Minimum Cost Homomorphisms to Semicomplete Multipartite Digraphs

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Abstract

For digraphs D and H, a mapping $f: V(D) \to V(H)$ is a *homomorphism of D to H* if $uv \in A(D)$ implies $f(u)f(v) \in A(H)$. For a fixed directed or undirected graph H and an input graph D, the problem of verifying whether there exists a homomorphism of D to H has been studied in a large number of papers. We study an optimization version of this decision problem. Our optimization problem is motivated by a real-world problem in defence logistics and was introduced recently by the authors and M. Tso.

Suppose we are given a pair of digraphs D, H and a cost $c_i(u)$ for each $u \in V(D)$ and $i \in V(H)$. The cost of a homomorphism f of D to H is $\sum_{u \in V(D)} c_{f(u)}(u)$. Let H be a fixed digraph. The minimum cost homomorphism problem for H, MinHOMP(H), is stated as follows: For input digraph D and costs $c_i(u)$ for each $u \in V(D)$ and $i \in V(H)$, verify whether there is a homomorphism of D to H and, if it does exist, find such a homomorphism of minimum cost. In our previous paper we obtained a dichotomy classification of the time complexity of MinHOMP(H) when H is a semicomplete digraph. In this paper we extend the classification to semicomplete k-partite digraphs, $k \geq 3$, and obtain such a classification for bipartite tournaments.

1 Introduction

In our terminology and notation, we follow [1, 5]. In this paper, directed (undirected) graphs have no parallel arcs (edges) or loops. The vertex (arc) set of a digraph G is denoted by $V(G)$ ($A(G)$). The vertex (edge) set of an undirected graph G is denoted by $V(G)$ ($E(G)$). A digraph D obtained from a complete k-partite (undirected) graph G by replacing every edge xy of G with arc xy, arc yx, or both xy and yx, is called a *semicomplete*

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k-partite digraph (or, *semicomplete multipartite digraph* when k is immaterial). The *partite sets* of D are the partite sets of G. A semicomplete k-partite digraph D is *semicomplete* if each partite set of D consists of a unique vertex. A *k-partite tournament* is a semicomplete k-partite digraph with no directed cycle of length 2. Semicomplete k-partite digraphs and its subclasses mentioned above are well-studied in graph theory and algorithms, see, e.g., [1].

For introductions to homomorphisms in directed and undirected graphs, see [1, 12, 14]. For digraphs D and H, a mapping $f: V(D) \rightarrow V(H)$ is a *homomorphism of D to H* if $uv \in A(D)$ implies $f(u)f(v) \in A(H)$. A homomorphism f of D to H is also called an *H-coloring* of D, and $f(x)$ is called a *color* of x for every $x \in V(D)$. We denote the set of all homomorphisms from D to H by $HOM(D, H)$.

For a fixed digraph H, the *homomorphism problem* HOMP(H) is to verify whether, for an input digraph D, there is a homomorphism of D to H (i.e., whether $HOM(D, H) \neq \emptyset$). The problem HOMP(H) has been studied for several families of directed and undirected graphs H, see, e.g., [12, 14]. The well-known result of Hell and Nešetřil [13] asserts that HOMP(H) for undirected graphs is polynomial time solvable if H is bipartite and it is NP-complete, otherwise.

Such a dichotomy classification for all digraphs is unknown and only partial classifications have been obtained; see [14]. For example, Bang-Jensen, Hell and MacGillivray [3] showed that HOMP(H) when H is a semicomplete digraph is polynomial time solvable if H has at most one cycle and HOMP(H) is NP-complete, otherwise. Bang-Jensen and Hell [2] proved that if a bipartite tournament H is a core, then HOMP(H) is polynomial time solvable when H has at most one cycle and HOMP(H) is NP-hard when H has at least two cycles. (A digraph H is a *core* if H does not contain no proper subdigraph H' such that there are both an H'-coloring of H and an H-coloring of H'.)

The authors of [9] introduced an optimization problem on H-colorings for undirected graphs H, MinHOMP(H) (defined below). The problem is motivated by a problem in defence logistics (see [9]) and can be viewed (see [9]) as an important special case of the valued constraint satisfaction problem recently introduced in [4]. In our previous paper [7], we obtained a dichotomy classification for the time complexity of MinHOMP(H) when H is a semicomplete digraph. In this paper, we extend that classification to obtain a dichotomy classification for semicomplete k-partite digraphs H, $k \geq 3$. We also obtain a classification of the complexity of MinHOMP(H) when H is a bipartite tournament. The case of arbitrary semicomplete bipartite digraphs is significantly more complicated and was recently solved in [8]. Another difficult solved case is that of undirected graphs (or, equivalently, of symmetric digraphs) [6]. A digraph D is symmetric if $xy \in A(D)$ implies $yx \in A(D)$. The general case of arbitrary digraphs remains a very hard and interesting open problem.

Suppose we are given a pair of digraphs D, H and a real cost $c_i(u)$ for each $u \in V(D)$ and $i \in V(H)$. The *cost* of a homomorphism f of D to H is $\sum_{u \in V(D)} c_{f(u)}(u)$. For a fixed digraph H, the *minimum cost homomorphism problem* MinHOMP(H) is formulated as follows. For an input digraph D and costs $c_i(u)$ for each $u \in V(D)$ and $i \in V(H)$, verify whether $HOM(D,H) \neq \emptyset$ and, if $HOM(D,H) \neq \emptyset$, find a homomorphism in $HOM(D,H)$ of minimum cost.

For a digraph G, if $xy \in A(G)$, we say that x *dominates* y and y is *dominated* by x (denoted by $x \rightarrow y$). The *out-degee* $d_G^+(x)$ (*in-degree* $d_G^-(x)$) of a vertex x in G is the number of vertices dominated by x (that dominate x). For sets $X, Y \subset V(G)$, $X \rightarrow Y$ means that $x \rightarrow y$ for each $x \in X,\ y \in Y$, but no vertex of Y dominates a vertex in X. A set $X \subseteq V(G)$ is *independent* if no vertex in X dominates a vertex in X. A *k-cycle*, denoted by $\vec{C}_k$, is a directed simple cycle with k vertices. A digraph H is an *extension* of a digraph D if H can be obtained from D by replacing every vertex x of D with a set S_x of independent vertices such that if $xy \in A(D)$ then $uv \in A(H)$ for each $u \in S_x,\ v \in S_y$.

The *underlying graph* $U(G)$ of a digraph G is the undirected graph obtained from G by disregarding all orientations and deleting one edge in each pair of parallel edges. A digraph G is *connected* if $U(G)$ is connected. The *components* of G are the subdigraphs of G induced by the vertices of components of $U(G)$. A digraph G is *strongly connected* if there is a path from x to y for every ordered pair of vertices $x, y \in V(G)$. A *strong component* of G is a maximal induced strongly connected subdigraph of G. A digraph G' is the *converse* of a digraph G if G' is obtained from G by reversing orientations of all arcs.

The rest of the paper is organized as follows. In Section 2, we give all polynomial time solvable cases of MinHOMP(H) when H is semicomplete k-partite digraph, $k \geq 3$, or a bipartite tournament. Section 3 is devoted to a full dichotomy classification of the time complexity of MinHOMP(H) when H is a semicomplete k-partite digraph, $k \geq 3$. A classification of the same problem for H being a bipartite tournament is proved in Section 4.

2 Polynomial Time Solvable Cases

In this section, we will apply the following theorem which was proved in [7] using a powerful result from [4].

Theorem 2.1 *Let H be a digraph and let there exist an ordering $\pi(1), \pi(2), \ldots, \pi(p)$ of the vertices of H satisfying the following Min-Max property: For any pair $\pi(i)\pi(k), \pi(j)\pi(s)$ of arcs in H, we have $\pi(\min\{i,j\})\pi(\min\{k,s\}) \in A(H)$ and $\pi(\max\{i,j\})\pi(\max\{k,s\}) \in A(H)$. Then MinHOMP($H$) is polynomial time solvable.*

The Min-Max property is closely related to a property of digraphs that has long been of interest [11]. We say that a digraph H has the *X-underbar property* if its vertices can be ordered $\pi(1), \pi(2), \ldots, \pi(p)$ so that $\pi(i)\pi(r), \pi(j)\pi(s) \in A(H)$ implies that $\pi(\min\{i,j\})\pi(\min\{r,s\}) \in A(H)$. It is interesting that the X-underbar property is sufficient to ensure that the list homomorphism problem for H has a polynomial solution [14].

Let TT_p denote the acyclic tournament on $p \geq 1$ vertices. Let $p \geq 3$ and let TT_p^- be a digraph obtained from TT_p by deleting the arc from the vertex of in-degree zero to the vertex of out-degree zero. In [7], we proved the following result for TT_p using Theorem 2.1. Thus, our proof is only for TT_p^-.

Lemma 2.2 *The problems* MinHOMP(TT_p) *and* MinHOMP(TT_p^-) *are polynomial time solvable for* $p \geq 1$ *and* $p \geq 3$*, respectively.*

Proof: Let $V(TT_p^-) = \{\pi(1), \pi(2), \ldots, \pi(p)\}$ and let $A(TT_p^-) = \{pi(i)\pi(j) : \ 1 \leq i < j \leq p, \ j - i < p - 1\}$. Let $\pi(i)\pi(k)$ and $\pi(j)\pi(s)$ be distinct arcs in TT_p^-. Observe that $\pi(\min\{i,j\})\pi(\min\{k,s\}) \neq \pi(1)\pi(p)$ and $\pi(\max\{i,j\})\pi(\max\{k,s\}) \neq \pi(1)\pi(p)$. Thus, $\pi(\min\{i,j\})\pi(\min\{k,s\})$ and $\pi(\max\{i,j\})\pi(\max\{k,s\})$ are arcs in TT_p^-. Therefore, MinHOMP(TT_p^-) is polynomial time solvable by Theorem 2.1. $\diamond$

Lemma 2.3 *Suppose that MinHOMP(H) is polynomial time solvable. Then, for each extension H' of H, MinHOMP(H') is also polynomial time solvable.*

Proof: Recall that we can obtain H' from H by replacing every vertex $i \in V(H)$ with a set S_i of independent vertices. Consider an H'-coloring h' of an input digraph D. We can reduce h' into an H-coloring of D as follows: if $h'(u) \in S_i$, then $h(u) = i$.

Let $u \in V(D)$. Assign $\min\{c_j(u) : \ j \in S_i\}$ to be a new cost $c_i(u)$ for each $i \in V(H)$. Observe that we can find an optimal H-coloring h of D with the new costs in polynomial time and transform h into an optimal H'-coloring of D with the original costs using the obvious inverse of the reduction described above. $\diamond$

In [7], we proved that MinHOMP(H) is polynomial time solvable when $H = \vec{C}_p$, $p \geq 2$. Combining this results with Lemmas 2.2 and 2.3, we immediately obtain the following:

Theorem 2.4 *If H is an extension of TT_p ($p \geq 1$), $\vec{C}_p$ ($p \geq 2$) or TT_p^- ($p \geq 3$), then MinHOMP(H) is polynomial time solvable.*

It seems that it is not possible to prove the following result by a straightforward application of Theorem 2.1. Nevertheless, our proof uses Theorem 2.1 in a somewhat indirect way via Lemma 2.2.

Theorem 2.5 *Let H be an acyclic bipartite tournament. Then MinHOMP(H) is polynomial time solvable.*

Proof: Let V_1, V_2 be the partite sets of H, let D be an input digraph and let $c_i(x)$ be the costs, $i \in V(H), \ x \in V(D)$. Observe that if D is not bipartite, then $HOM(D, H) = \emptyset$, so we may assume that D is bipartite. We can check whether D is bipartite in polynomial time. Let U_1, U_2 be the partite sets of D.

To prove that we can find a minimum cost H-coloring of D in polynomial time, it suffices to show that we can find a minimum cost H-coloring f of D such that $f(U_1) \subseteq V_1$ and $f(U_2) \subseteq V_2$. Indeed, if D is connected, to find a minimum cost H-coloring of D we can choose from a minimum cost H-coloring f with $f(U_1) \subseteq V_1$ and $f(U_2) \subseteq V_2$ and a minimum cost H-coloring h of D with $h(U_1) \subseteq V_2$ and $h(U_2) \subseteq V_1$. If D is not connected, we can find a minimum cost H-coloring of each component of D separately.

To force $f(U_1) \subseteq V_1$ and $f(U_2) \subseteq V_2$ for each H-coloring f, it suffices to modify the costs such that it is too expensive to assign any color from V_j to a vertex in U_{3-j}, $j = 1, 2$. Let $M = |V(D)| \cdot \max\{c_i(x) : \ i \in V(H), \ x \in V(D)\} + 1$ and replace $c_i(x)$ by $c_i(x) + M$ for each pair $x \in U_j$, $i \in V_{3-j}$, $j = 1, 2$.

Observe that the vertices of H can be ordered $i_1, i_2, \ldots, i_p$ such that i_k is an arbitrary vertex of in-degree zero in $H - \{i_1, i_2, \ldots, i_{k-1}\}$ for every $k \in \{1, 2, \ldots, p\}$. Thus, H is a subdigraph of TT_p with vertices $i_1, i_2, \ldots, i_p$ ($i_s i_t \in A(TT_p)$ if and only if $s < t$). Observe that

$$\{f \in HOM(D, H) : f(U_j) \subseteq V_j, \ j = 1, 2\} = \{f \in HOM(D, TT_p) : f(U_j) \subseteq V_j, \ j = 1, 2\}.$$

Thus, to solve MinHOMP(H) with the modified costs it suffices to solve MinHOMP(TT_p) with the same costs (this will solve MinHOMP(H) under the assumption $f(U_j) \subseteq V_j$). We can solve the latter in polynomial time by Lemma 2.2. $\diamond$

3 Classification for semicomplete k-partite digraphs, $k \geq 3$

The following lemma allows us to prove that MinHOMP(H) is NP-hard when MinHOMP(H') is NP-hard for an induced subdigraph H' of H.

Lemma 3.1 *[7] Let H' be an induced subdigraph of a digraph H. If MinHOMP(H') is NP-hard, then MinHOMP(H) is also NP-hard.*

The following lemma is the NP-hardness part of the main result in [7].

Lemma 3.2 *Let H be a semicomplete digraph containing a cycle and let $H \notin \{\vec{C}_2, \vec{C}_3\}$. Then MinHOMP($H$) is NP-hard.*

The following lemma was proved in [10]. The digraph H_1 from the lemma is depicted in Fig. 1 (a).

Lemma 3.3 *Let H_1 be a digraph obtained from $\vec{C}_3$ by adding an extra vertex dominated by two vertices of the cycle and let H be H_1 or its converse. Then* HOMP(H) *is NP-complete.*

We need two more lemmas for our classification. The digraph H' from the next lemma is depicted in Fig. 1 (b).

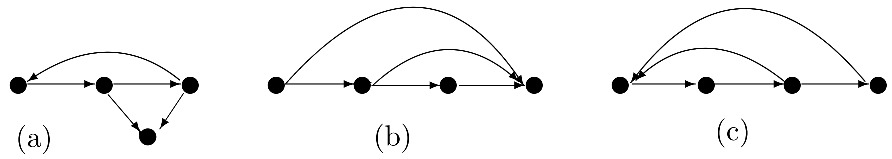

Figure 1: Graphs used in Lemmas 3.3, 3.4 and 3.5

Lemma 3.4 *Let H' be given by $V(H') = \{1, 2, 3, 4\}$, $A(H') = \{12, 23, 34, 14, 24\}$ and let H be H' or its converse. Then MinHOMP(H) is NP-hard.*

Proof: Let $H = H'$. We reduce the maximum independence set problem (MISP) to MinHOMP(H). Let G be an arbitrary graph without isolated vertices. We construct a digraph D from G as follows: every vertex of G belongs to D and, for each pair x, y of adjacent vertices of G, we add to D new vertices u_{xy} and v_{xy} together with arcs $u_{xy}x, u_{xy}v_{xy}, v_{xy}y$. (No edge of G is in D.) Let n be the number of vertices in D. Let x, y be an adjacent pair of vertices in G. We set $c_3(x) = c_3(y) = c_i(u_{xy}) = c_i(v_{xy}) = 1$ for $i = 1, 2, 3$, $c_4(x) = c_4(y) = n + 1$ and $c_j(x) = c_j(y) = c_4(u_{xy}) = c_4(v_{xy}) = n^2 + n + 1$ for $j = 1, 2$.

Consider a mapping $f : V(D) \rightarrow V(H)$ such that $f(z) = 4$ for each $z \in V(G)$ and $f(u_{xy}) = 1$, $f(v_{xy}) = 2$ for each pair x, y of adjacent vertices of G. Observe that f is an H-coloring of D of cost smaller than $n^2 + n + 1$.

Consider now a minimum cost H-coloring h of D. Let x, y be a pair of adjacent vertices in G. Due to the values of the costs, h can assign x, y only colors 3 and 4 and u_{xy}, v_{xy} only colors 1,2,3. The coloring can assign u_{xy} either 1 or 2 as otherwise v_{xy} must be assigned color 4. If u_{xy} is assigned 1, then v_{xy}, y, x must be assigned 2, (3 or 4) and 4, respectively. If u_{xy} is assigned 2, then v_{xy}, y, x must be assigned 3,4 and (3 or 4), respectively. In both cases, only one of the vertices x and y can receive color 3. Since h is optimal, the maximum number of vertices in D that it inherited from G must be assigned color 3. This number is the maximum number of independent vertices in G. Since MISP is NP-hard, so is MinHOMP(H). ◇

The digraph H from the next lemma is depicted in Fig. 1 (c).

Lemma 3.5 *Let H be given by $V(H) = \{1, 2, 3, 4\}$, $A(H) = \{12, 23, 31, 34, 41\}$. Then MinHOMP(H) is NP-hard.*

Proof: We will reduce the maximum independent set problem to MinHOMP(H). However before we do this we consider a digraph $D^{gadget}(u, v)$ defined as follows (see Fig. 2):

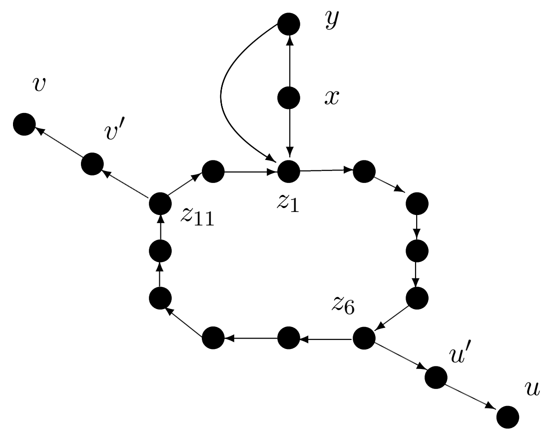

Figure 2: Gadget for Lemma 3.5

$V(D^{gadget}(u,v)) = \{x, y, u', u, v', v, z_1, z_2, \ldots, z_{12}\}$ and

$$A(D^{gadget}(u,v)) = \{xy, xz_1, yz_1, z_6u', u'u, z_{11}v', v'v, z_1z_2, z_2z_3, z_3z_4, \ldots, z_{11}z_{12}, z_{12}z_1\}$$

Observe that in any homomorphism f of $D^{gadget}(u,v)$ to H we must have $f(z_1) = 1$ since vertices x, y, z_1 can only map to 3,4,1, respectively. This implies that $(f(z_1), f(z_2), \ldots, f(z_{12}))$ has to coincide with one of the following two sequences:

$$(1,2,3,1,2,3,1,2,3,1,2,3) \text{ or } (1,2,3,4,1,2,3,4,1,2,3,4).$$

If the first sequence is the actual one, then we have $f(z_6) = 3$, $f(u') \in \{1,4\}$, $f(u) \in \{1,2\}, f(z_{11}) = 2$, $f(v') = 3$ and $f(v) \in \{1,4\}$. If the second sequence is the actual one, then we have a symmetrical situation $f(z_6) = 2$, $f(u') = 3$, $f(u) \in \{1,4\}, f(z_{11}) = 3$, $f(v') \in \{1,4\}$ and $f(v) \in \{1,2\}$. Notice that we cannot assign color 2 to both u and v in a homomorphism.

Let G be a graph. Construct a digraph D as follows. Start with $V(D) = V(G)$ and, for each edge $uv \in E(G)$, add a distinct copy of $D^{gadget}(u,v)$ to D (notice that u and v are not copied but shared among gadgets). Note that the vertices in $V(G)$ form an independent set in D and that $|V(D)| = |V(G)| + 16|E(G)|$.

Let all costs $c_i(t) = 1$ for $t \in V(D)$ apart from $c_j(p) = 2$ for all $p \in V(G)$ and $j \in \{1,4\}$. Clearly, a minimum cost H-coloring h of D must aim at assigning as many vertices of $V(G)$ in D a color different from 1 and 4. However, if pq is an edge in G, by the arguments above, h cannot assign color 2 to both p and q; h can assign color 2 to either p or q (or neither). Thus, a minimum cost homomorphism of D to H corresponds to a maximum independent set in G and vise versa (the vertices of a maximum independent set are assigned color 2 and all other vertices in $V(G)$ are assigned color 1). $\diamond$

Theorem 3.6 *Let H be a semicomplete k-partite digraph, $k \geq 3$. If H is an extension of TT_k, $\vec{C}_3$ or TT^-_{k+1}, then MinHOMP(H) is polynomial time solvable. Otherwise, MinHOMP(H) is NP-hard.*

Proof: Since H is a semicomplete k-partite digraph, $k \geq 3$, if H has a cycle, then there can be three possibilities for the length of a shortest cycle in H: 2,3 or 4. Thus, we consider four cases: the above three cases and the case when H is acyclic.

Case 1: H has a 2-cycle C. Let i, j be vertices of C. The vertices i, j together with a vertex from a partite set different from those where i, j belong to form a semicomplete digraph with a 2-cycle. Thus, by Lemmas 3.1 and 3.2, MinHOMP(H) is NP-hard.

Case 2: A shortest cycle C of H has three vertices ($C = ijli$). If H has at least four partite sets, then MinHOMP(H) can be shown to be NP-hard similarly to Case 1. Assume that H has three partite sets and that MinHOMP(H) is not NP-hard. Let V_1, V_2 and V_3 be partite sets of H such that $i \in V_1$, $j \in V_2$ and $l \in V_3$. Consider a vertex $s \in V_1$ outside C. If s is dominated by j and l or dominates j and l, then MinHOMP(H) is NP-hard by Lemmas 3.1 and 3.3, a contradiction. If $j \rightarrow s \rightarrow l$, then MinHOMP($H$) is NP-hard by Lemmas 3.1 and 3.5, a contradiction. Thus, $l \rightarrow s \rightarrow j$. Similar arguments show that $l \rightarrow V_1 \rightarrow j$. Consider $p \in V_2$. Similar arguments show that $p \rightarrow V_1 \rightarrow j$ and moreover $V_3 \rightarrow V_1 \rightarrow j$. Again, similarly we can prove that $V_3 \rightarrow V_1 \rightarrow V_2$, i.e., H is an extension of $\vec{C}_3$.

Case 3: A shortest cycle C of H has four vertices ($C = ijsti$). Since C is a shortest cycle, i, s belong to the same partite set, say V_1, and j, t belong to the same partite set, say V_2. Since H is not bipartite, there is a vertex l belonging to a partite set different from V_1 and V_2. Since H has no cycle of length 2 or 3, either l dominates $V(C)$ or $V(C)$ dominates l. Consider the first case ($l \rightarrow V(C)$) as the second one can be tackled similarly. Let H' be the subdigraph of H induced by the vertices l, i, j, s. Observe now that MinHOMP(H) is NP-hard by Lemmas 3.1 and 3.4.

Case 4: H has no cycle. Assume that MinHOMP(H) is not NP-hard, but H is not an extension of an acyclic tournament. The last assumption implies that there is a pair of nonadjacent vertices i, j and a distinct vertex l such that $i \rightarrow l \rightarrow j$. Let s be a vertex belonging to a partite set different from the partite sets which i and l belong to. Without loss of generality, assume that at least two vertices in the set $\{i, j, l\}$ dominate s. If all three vertices dominate s, then by Lemmas 3.1 and 3.4, MinHOMP(H) is NP-hard, a contradiction. Since H is acyclic, we conclude that $\{i, l\} \rightarrow s \rightarrow j$. Let V_1 be the partite set of i and j. Similar arguments show that for each vertex $t \in V(H) - V_1$, $i \rightarrow t \rightarrow j$. By considering a vertex $p \in V_1 - \{i, j\}$ and using arguments similar to the ones applied above, we can show that either $p \rightarrow (V(H) - V_1)$ or $(V(H) - V_1) \rightarrow p$. This implies that we can partition V_1 into V_1' and V_1'' such that $V_1' \rightarrow (V(H) - V_1) \rightarrow V_2''$. This structure of H implies that there is no pair a, b of nonadjacent vertices in $V(H) - V_1$ such that $a \rightarrow c \rightarrow b$ for some vertex $c \in V(H)$ since otherwise the problem is NP-hard by Lemmas 3.1 and 3.4. Thus, the subdigraph $H - V_1$ is an extension of an acyclic tournament and, therefore, H is an extension of TT^-_{k+1}. $\diamond$

4 Classification for bipartite tournaments

The following lemma can be proved similarly to Lemma 3.3.

Lemma 4.1 *Let H_1 be given by $V(H_1) = \{1, 2, 3, 4, 5\}$, $A(H_1) = \{12, 23, 34, 41, 15, 35\}$ and let H be H_1 or its converse. Then MinHOMP(H) is NP-hard.*

Now we can obtain a dichotomy classification for MinHOMP(H) when H is a bipartite tournament.

Theorem 4.2 *Let H be a bipartite tournament. If H is acyclic or an extension of a 4-cycle, then MinHOMP(H) is polynomial time solvable. Otherwise, MinHOMP(H) is NP-hard.*

Proof: If H is an acyclic bipartite tournament or an extension of a 4-cycle, then MinHOMP(H) is polynomial time solvable by Theorems 2.5 and 2.4. We may thus assume that H has a cycle C, but H is not an extension of a cycle. We have to prove that MinHOMP(H) is NP-hard.

Let C be a shortest cycle of H. Since H is a bipartite tournament, we have $|V(C)| = 4$. Thus, we may assume, without loss of generality, that $C = i_1i_2i_3i_4i_1$, where i_1, i_3 belong to a partite set V_1 of H and i_2, i_4 belong to the other partite set V_2 of H.

We may assume that any vertex in V_1 dominates either i_2 or i_4 and is dominated by the other vertex in $\{i_2, i_4\}$, as otherwise we are done by Lemmas 4.1 and 3.1. Analogously any vertex in V_2 dominates exactly one of the vertices in $\{i_1, i_3\}$. Therefore, we may partition the vertices in H into the following four sets.

$$J_1 = \{j_1 \in V_1 : \ i_4 \to j_1 \to i_2\} \qquad J_2 = \{j_2 \in V_2 : \ i_1 \to j_2 \to i_3\}$$
$$J_3 = \{j_3 \in V_1 : \ i_2 \to j_3 \to i_4\} \qquad J_4 = \{j_4 \in V_2 : \ i_3 \to j_4 \to i_1\}$$

If $q_2q_1 \in A(H)$, where $q_j \in J_j$ for $j = 1, 2$ then we are done by Lemmas 4.1 and 3.1 (consider the cycle $q_1i_2i_3i_4q_1$ and the vertex q_2 which dominates both q_1 and i_3). Thus, $J_1 \to J_2$ and analogously we obtain that $J_2 \to J_3 \to J_4 \to J_1$, so H is an extension of a cycle, a contradiction. $\diamond$

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