Parameterized Complexity for Graph Linear Arrangement Problems

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Linear Arrangements

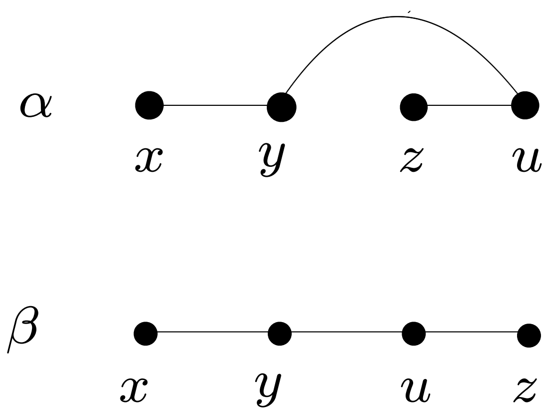
A linear arrangement of a graph $G = (V, E)$ is a one-to-one mapping $\alpha : V \to \{1, \ldots, |V|\}$. The length of an edge $uv \in E$ relative to α is

$$\lambda_\alpha(uv) = |\alpha(u) - \alpha(v)|.$$

The cost $\mathsf{c}(\alpha, G)$ of a linear arrangement α is

$$\mathsf{c}(\alpha, G) = \sum_{e \in E} \lambda_\alpha(e).$$

Linear arrangements of minimal cost are optimal; $\mathsf{ola}(G)$ denotes the cost of an optimal linear arrangement of G.



Linear arrangements α and β with

$c(\alpha, P_4) = 4$ and $c(\beta, P_4) = 3$

Linear Arrangement Problem (LAP)

It is the problem of deciding, given a graph G and an integer k, whether ola$(G) \leq k$.

LAP is a classical NP-complete problem (Garey and Johnson, 1979).

Goldberg and Klipker (1976) were the first to obtain a polynomial-time algorithm for computing optimal linear arrangements of trees. Faster algorithms for trees were obtained by Shiloach (1979) and Chung (1984).

Fixed Parameter Tractability: Definitions

A parameterized problem Π can be considered as a set of pairs (I, k) where I is the problem instance and k (usually an integer) is the parameter. Π is called fixed-parameter tractable (FPT) if membership of (I, k) in Π can be decided in time $O(f(k)|I|^c)$, where $|I|$ is the size of I, $f(k)$ is a computable function, and c is a constant independent from k and I.

A reduction to problem kernel (or kernelization) is a polynomial-time many-to-one transformation from the parameterized problem to itself, such that (i) (I, k) is reduced to (I', k') with $k' \leq ck$, $|I'| \leq g(k)$, for some constant c and some computable function g, and (ii) $(I, k) \in \Pi$ if and only if $(I', k') \in \Pi$. Here, (I', k') is called the problem kernel.

Fixed Parameter Tractability: Approaches

FPT: one of the approaches to solve NP-hard problems

FPT applicability: bioinformatics (e.g., M.A. Langston, clique computation via vertex cover)

FPT applicability: preprocessing rules for exact and approx. computations

Well-known FPT approaches:

(a) Reduction to problem kernel (lots)

(b) Bounded search trees (even more)

(c) Color-coding

Parameterized LAP

The following is a straightforward way to parameterize LAP (Fernau, 2005, Serna and Thilikos, 2005):

Parameterized LAP
Instance: A graph G.
Parameter: A positive integer k.
Question: Does G have a linear arrangement of cost at most k?

An edge has length at least 1 in any LA. Thus, for a graph G with m edges we have $\text{ola}(G) \geq m$. Consequently, parameterized LAP is FPT by trivial reasons.

LAPAGV

Consider the net cost $\mathsf{nc}(\alpha, G)$ of a linear arrangement α defined as follows: $\mathsf{nc}(\alpha, G) = \sum_{e \in E}(\lambda_\alpha(e) - 1)$. The net cost of an optimal LA of G is $\mathsf{ola}^+(G) = \mathsf{ola}(G) - m$. The following parameterization of LAP is due to (Fernau, 2005) who asked whether LAPAGV is FPT ?

LA parameterized above guaranteed value (LAPAGV)
Instance: A graph G.
Parameter: A positive integer k.
Question: Does G have a linear arrangement of net cost at most k?

Lemmas

Lemma 1. *Let* $G_1, \ldots, G_p$ *be the connected components of a graph* G. *Then* $\mathrm{ola}^+(G) = \sum_{i=1}^{p} \mathrm{ola}^+(G_i)$.

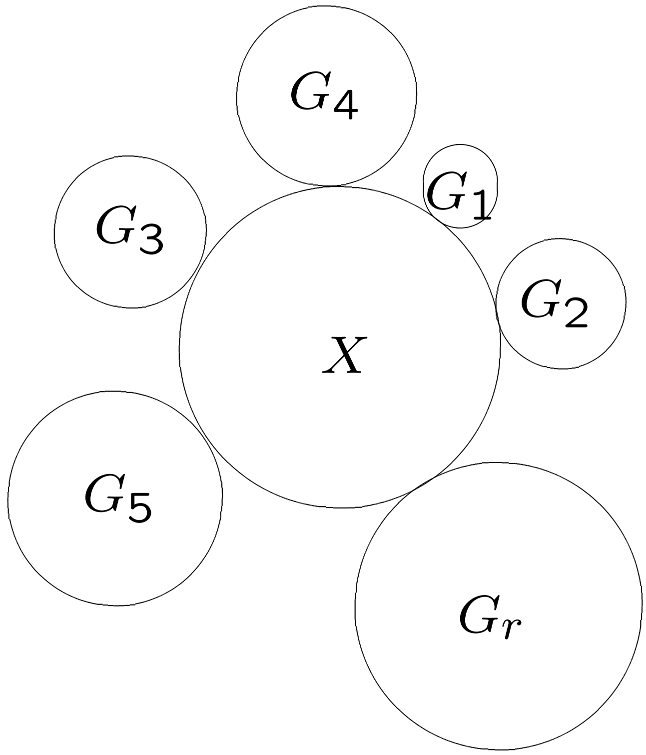
Lemma 2. *If* G *is a connected bridgeless graph of order* $n \geq 1$, *then* $\mathrm{ola}^+(G) \geq (n-1)/2$.

Proof: Assume $n \geq 3$; G is 2-edge-connected. Let α be an optimal linear arrangement of G and $u = \alpha^{-1}(1)$, $w = \alpha^{-1}(n)$.

Since G is 2-edge-connected, by Menger's Theorem there are two paths P, P' between u to w such that $E(P) \cap E(P') = \{u, w\}$. Let $G' = P \cup P'$. We can prove $|E(G')| \leq 3(n-1)/2$. We obtain $\mathrm{ola}^+(G) = \mathrm{nc}(\alpha, G) \geq \mathrm{nc}(\alpha, G') \geq 2(n-1) - |E(G')| \geq (n-1)/2$.

Lemma 3

Lemma 3. *Let* G *be a connected graph. Let* X *be a vertex set of* G *such that* $G[X]$ *is connected and let* $G - X$ *have connected components* $G_1, G_2, \ldots, G_r$ *with* $n_1, n_2, \ldots, n_r$ *vertices, respectively, such that* $n_1 \leq n_2 \leq \ldots \leq n_r$. *Then* $\mathrm{ola}^+(G) \geq \mathrm{ola}^+(G[X]) + \sum_{i=1}^{r-2} n_i$.



$$\mathsf{ola}^{+}(G) \geq \mathsf{ola}^{+}(G[X]) + (n_1 + n_2 + n_3 + n_4)$$

Proof: We call a vertex $u \in V(G)$ α-special if $G - u$ is connected and $\alpha(u) \notin \{1, n\}$. Let α be an optimal LA of G. Assume $r \geq 3$. Each non-trivial G_i has a pair u_i, v_i of distinct vertices such that $G_i - u_i$ and $G_i - v_i$ are connected. If G_i is trivial, then set $u_i = v_i$. Since $r \geq 3$, for some $j \in \{1, 2, \ldots, r\}$, $\alpha(u_j) \notin \{1, n\}$ and $\alpha(v_j) \notin \{1, n\}$. Now we claim that there is a vertex $u \in V(G_j)$ such that $G - u$ is connected. Indeed, we set $u = u_j$ if there are edges between v_j and $G[X]$, we set $u = v_j$, otherwise.

We have proved that G has an α-special vertex u not in X. Note: $\text{ola}^+(G) \geq \text{ola}^+(G - u) + 1$ for an α-special vertex u of G. Procedure: while $G - X$ has a least three components, choose an α-special vertex $u \notin X$ of G and replace G with $G - u$.

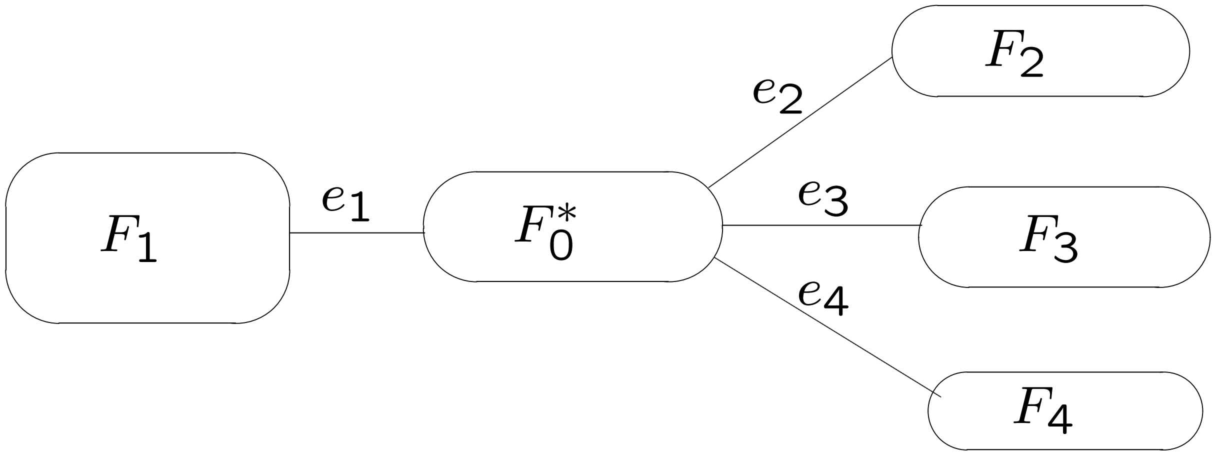
Lemma 4

A bridge e of G is k-separating if both components of $G - e$ have more than k vertices.

Lemma 4. *Let k be a positive integer and let G be a connected graph with n vertices with $ola^+(G) \leq k$. Then either G has a k-separating bridge or $n \leq 4k + 1$.*

Proof: If G is a bridgeless graph, by Lemma 2, $n \leq 2k+1$. Assume: G has a bridge. Choose a bridge e_1 with maximal $\min\{|V(F_1)|, |V(F_0)|\}$, where F_1, F_0 are the components of $G - e_1$. Assume: $|V(F_1)| \leq |V(F_0)|$. Since e_1 is not a k-separating bridge, $|V(F_1)| \leq k$. Let F_0^* the bridgeless component of F_0 containing a vertex incident to e_1. If $F_0 = F_0^*$ then $|V(F_0)| \leq 2k+1$ and we are done; hence we assume that $F_0 \neq F_0^*$.

Let $e_2, \ldots, e_r$ denote the bridges of F_0 that are incident to vertices in F_0^*. Moreover, let $F_2, \ldots, F_r$ denote the corresponding connected components of $F_0 - V(F_0^*)$.



F_1
e_1
F_0^*
e_2
e_3
e_4
F_2
F_3
F_4

Assume: $|V(F_2)| \geq |V(F_3)| \geq \ldots \geq |V(F_r)|$. Easy to see: $|V(F_1)| \geq |V(F_2)|$. By Lemma 3, $\text{ola}^+(G) \geq \text{ola}^+(F_0^*) + \sum_{i=3}^{r} |V(F_i)|$. Thus, $\sum_{i=3}^{r} |V(F_i)| \leq k - \text{ola}^+(F_0^*)$. Since $|V(F_2)| \leq |V(F_1)| \leq k$ and, by Lemma 2, $|V(F_0^*)| \leq 2 \cdot \text{ola}^+(F_0^*) + 1$, we obtain that

$$n = |V(F_0^*)| + \sum_{i=1}^{r} |V(F_i)| \leq (2 \cdot \text{ola}^+(F_0^*) + 1) + (3k - \text{ola}^+(F_0^*)) = 3k + \text{ola}^+(F_0^*) + 1 \leq 4k + 1.$$

Suppressing Lemma

Let G be a graph and let v be a vertex of degree 2 of G. Let vu_1, vu_2 denote be the edges incident with v. Assume that $u_1u_2 \notin E(G)$. We obtain a graph G' from G by removing v (and the edges vu_1, vu_2) from G and adding instead the edge u_1u_2. We say that G' is obtained from G by suppressing vertex v. Furthermore, if the two edges incident with v are k-separating bridges for some positive integer k, then we say that v is k-suppressible.

Lemma 5. *Let G be a connected graph and let v be an* $\text{ola}^+(G)$*-suppressible vertex of G. Then* $\text{ola}^+(G) = \text{ola}^+(G')$ *holds for the graph G' obtained from G by suppressing v.*

Kernel Theorems

Theorem 1. *Let k be a positive integer, and let G be a connected graph without k-suppressible vertices. If* $\text{ola}^+(G) \leq k$, *then G has at most* $5k+2$ *vertices and at most* $6k+1$ *edges.*

Theorem 2. *Let $f(n,m)$ be the time sufficient for checking whether* $\text{ola}^+(G) \leq k$ *for a connected graph G with n vertices and m edges. Then* $f(n,m) = O(m+n+f(5k+2, 6k+1))$.

Theorem 3. *We have* $f(5k+2, 6k+1) = O(5.88^k)$.

Serna-Thilikos Problems (2005)

Vertex Average Min Linear Arrangement

Instance: A graph G.

Parameter: A positive integer k.

Question: Does G have an LA of cost $\leq k|V(G)|$?

Edge Average Min Linear Arrangement

Instance: A graph G.

Parameter: A positive integer k.

Question: Does G have an LA of cost $\leq k|E(G)|$?

Serna-Thilikos Question

Q.: Are the problems FPT ?

Theorem 4. *For each fixed* $k \geq 2$*, both problems are* NP*-complete.*

A.: No, unless P=NP.

Serna-Thilikos Problem on Graph Profile (2005)

For an LA α of $G = (V, E)$, its profile prf$(\alpha, G) =$

$\sum_{v \in V}$ max$\{\alpha(v) - \alpha(u) : u \in N[v]\}$.

Vertex Average Profile
Instance: A graph G.
Parameter: A positive integer k.
Question: Does G have an LA of profile $\leq k|V(G)|$?

Theorem 5. *For each fixed $k \geq 2$, the above problem is* NP-*complete.*