $V_2 = \{x_2, y_2\}$, $\ldots$, $V_l = \{x_l, y_l\}$ such that the arcs

$$x_0 y_0, x_0 x_1, x_1 x_2, \ldots, x_{l-1} x_l, y_0 y_1, y_1 y_2, \ldots, y_{l-1} y_l$$

are the only forward arcs in the ordering $\beta = x_0, y_0, x_1, y_1, \ldots, x_l, y_l$ and $x_0 y_0 x_0$ is the only cycle of length two in D_l.

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Observe that the diameter of D_l is $l + 1$ as $\operatorname{dist}_{D_l}(x_0, y_l) = \operatorname{dist}_{D_l}(y_0, x_l) = l + 1$ and that the diameter of both $D - x_0 y_0$ and $D - y_0 x_0$ is $l + 2$ (while a shortest (x_0, y_l)-path in D_l uses only forward arcs of β, a shortest (x_0, y_l)-path in $D_l - x_0 y_0$ must use a backward arc). It is not difficult to see that this example can also be modified to semicomplete k-partite digraphs with fixed $k \geq 3$ (there will be fewer backward arcs).

In view of this example and Theorems A and B, we suspect that the following conjecture is correct.

Conjecture 2.1 *There is an absolute constant c such that for every strong semicomplete multipartite digraph D, we have $\operatorname{diam}_{\min}(D) \leq \operatorname{diam}(D) + c$.*

The following two theorems provide further support to this conjecture. Theorem 2.2 shows that the conjecture is true for strong semicomplete multipartite digraphs with a bounded (from above by an absolute constant) number of cycles of length two; Theorem 2.3 indicates that the conjecture is correct for extended semicomplete digraphs. (Theorem 2.3 is also of independent interest as a generalization of Theorem A.)

Theorem 2.2 *If D is a strong semicomplete multipartite digraph, $D \neq \overleftrightarrow{K}_{1,n-1}$, containing a cycle of length two, then for every cycle xyx of length two in D, we have*

$$\min\{\operatorname{diam}(D - xy), \operatorname{diam}(D - yx)\} \leq max\{5, diam(D) + 1\}.$$

Proof: Assume that the theorem is false and that D is a counter-example to the theorem. By Theorem B, we may assume that D is not bipartite. Let xyx be a cycle of length two in D such that

$$\min\{\operatorname{diam}(D - xy), \operatorname{diam}(D - yx)\} > \max\{5, \operatorname{diam}(D) + 1\}.$$

Therefore, there exist vertices $s_{xy}, t_{xy}, s_{yx}, t_{yx}$ in D such that

$$\operatorname{dist}_{D-xy}(s_{xy}, t_{xy}) > \max\{5, \operatorname{diam}(D) + 1\} \text{ and } \operatorname{dist}_{D-yx}(s_{yx}, t_{yx}) > \max\{5, \operatorname{diam}(D) + 1\}.$$

Let $P = p_0 p_1 \ldots p_l$ be an (s_{xy}, t_{xy})-path in D of minimum length (in particular, $l \leq \operatorname{diam}(D)$) and let $Q = q_0 q_1 \ldots q_m$ be an (s_{yx}, t_{yx})-path in D of minimum length (in particular, $m \leq \operatorname{diam}(D)$). Let ρ and η be defined such that $xy = p_\rho p_{\rho+1}$ and $yx = q_\eta q_{\eta+1}$.

We now consider the following cases, which exhaust all possibilities. Indeed,

$$(\rho, \eta) \in [0, l-1] \times [0, m-1] = [0, l-2] \times [0, m-2] \cup [1, l-1] \times [1, m-1] \cup \{(0, m-1)\} \cup \{(l-1, 0)\},$$

where $[p, q] = \{p, p+1, ..., q\}$.

Case 1: $\rho + 1 < l$, $\eta + 1 < m$. The vertex $p_{\rho+2}$ must belong to the same partite set as x, since otherwise $y \to p_{\rho+2} \to x$ (by the minimality of l). This would give us a contradiction against $\mathrm{dist}_{D-yx}(s_{yx}, t_{yx}) > \max\{5, \mathrm{diam}(D) + 1\}$. Analogously, $q_{\eta+2}$ belongs to the same partite set as y. Since $p_{\rho+2}$ belongs to the same partite set as x and $q_{\eta+2}$ belongs to the same partite set as y, $p_{\rho+2}$ and $q_{\eta+2}$ are adjacent. If $q_{\eta+2} \to p_{\rho+2}$, then $\mathrm{dist}_{D-xy}(s_{xy}, t_{xy}) = \mathrm{dist}_D(s_{xy}, t_{xy})$, a contradiction. Analogously, if $p_{\rho+2} \to q_{\eta+2}$, then we also get a contradiction.

Case 2: $\rho > 0$, $\eta > 0$. This case can be transformed into Case 1 by considering the converse of D.

Case 3: $\rho = 0$, $\eta + 1 = m$. Since D is not bipartite, there exists a vertex in a partite set different from those of x and y. Let w be such a vertex. Clearly, if $x \to w \to y$ or $y \to w \to x$, then we would get a contradiction. Thus, either $\{x, y\} \to w$ or $w \to \{x, y\}$. Assume that $\{x, y\} \to w$ (the other case can be handled analogously).

Let $R = r_0 r_1 \ldots r_t$ be a shortest path from w to p_l in D. Clearly, R must use the arc xy, since otherwise there is a path from $x = p_0$ to p_l of length at most $\mathrm{diam}(D) + 1$ in $D - xy$, a contradiction. Let $r_i = x$ and observe that $i > 1$.

If $\eta = 0$, then the path $y r_0 r_1 \ldots r_i$ is a path from q_0 to q_m of length at most $\mathrm{diam}(D) + 1$ in $D - xy$, a contradiction. Therefore $\eta \geq 1$. Either r_i or r_{i-1} is adjacent to $q_{\eta-1}$, and clearly the arc must go from r_i or r_{i-1} to $q_{\eta-1}$. If $r_i \to q_{\eta-1}$, then we have a path of length 2 from x to y, a contradiction, so $r_{i-1} \to q_{\eta-1}$. However, now the path

$$x r_0 r_1 \ldots r_{i-1} q_{\eta-1} r_{i+1} r_{i+2} \ldots r_t$$

is a (p_0, p_l)-path of length at most $\mathrm{diam}(D) + 1$ in $D - xy$, a contradiction.

Case 4: $\eta = 0$, $\rho + 1 = l$. This case can be transformed into Case 3 by considering the converse of D. $\qquad\square$

Theorem 2.3 *If T is a strong semicomplete digraph with n vertices, $n \geq 3$, and $s_i \geq 1$ for $i = 1, 2, \ldots, n$, then*

$$\mathrm{diam}_{\min}(T(s_1, s_2, \ldots s_n)) \leq max\{3, \mathrm{diam}(T)\}.$$

Proof: By Theorem A, there exists an orientation T' of T such that

$$\mathrm{diam}(T') \leq \max\{3, \mathrm{diam}(T)\}.$$

Let $D = T(s_1, s_2, \ldots, s_n)$ and $D' = T'(s_1, s_2, \ldots, s_n)$. We will show that D' is an orientation of D such that

$$\mathrm{diam}(D') \leq \max\{3, \mathrm{diam}(T')\} \leq \max\{3, \mathrm{diam}(T)\},$$

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which would complete the proof.

Let $V_1, V_2, ..., V_n$ be the partite sets of the semicomplete n-partite digraph D; $|V_i| = s_i$ for $i = 1, 2, ..., n$. Let x and y be arbitrary vertices in D' and assume that $x \in V_i$ and $y \in V_j$. If $i \neq j$, then the path from i to j of length at most $\text{diam}(T')$ in T' can be easily transformed into a path from x to y of length at most $\text{diam}(T')$ in D'. So, we may assume that $i = j$. Let V' be the set of all the vertices dominated by V_i in D' and V'' the set of all the vertices dominating V_i. Clearly, $V(D') = V_i \cup V' \cup V''$ and there must be an arc uw from V' to V'', as otherwise T' would not be strong. Now the path $xuwy$ is a path of length 3 in D', so we are done. $\qquad\square$

The sharpness of the bound of this theorem follows from the sharpness of the bound of Theorem A. Theorem 2.3 also generalizes the following result.

Theorem C [5, 14] *If G is a complete k-partite (undirected) graph and $k \geq 3$, then $\text{diam}_{\min}(G) \leq 3$, and the bound is sharp.*

3   Extended digraphs

The aim of this section is to provide a generalization to the following theorem as well as to some other results in [12].

Theorem D [12] *For a connected graph G of order $n \geq 3$ and integers $s_1, s_2, ..., s_n \geq 2$, we have $\text{diam}(G) \leq \text{diam}_{\min}(G(s_1, s_2, ..., s_n)) \leq \text{diam}(G) + 2$.*

Theorem D is valid for $n \geq 3$. The case $n = 2$ is covered in the following result.

Theorem E [19] *If $s_1 \geq s_2 \geq 2$, then $\text{diam}_{\min}(K_{s_1,s_2}) = 3$ for $s_1 \leq \binom{s_2}{\lfloor s_2/2 \rfloor}$, and $\text{diam}_{\min}(K_{s_1,s_2}) = 4$, otherwise.*

Theorem 3.1 *Let H be a strong digraph of order $n \geq 3$ and let $D = H(s_1, s_2, ..., s_n)$ with $s_i \geq 2$, $1 \leq i \leq n$. Then $\text{diam}(H) \leq \text{diam}_{\min}(D) \leq \text{diam}(H) + 2$.*

Theorem 3.1 can be proved similarly to Theorem D with only some minor alterations and additions needed. However, we provide a proof of Theorem 3.1 for the sake of completeness.

The requirement $n \geq 3$ is important as one can see from Theorem E ($\text{diam}(K_2) = 1$, but $\text{diam}_{\min}(K_{s,2}) = 4$ for $s \geq 3$). Clearly, $\text{diam}(H) \leq \text{diam}(D')$ for every orientation D' of D. To prove the more difficult part of the inequality in Theorem 3.1, we will use the following lemma.

Lemma 3.2 *Let t_i, s_i be integers such that $2 \leq t_i \leq s_i$ for $1 \leq i \leq n$ and let H be a strong digraph with vertices $1, 2, ..., n$, $n \geq 3$. If the digraph $D' = H(t_1, t_2, ..., t_n)$ admits an orientation F' in which every vertex $v = (p, i)$, such that i belongs to a cycle in H of length two, lies on a cycle C_v of length not exceeding m, then $D = H(s_1, s_2, ..., s_n)$ has an orientation F with diameter at most $\max\{m, \mathrm{diam}(F')\}$.*

Proof: Given an orientation F' of D', we define an orientation F of D as follows. We have $(p, i) \rightarrow (q, j)$ in F if and only if one of the following holds:

(a) $p < t_i$, $q < t_j$ and $(p, i) \rightarrow (q, j)$ in F'.

(b) $p < t_i$, $q \geq t_j$ and $(p, i) \rightarrow (t_j, j)$ in F'.

(c) $p \geq t_i$, $q < t_j$ and $(t_i, i) \rightarrow (q, j)$ in F'.

(d) $p \geq t_i$ and $q \geq t_j$ and $(t_i, i) \rightarrow (t_j, j)$ in F'.

Let $u = (p, i)$ and $v = (q, j)$ be a pair of distinct vertices in F. If $i \neq j$, then it is clear that $\mathrm{dist}_F(u, v) \leq \mathrm{diam}(F')$ (we can use obvious modifications of the corresponding paths in F'). We have the same result if $i = j$ but $p < t_i$ or $q < t_j$. Assume that $i = j$, $p \geq t_i$ and $q \geq t_j$. If i belongs to a cycle in H of length two, then using the cycle C_u we conclude that $\mathrm{dist}_F(u, v) \leq m$. If i belongs to no cycle in H of length two, then since u, v dominate and are dominated by the same vertices and since $\mathrm{dist}_F((1, i), (2, i)) \leq \mathrm{diam}(F)$, we have $\mathrm{dist}((p, i), (q, i)) \leq \mathrm{diam}(F)$. $\qquad \square$

Proof of Theorem 3.1: We prove that there exists an orientation D' of D such that $\mathrm{diam}(D') \leq \mathrm{diam}(H) + 2$. If $\mathrm{diam}(H) = 1$, then this claim follows from Theorem C. Thus, we may assume that $\mathrm{diam}(H) \geq 2$.

Define an orientation F' of $H(t_1, t_2, ..., t_n)$, where every $t_i = 2$, as follows:

$$(1, i) \rightarrow (1, j) \rightarrow (2, i) \rightarrow (2, j) \rightarrow (1, i) \text{ if and only if } i < j. \tag{1}$$

Let $u = (p, i)$ and $v = (q, j)$ be a pair of distinct vertices in F'. We show that $\mathrm{dist}_{F'}(u, v) \leq \mathrm{diam}(H) + 2$. Suppose that $i k_1 k_2 \ldots k_s j$ is a path of length $s + 1 = \mathrm{dist}_H(i, j)$ in H. Then the path $Q = (p, i)(k_1^*, k_1)(k_2^*, k_2) \ldots (k_s^*, k_s)(j^*, j)$, where $x^* = 1$ or 2, is of length $\mathrm{dist}_H(i, j)$ in F'. If $j^* = q$, then the last inequality follows. Otherwise, i.e. $j^* \neq q$, the path $Q(3 - k_s^*, k_s)(q, j)$ is of length $\mathrm{dist}_H(i, j) + 2$ in F'. Thus, $\mathrm{dist}_{F'}(u, v) \leq \mathrm{diam}(H) + 2$. Hence, $\mathrm{diam}(F') \leq \mathrm{diam}(H)$. By (1), every vertex (p, i) of F', such that i lies on a cycle in H of length two, belongs to a cycle of length 4. Now this theorem follows from Lemma 3.2. $\qquad \square$

The following result as well as its corollary can be proved similarly to Theorem 2 and its corollary in [12].

Theorem 3.3 *Let H be a strong digraph of order $n \geq 3$ and let $D = H(s_1, s_2, ..., s_n)$ with $s_i \geq 4$, $1 \leq i \leq n$ be of diameter at least four. Then $\mathrm{diam}_{\min}(D) = \mathrm{diam}(H)$.*

Corollary 3.4 *Let H be a strong digraph of order $n \geq 3$ and let $D = H(s_1, s_2, ..., s_n)$ with $s_i \geq 4$, $1 \leq i \leq n$ be of diameter at least three. Then $\mathrm{diam}_{\min}(D) \leq \mathrm{diam}(H) + 1$.*

Some other results in [12] can also be generalized to orientations of digraphs (the results on cycles). Finally, we would like to suggest the following generalization of the conjecture in [12] (the conjecture in [12] is the same as Conjecture 3.5, but only for connected undirected graphs). The conjecture in [12] seems quite difficult; it was settled for trees in [20].

Conjecture 3.5 *Let H be a strong digraph of order $n \geq 3$ and let $D = H(s_1, s_2, ..., s_n)$ with $s_i \geq 2$, $1 \leq i \leq n$ be of diameter at least three. Then $\mathrm{diam}_{\min}(D) \leq \mathrm{diam}(H) + 1$.*

This conjecture is correct for semicomplete digraphs H, see Theorem 2.3, and $H = \overleftrightarrow{T}$, where T is a tree [20].

Acknowledgements

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