Introduction to the Minimum Cost Homomorphism Problem for Directed and Undirected Graphs

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Abstract

For digraphs D and H, a mapping $f: V(D) \to V(H)$ is a homomorphism of D to H if $uv \in A(D)$ implies $f(u)f(v) \in A(H)$. If, moreover, each vertex $u \in V(D)$ is associated with costs $c_i(u), i \in V(H)$, then the cost of the homomorphism f is $\sum_{u \in V(D)} c_{f(u)}(u)$. For each fixed digraph H, we have the *minimum cost homomorphism problem for* H. The problem is to decide, for an input graph D with costs $c_i(u)$, $u \in V(D), i \in V(H)$, whether there exists a homomorphism of D to H and, if one exists, to find one of minimum cost. Minimum cost homomorphism problems encompass (or are related to) many well studied optimization problems.

1 Introduction, Terminology and Notation

The purpose of this paper is twofold: (a) To give an introduction to a new area, the minimum cost homomorphism problem for graphs; (b) To initiate the study of the problem for digraphs with possible loops and to prove new results for such digraphs. Notice that our proofs of the new results illustrate approaches known from the literature.

In most of the literature on digraphs, only digraphs without loops and multiple arcs are studied. This is justified by the fact that multiple arcs and especially loops play no role when we investigate path and cycle structure of digraphs and several other central topics. However, digraphs with possible loops appear naturally in many applications such as digraphs of relations, automata, Markov chains, etc. In homomorphism problems for undirected graphs, graphs with loops are often investigated [6, 12, 14] (multiple edges are of no interest for such problems). Nevertheless, loops are usually not taken into consideration when digraph homomorphisms are studied [12, 14] (one notable exception is

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[4]). We think the situation should be changed and we initiate research of the minimum cost homomorphism problem for digraphs with possible loops.

In this paper, we consider only directed (undirected) graphs that do not have multiple arcs (edges), but may have loops. If a directed (undirected) graph G has no loops, we call G *loopless*. If a directed (undirected) graph G has a loop at every vertex, we call G *reflexive*. When we wish to stress that a family of digraphs contains digraphs with loops we will say digraphs *with possible loops (w.p.l.)* For an undirected graph H, $V(H)$ and $E(H)$ denote its vertex and edge sets, respectively. For a digraph H, $V(H)$ and $A(H)$ denote its vertex and arc sets, respectively.

For directed (undirected) graphs G and H, a mapping $f: V(G) \rightarrow V(H)$ is a *homomorphism of G to H* if uv is an arc (edge) implies that $f(u)f(v)$ is an arc (edge). A homomorphism f of G to H is also called an *H-coloring* of G, and $f(x)$ is called the *color* of the vertex x in G. We denote the set of all homomorphisms from G to H by $HOM(G,H)$. Recent treatments of homomorphisms in directed and undirected graphs can be found in [12, 14]. Let H be a fixed directed or undirected graph. The *homomorphism problem* for H asks whether a directed or undirected input graph G admits a homomorphism to H. The *list homomorphism problem* for H asks whether a directed or undirected input graph G with lists (sets) $L_u \subseteq V(H), u \in V(G)$ admits a homomorphism f to H in which $f(u) \in L_u$ for each $u \in V(G)$.

Suppose G and H are directed (or undirected) graphs, and $c_i(u)$, $u \in V(G)$, $i \in V(H)$ are real *costs*. The *cost of a homomorphism* f of G to H is $\sum_{u \in V(G)} c_{f(u)}(u)$. If H is fixed, the *minimum cost homomorphism problem*, MinHOM(H), for H is the following optimization problem. Given an input graph G, together with costs $c_i(u)$, $u \in V(G)$, $i \in V(H)$, we wish to find a minimum cost homomorphism of G to H, or state that none exists.

The minimum cost homomorphism problem was introduced in [10], where it was motivated by a real-world problem in defence logistics. We believe it offers a practical and natural model for optimization of weighted homomorphisms. The problem's special cases include the list homomorphism problem [12, 14] and the general optimum cost chromatic partition problem, which has been intensively studied [11, 15, 16], and has a number of applications [19, 21].

By *directed path (cycle)* we mean a simple directed path (cycle) (i.e., with no self-crossing). We assume that a directed cycle has at least two vertices. In particular, a loop is not a cycle. A directed cycle with k vertices is called a *directed k-cycle* and denoted by $\vec{C}_k$. A digraph H is *cyclic* if H has a cycle and *acyclic* if H has no cycle.

Let H be a digraph. The *converse* of H is the digraph obtained from H by replacing every arc xy with the arc yx. If xy is an arc of a digraph H, we will say that x *dominates* y, y *is dominated by* x and denote it by $x \rightarrow y$. For a pair X, Y of vertex sets of a digraph H, we define $X \times Y = \{xy : x \in X, y \in Y\}$; $X \rightarrow Y$ means that $x \rightarrow y$ for each $x \in X, y \in Y$.

The *underlying graph* $UN(H)$ is the undirected graph obtained from H by disregarding all orientations and deleting an edge in each pair of parallel edges.

An *empty digraph* is a digraph with no arcs. A loopless digraph D is a *tournament* (*semicomplete digraph*) if there is exactly one arc (at least one arc) between every pair of vertices. We will consider *tournaments with possible loops (w.p.l.)*, i.e., digraphs obtained from tournaments by appending some number of loops (possibly zero loops). A *k-partite tournament* (*semicomplete k-partite digraph*) is a digraph obtained from a complete k-partite graph by replacing every edge xy with one of the two arcs xy, yx (with at least one of the arcs xy, yx). We will also consider k-partite tournaments w.p.l. An acyclic semicomplete digraph on p vertices with no loops is denoted by TT_p and called a *transitive tournament*. The vertices of a transitive tournament TT_p can be labelled $1, 2, \ldots, p$ such that $ij \in A(TT_p)$ if and only if $1 \le i < j \le p$. By TT_p^- ($p \ge 2$), we denote TT_p without the arc $1p$.

Let H be a loopless digraph with vertices $x_1, x_2, \ldots, x_p$ and let $S_1, S_2, \ldots, S_p$ be digraphs. Then the *composition* $H[S_1, S_2, \ldots, S_p]$ is the digraph obtained from H by replacing x_i with S_i for each $i = 1, 2, \ldots, p$. In other words,

$$V(H[S_1, S_2, \ldots, S_p]) = V(S_1) \cup V(S_2) \cup \ldots \cup V(S_p) \text{ and}$$

$$A(H[S_1, S_2, \ldots, S_p]) = \cup\{V(S_i) \times V(S_j) : \ x_ix_j \in A(H), \ 1 \le i \ne j \le p\} \cup (\cup_{i=1}^{p} A(S_i)).$$

If every S_i is an empty digraph, the composition $H[S_1, S_2, \ldots, S_p]$ is called an *extension* of H.

2 Polynomial Time Solvable Cases

In the three subsections of this section, we describe various approaches used in the literature to prove that MinHOM(H) is polynomial time solvable for a certain directed or undirected graph H. While the approaches described in Subsection 2.1 are elementary, the methods described in the other two subsections are more sophisticated.

2.1 Directed Cycles and Extensions

Recall that $\vec{C}_k$ denotes a directed cycle on k vertices, $k \ge 2$; let $V(\vec{C}_k) = \{1, 2, \ldots, k\}$ and $A(\vec{C}_k) = \{12, 23, \ldots, (k-1)k\} \cup \{k1\}$. One can check whether $HOM(D, \vec{C}_k) \ne \emptyset$ using the following algorithm $\mathcal{A}$ from Section 1.4 of [14]. First, we may assume that D is connected (i.e., its underlying undirected graph is connected) as otherwise $\mathcal{A}$ can be applied to each component of D separately. Choose a vertex x of D and assign it color 1. Assign every out-neighbor of x color 2 and each in-neighbor of x color k. For every vertex y with color i, we assign every out-neighbor of y color $i + 1$ modulo k and every in-neighbor of y color

$i-1$ modulo k. We have $HOM(D, \vec{C}_k) \neq \emptyset$ if and only if no vertex is assigned different colors.

To solve MinHOM(H) for $H = \vec{C}_k$, choose an initial vertex x in each component D' of D (a component of its underlying graph). Using the algorithm $\mathcal{A}$ from the previous paragraph, we can check whether each D' admits an H-coloring. If the coloring of D' exists, we compute the cost of this coloring and compute the costs of the other $k-1$ H-colorings when x is colored $2, 3, \dots, k$, respectively. Thus, we can find a minimum cost homomorphism in $HOM(D', H)$. Thus, in polynomial time, we can obtain a H-coloring of the whole digraph D of minimum cost. In other words, we have the following lemma, which was first proved in [7].

Lemma 2.1 *For $H = \vec{C}_k$, $k \geq 2$, MinHOM(H) is polynomial time solvable.*

The following simple lemma proved in [8] is useful when dealing with extensions of certain digraphs.

Lemma 2.2 *Let H be a loopless digraph. If MinHOM(H) is polynomial time solvable then, for each extension H' of H, MinHOM(H') is also polynomial time solvable.*

Proof: Recall that we can obtain H' from H by replacing every vertex $i \in V(H)$ with an empty digraph S_i. Consider an H'-coloring h' of an input digraph D. We can reduce h' into an H-coloring of D as follows: if $h'(u) \in S_i$, then $h(u) = i$.

Let $u \in V(D)$. Assign $\min\{c_j(u) : j \in S_i\}$ to be a new cost $c_i(u)$ for each $i \in V(H)$. Observe that we can find an optimal H-coloring h of D with the new costs in polynomial time and transform h into an optimal H'-coloring of D with the original costs using the obvious inverse of the reduction described above. $\diamond$

Corollary 2.3 *Let H be an extension of a directed cycle. Then MinHOM(H) is polynomial time solvable.*

2.2 Min-Max Ordering Theorem

Let H be a digraph and let $v_1, v_2, \dots, v_p$ be an ordering of vertices of H. Let $e = v_i v_r$ and $f = v_j v_s$ be two arcs in H. The pair $v_{\min\{i,j\}} v_{\min\{s,r\}}$ ($v_{\max\{i,j\}} v_{\max\{s,r\}}$) is called the *minimum* (*maximum*) of the pair e, f. (The minimum (maximum) of two arcs is not necessarily an arc.) An ordering $v_1, v_2, \dots, v_p$ is a *Min-Max ordering of $V(H)$* if both minimum and maximum of every two arcs in H are in $A(H)$. Two arcs $e, f \in A(H)$ are called a *non-trivial pair* if $\{e, f\} \neq \{g', g''\}$, where g' (g'') is the minimum (maximum) of e, f. Clearly, to check that an ordering is Min-Max, it suffices to verify that the minimum and maximum of every non-trivial pair of arcs are arcs, too.

The following approach is based on some results for the *valued constraint satisfaction problem (VCSP)* [2, 3]. Let Z be the set consisting of of all nonnegative integers and ∞, and let Φ be a set of functions $\phi:\ W^{r(\phi)}\rightarrow Z$, where $r(\phi)$ is the arity of ϕ. An instance $\mathcal{I}$ of VCSP(Φ) is a triple (V, W, C), where V is a finite set of *variables*, which are to be assigned values from the set W, and C is a set of (valued) constraints. Each element of C is a pair $c = (\sigma, \phi)$, where σ is a $|\sigma|$-tuple of variables and $\phi:\ W^{|\sigma|}\rightarrow Z$ is a (cost) function, $\phi \in \Phi$. An *assignment* for $\mathcal{I}$ is a mapping s from V to W. The *cost* of s is defined as follows: $c_{\mathcal{I}}(s) = \sum_{((v_1,\ldots,v_m),\phi)\in C} \phi(s(v_1), \ldots, s(v_m))$. An *optimal solution* of $\mathcal{I}$ is an assignment s of minimum cost.

Let W be a totally ordered set. A binary function $\phi:\ W^2\rightarrow Z$ is called *submodular* if, for all $x, y, u, v \in W$, we have

$$\phi(\min\{x, u\}, \min\{y, v\}) + \phi(\max\{x, u\}, \max\{y, v\}) \leq \phi(x, y) + \phi(u, v).$$

The following theorem is the main 'positive' result in [3].

Theorem 2.4 *For each Φ consisting of some unary functions and some binary submodular functions, VCSP(Φ) can be solved in time $O(|V|^3|W|^3)$.*

The following theorem was proved in [7] for loopless digraphs. In fact, the same proofs is valid for digraphs with possible loops. We give the proof from [7] for the sake of completeness. In fact, the theorem can be proved directly using a transformation from MinHOM(H) to the minimum cut problem in a special flow network similar to networks used in [3, 6, 9, 17].

Theorem 2.5 *Let H be a digraph and let an ordering $1, 2, \ldots, p$ of $V(H)$ is a Min-Max ordering, i.e., for any pair ik, js of arcs in H, we have $\min\{i, j\}\min\{k, s\} \in A(H)$ and $\max\{i, j\}\max\{k, s\} \in A(H)$. Then MinHOM($H$) is polynomial time solvable.*

Proof: Let $1, 2, \ldots, p$ be a Min-Max ordering of vertices of H. The Min-Max property of the ordering ensures that the binary function ϕ, defined by $\phi(i, j) = 0$ if $ij \in A(H)$ and $\phi(i, j) = \infty$ otherwise, is submodular. We will reduce MinHOM(H) to VCSP(Φ), where Φ satisfies the conditions of Theorem 2.4. Let $\phi_u(i) = c_i(u)$ for all $u \in V(D)$ and $i \in V(H)$. Let $V = V(D)$ and $W = V(H)$. An assignment is an arbitrary function f from $V(D)$ to $V(H)$. Let $C = C' \cup C''$, where $C' = \{(u, \phi_u):\ u \in V(D)\}$ (for a fixed u, ϕ_u is a unary function from $V(H)$ to Z) and $C'' = \{((u, v), \phi_{uv}):\ uv \in A(D)\}$, where each $\phi_{uv} = \phi$. Since each ϕ_{uv} is submodular, $\Phi = \{\phi_u:\ u \in V(D)\} \cup \{\phi_{uv}:\ uv \in A(D)\}$ satisfies the conditions of Theorem 2.4.

Let $\mathcal{I}$ be an instance of the above-constructed VCSP(Φ). It remains to observe that, if an assignment f is an H-coloring of D, then

$$c_{\mathcal{I}}(f) = \sum_{u\in V(D)} \phi_u(f(u)) + \sum_{uv\in A(D)} \phi_{uv}(f(u), f(v)) = \sum_{u\in V(D)} c_{f(u)}(u),$$

which is the cost of f in MinHOM(H) (an integer), and if f is not an H-coloring, then $c_{\mathcal{I}}(f) = \infty$. Thus, by solving VCSP($\Phi$) we will determine whether $HOM(H) \neq \emptyset$, and find an optimal $h \in HOM(H)$, if $HOM(H) \neq \emptyset$. $\diamond$

2.3 Applications of Min-Max Ordering Theorem

We start with the following result from [8], which has an almost trivial proof provided we apply Theorem 2.5. It seems it is not easy to show directly (without the use of Theorem 2.5) that MinHOM(TT_p^-) is polynomial time solvable.

Theorem 2.6 *If H is TT_p ($p \geq 1$) or TT_p^- ($p \geq 3$), then MinHOM(H) is polynomial time solvable.*

Proof: The first case is trivial. To show the case $H = TT_p^-$, label the vertices of TT_p^- by $1, 2, \ldots p$ such that $ij \in A(TT_p^-)$ if and only if $1 \leq i < j \leq p$, but $ij \neq 1p$. Observe that $1, 2, \ldots, p$ is a Min-Max ordering since $1p$ can be neither the minimum nor the maximum of a non-trivial pair of arcs. $\diamond$

While studying homomorphisms, we can view an undirected graph G as a directed graph by replacing every edge xy of G by the pair xy, yx of arcs. This way we can define Min-Max orderings for undirected graphs and apply Theorem 2.5.

Consider two families of undirected graphs. A graph H with vertices $\{1, 2, \ldots, p\}$ is called *proper interval* if there is an inclusion-free family $\{I_1, I_2, \ldots, I_p\}$ of intervals on the real line such that $ij \in E(H)$ ($1 \leq i, j \leq p$) if and only if $I_i \cap I_j \neq \emptyset$. The definition implies that each vertex of a proper interval graph has a loop. A bipartite graph H with partite sets $P = \{1', 2', \ldots, p'\}$ and $Q = \{1'', 2'', \ldots q''\}$ is called a *proper interval bigraph* if there are two inclusion-free families $\{I_1, I_2, \ldots I_p\}$, $\{J_1, J_2, \ldots, J_q\}$ of intervals on the real line such that $i'j'' \in E(H)$ if and only if $I_i \cap J_j \neq \emptyset$.

The following result was proved in [13, 20].

Theorem 2.7 *The vertices of a reflexive graph H has a Min-Max ordering if and only if it is a proper interval graph.*

A similar result for loopless graphs contains proper interval bigraphs rather than proper interval graphs [9]. These theorems and Theorem 2.5 imply the following:

Corollary 2.8 *[9] Let H be a graph in which every component is either a proper interval graph or a proper interval bigraph. Then MinHOM(H) is polynomial time solvable.*

This result is of importance due to the following 'opposite' result:

Theorem 2.9 *[9] Let H be a graph in which at least one component is neither a proper interval graph nor a proper interval bigraph. Then MinHOM(H) is NP-hard.*

Notice that if a component of H has a pair of vertices one with a loop and the other without a loop, MinHOM(H) is NP-hard due to Lemma 4.2.

2.4 k-Min-Max Ordering

A collection $V_1, V_2, \dots, V_k$ of subsets of a set V is called a *k-partition* of V if $V = V_1 \cup V_2 \cup \dots \cup V_k$, and $V_i \cap V_j = \emptyset$ provided $i \neq j$.

Let $H = (V, A)$ be a loopless digraph and let $k \geq 2$ be an integer. We say that H has a *k-Min-Max ordering* if there is a k-partition of V into subsets $V_1, V_2, \dots V_k$ and there is an ordering $v_1^i, v_2^i, \dots, v_{\ell(i)}^i$ of V_i for each i such that

(i) Every arc of H is an arc from V_i to V_{i+1} for some $i \in \{1, 2, \dots, k\}$,

(ii) $v_1^i, v_2^i, \dots, v_{\ell(i)}^i v_1^{i+1} v_2^{i+1}, \dots, v_{\ell(i+1)}^{i+1}$ is a Min-Max ordering of the subdigraph of H induced by $V_i \cup V_{i+1}$ for each $i \in \{1, 2, \dots, k\}$,

where all indices $i + 1$ are taken modulo k.

Note that if H is a loopless strong digraph in which the greatest common divisor of all cycle lengths is k, then $V(H)$ has a k-partition, $k \geq 2$, satisfying (i) (see Theorem 10.5.1 in [1]). A simple example of a digraph having a k-Min-Max ordering, but no Min-Max ordering, is an extension of $\vec{C}_k$. To see than an extension H of $\vec{C}_k$ has no Min-Max ordering, consider an ordering $1, 2, \dots, p$ of the vertices of H, and an arc leaving 1 and an arc coming into 1. The minimum arc of the two arcs is the loop 11 not in H.

The following theorem from [9] establishes usefulness of k-Min-Max orderings. The proof of Theorem 2.10 in [9] is based on a reduction from MinHOM(H) to the minimum cut problem in a special network. The reduction is somewhat similar to the ones used in in [17] and [3].

Theorem 2.10 *If a loopless digraph H has a k-Min-Max ordering, then MinHOM(H) is polynomial time solvable.*

Consider the following simple application of Theorem 2.10.

Proposition 2.11 *Let H' be a directed k-cycle and let H be obtained from H' by appending a set S of $s \leq k$ new vertices each dominated by exactly one vertex of H' such that every vertex of H' dominates at most one vertex of S. Then MinHOM(H) is polynomial time solvable.*

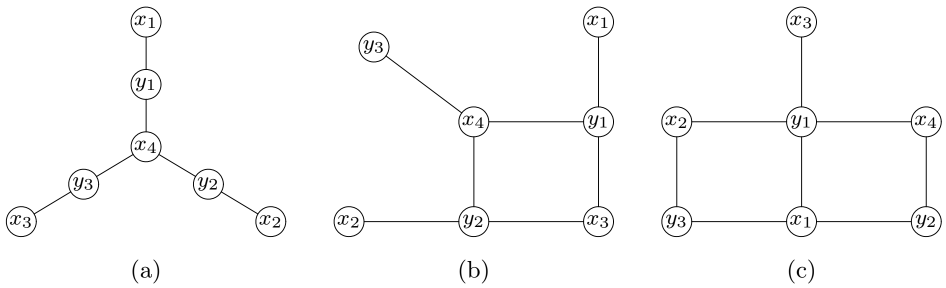

Figure 1: A bipartite claw (a), a bipartite net (b) and a bipartite tent (c).

The proof of Proposition 2.11 that uses Theorem 2.10 is trivial. However, it is not so obvious how to prove Proposition 2.11 without using Theorem 2.10. It is interesting to compare Proposition 2.11 with Theorem 4.5.

Theorem 2.10 allows us to prove much more difficult results such as Theorem 2.12 formulated below. We will start from a number of definitions. A bipartite graph H with vertices $x_1, x_2, x_3, x_4, y_1, y_2, y_3$ is called

a bipartite claw if its edge set $E(H) = \{x_4y_1, y_1x_1, x_4y_2, y_2x_2, x_4y_3, y_3x_3\}$;

a bipartite net if its edge set $E(H) = \{x_1y_1, y_1x_3, y_1x_4, x_3y_2, x_4y_2, y_2x_2, y_3x_4\}$;

a bipartite tent if its edge set $E(H) = \{x_1y_1, x_1y_2, x_1y_3, x_2y_1, x_2y_3, x_3y_1, x_4y_1, x_4y_2\}$.

The graphs are depicted in Figure 2.4.

Let us introduce five special digraphs for which the minimum homomorphism problem is NP-hard [9]. The digraph C'_4 has vertex set $\{x_1, x_2, y_1, y_2\}$ and arc set

$$\{x_1y_1, y_1x_2, x_2y_2, y_2x_1, y_1x_1\}.$$

The digraph C''_4 has the same vertex set, but its arc set is $A(C'_4) \cup \{x_2y_1\}$. The digraph H^* has vertex set $\{x_1, x_2, y_1, y_2, y_3\}$ and arc set $\{x_1y_1, y_1x_2, x_2y_2, y_2x_1, x_1y_3, x_2y_3\}$.

Let N_1 be a digraph with $V(N_1) = \{x_1, x_2, x_3, y_1, y_2, y_3\}$ and

$$A(N_1) = \{x_1y_1, y_1x_1, x_2y_2, y_2x_2, x_3y_3, y_3x_3, y_1x_2, y_1x_3, x_1y_2, x_1y_3, x_3y_2, x_2y_3\}.$$

Let N_2 be a digraph with $V(N_2) = \{x_1, x_2, x_3, y_1, y_2, y_3\}$ and

$$A(N_2) = \{x_1y_1, x_2y_2, y_2x_2, x_3y_3, y_3x_3, y_1x_2, y_1x_3, x_1y_2, x_1y_3, x_3y_2, x_2y_3\}.$$

Let H be a bipartite digraph with partite sets P, Q. Then $H^{\rightarrow}$ ($H^{\leftarrow}$, $H^{\leftrightarrow}$) denotes a bipartite digraph obtained from H by deleting all arcs from Q to P (a bipartite digraph obtained from H by deleting all arcs from P to Q, a bipartite digraph obtained from H by deleting all arcs not in directed 2-cycles).

A digraph H belongs to the family $\mathcal{HFORB}$ if H or its converse is isomorphic to one of the five digraphs above or $UN(H^s)$ is isomorphic to bipartite claw, bipartite net, bipartite tent or even cycle with at least 6 vertices, where $s \in \{\rightarrow, \leftarrow, \leftrightarrow\}$.

The family $\mathcal{HFORB}$ is of importance due to the following two results proved in [9].

Theorem 2.12 *If a semicomplete bipartite digraph H does not contain a digraph from $\mathcal{HFORB}$ as an induced subdigraph, then MinHOM(H) is polynomial time solvable.*

Theorem 2.13 *The problem MinHOM(H) is NP-hard if $H \in \mathcal{HFORB}$.*

3 Known Dichotomies

In this paper we assume that P≠NP as otherwise the dichotomies below are of no interest. Corollary 2.8 and Theorem 2.9 provide a complete dichotomy of the computational complexity of MinHOM(H) when H is an undirected graph. Theorems 2.12 and 2.13 give a dichotomy when H is a semicomplete bipartite digraph. Interestingly, the following dichotomy for semicomplete k-partite digraphs ($k \geq 3$) is less complicated than the one for semicomplete bipartite digraphs in both formulation and proof.

Theorem 3.1 *Let H be a semicomplete k-partite digraph, $k \geq 3$. If H is an extension of TT_k, $\vec{C}_3$ or TT_{k+1}^-, then MinHOM(H) is polynomial time solvable. Otherwise, MinHOM(H) is NP-hard.*

The 'polynomial part' of this theorem follows immediately from Lemmas 2.1 and 2.2 and Theorem 2.6.

4 New Dichotomies

The following lemma is an obvious basic observation often used to obtain dichotomies.

Lemma 4.1 *[7] Let H' be an induced subdigraph of a digraph H. If MinHOM(H') is NP-hard, then MinHOM(H) is also NP-hard.*

4.1 Directed Cycles w.p.l.

NP-completeness of the independence set problem for graphs [5] is often used to prove NP-hardness of MinHOM(H) for certain digraphs H. The next lemma gives a simple example of this reduction.

Lemma 4.2 *If $V(H) = \{1, 2\}$ and $A(H) = \{12, 21, 22\}$, then MinHOM(H) is NP-hard.*

Proof: Let D be an input of MinHOM(H), let D be a symmetric digraph (i.e., $xy \in A(D)$ implies $yx \in A(D)$) and let G be the underlying graph of D. Set $c_1(x) = 0$ and $c_2(x) = 1$ for each vertex $x \in V(D)$. Clearly, $h(x) = 2$ for each $x \in V(D)$ defines a homomorphism of D to H. Let f be a minimum cost homomorphism of D to H. Then $f(x) = f(y) = 1$ implies that x and y are not adjacent in G. Hence $f^{-1}(1)$ is a maximum size independent set in G.

Let I is a maximum size independent set in G. Then a homomorphism g of D to H such that $g(x) = 1$ if and only if $x \in I$ is of minimum cost. Since the maximum size independent set problem is NP-hard, MinHOM(H) is NP-hard as well. $\diamond$

We have seen that MinHOM($\vec{C}_2$) is polynomial time solvable. The above lemma shows that if we add just one loop to $\vec{C}_2$, we get an NP-hard case. It is easy to see that addition of two loops returns us to polynomial time solvable cases. Similar changes in the complexity are not valid for $\vec{C}_k$, $k \geq 3$, as shown in the next lemma.

Lemma 4.3 *Let a digraph H is obtained from $\vec{C}_k$, $k \geq 3$, by adding at least one loop. Then MinHOM(H) is NP-hard.*

Proof: Assume that kk is a loop.

Let G be a loopless digraph with n vertices. Construct a digraph D as follows: $V(D) = \{x_1, x_2, \ldots, x_{k-1} : \ x \in V(G)\}$ and $A(D) = \{x_1x_2, x_2x_3, \ldots, x_{k-2}x_{k-1} : \ x \in V(G)\} \cup \{x_{k-1}y_1 : \ xy \in A(G)\}$. For each $x \in V(G)$, set the costs for vertices of D as follows: $c_i(x_i) = 0$ for each $i = 1, 2, \ldots, k-1$, $c_j(x_i) = (k-1)n + 1$ for each $i = 1, 2, \ldots, k-1$ and $j \in \{1, 2, \ldots k-1\} - \{i\}$, and $c_k(x_i) = 1$ for each $i = 1, 2, \ldots, k-1$.

Clearly, $h(x_i) = k$ for each $x \in V(D)$ and $i = 1, 2, \ldots, k-1$ defines a homomorphism of D to H. Let f be a minimum cost homomorphism of D to H. It follows from the fact that the cost of h is $(k-1)n$ that $f(x_i) \in \{i, k\}$ for each $x \in V(D)$ and $i = 1, 2, \ldots, k-1$. Thus, for every vertex $x \in V(G)$ we have either $f(x_i) = i$ or $f(x_i) = k$ for each $i = 1, 2, \ldots, k-1$.

Let $f(x_1) = f(y_1) = 1$, where x, y are distinct vertices of G. If $xy \in A(G)$, then $x_{k-1}y_1 \in A(D)$, which is a contradiction since $f(x_{k-1}) = k-1$. Thus, x and y are non-adjacent in G. Hence, $I = \{x \in V(G) : \ x_1 \in f^{-1}(1)\}$ is an independent set in G. Observe that the cost of f is $(k-1)(n - |I|)$.

Conversely, if I is an independent set in G, we obtain a homomorphism g of D to H by fixing $g(x_i) = i$, $i = 1, 2, \ldots, k-1$, for every $x \in I$ and $g(x_i) = k$, $i = 1, 2, \ldots, k-1$, for every $x \in V(G) - I$. Observe that the cost of g is $(k-1)(n - |I|)$. Hence a homomorphism g of D to H is of minimum cost if and only if the corresponding independent set I is of maximum size in G. Since the maximum size independent set problem is NP-hard, MinHOM(H) is NP-hard as well. $\diamond$

Our first dichotomy follows directly from Lemmas 2.1, 4.2 and 4.3 and a simple observation that MinHOM($\vec{C}_2^*$) is polynomial time solvable, where $\vec{C}_2^*$ is the digraph obtained from a directed 2-cycle by adding loop to each vertex.

Proposition 4.4 *Let H be a digraph obtained from $\vec{C}_k$, $k \geq 2$, by possibly adding loops. Then MinHOM(H) is polynomial time solvable if $H \in \{\vec{C}_2^*\} \cup \{\vec{C}_k : \ k \geq 2\}$. Otherwise, MinHOM(H) is NP-hard.*

4.2 Tournaments w.p.l.

A special case of the following lemma was proved in [7]. Our proof is a modification of the proof in [7].

Lemma 4.5 *Let H' be a digraph obtained from $\vec{C}_k$, $k \geq 2$, by adding an extra vertex $k+1$ dominated by at least two vertices of the cycle and let H'' is the digraph obtained from H' by adding the loop at vertex $k+1$. Let H be H' or its converse or H'' or its converse. Then MinHOM(H) is NP-hard.*

Proof: Let $V(\vec{C}_k) = \{1, 2, \dots, k\}$ and $A(\vec{C}_k) = \{12, 23, \dots, (k-1)k\} \cup \{k1\}$. Let $H = H'$ or H''. Without loss of generality, we may assume that vertices 1 and ℓ dominate vertex $k+1$. We will reduce the maximum independent set problem to MinHOM(H). Let G be a graph. Construct a digraph D as follows:

$$V(D) = V(G) \cup \{v_i^e : \ e \in E(G) \ i \in V(H)\}, \ A(D) = A_1 \cup A_2, \text{ where}$$

$$A_1 = \{v_1^e v_2^e, v_2^e v_3^e, \dots v_{k-1}^e v_k^e, v_k^e v_1^e : \ e \in E(G)\}$$

and

$$A_2 = \{v_1^{uv} u, v_{k+1}^{uv} u, v_\ell^{uv} v, v_{k+1}^{uv} v : \ uv \in E(G)\}.$$

Let all costs $c_i(t) = 1$ for $t \in V(D)$ apart from $c_{k+1}(p) = 2$ for all $p \in V(G)$.

Consider a minimum cost homomorphism f of D to H. By the choice of the costs, f assigns the maximum possible number of vertices of G (in D) a color different from $k+1$. However, if pq is an edge in G, by the definition of D, f cannot assign colors different from $k+1$ to both p and q. Indeed, if both p and q were assigned colors different from $k+1$, then the existence of arcs $v_{k+1}^{pq} p$ and $v_{k+1}^{pq} q$ would imply that they would be assigned the same color, which however is impossible by the existence of arcs $v_1^{pq} p$ and $v_\ell^{pq} q$. Observe that f may assign exactly one of the vertices p, q color $k+1$ and the other a color different from $k+1$. Also f may assign both of them color $k+1$. Thus, G has a maximum independent set with α vertices if and only if D has a minimum cost H-coloring of cost $|E(G)| \cdot |V(H)| + 2|V(G)| - \alpha$. This reduces the maximum independent set problem to MinHOM(H). $\diamond$

The proof of the following theorem is similar to the proof of the 'loopless version' of the theorem in [8]. The only significant difference is the use of Lemma 4.3.

Theorem 4.6 *Let T be a tournament w.p.l. If H is an acyclic tournament w.p.l. or $H = \vec{C}_3$, then MinHOM(H) is polynomial time solvable. Otherwise, MinHOM(H) is NP-hard.*

Proof: The polynomial time solvability part follows from Lemma 2.1 and Theorem 2.6. Now assume that H is not an acyclic tournament and $H \neq \vec{C}_3$. Suppose MinHOM(H) is not NP-hard. Since H is a cyclic tournament, it is easy to show that H contains a directed 3-cycle C. By Lemmas 4.1 and 4.3, C has no loops, and, thus, there is a vertex $i \in V(H)$ not on C.

Assume that the subdigraph L of H induced by $V(C) \cup \{i\}$ is strong. Assume that i belongs to a directed 3-cycle. Then, if i has a loop, MinHOM(H) is NP-hard, a contradiction. Thus, i has no loop. However, MinHOM(H) is NP-hard by Lemmas 4.1 and 4.5, a contradiction. $\diamond$

4.3 Cyclic Multipartite Tournaments w.p.l.

Lemma 4.7 *Let H' be given by $V(H') = \{1,2,3,4\}$, $A(H') = \{12,23,31,34,41\}$. Let H'' be obtained from H' by adding the loop 44. If H is H' or H'', then MinHOM(H) is NP-hard.*

Proof: Since the case $H = H'$ was proved in [8], we will consider only the case $H = H''$. We will reduce the maximum independent set problem to MinHOM(H). Consider a digraph $T(u,v)$ defined as follows: $V(T(u,v)) = \{x, y, u', u, v', v, z_1, z_2, \dots, z_{12}\}$,

$$A(T(u,v)) = \{xy, xz_1, yz_1, z_6u', u'u, z_{11}v', v'v, z_1z_2, z_2z_3, z_3z_4, \dots, z_{11}z_{12}, z_{12}z_1\}.$$

Let G be a graph with n vertices and m edges. Construct a digraph D as follows. Start with $V(D) = V(G)$ and, for each edge $uv \in E(G)$, add a distinct copy of $T(u,v)$ to D. Note that the vertices in $V(G)$ form an independent set in D and that $|V(D)| = n + 16m$.

Let $M = 2n + 16m$. Let the cost $c_i(t) = 1$ for each $t \in V(D)$ and $i = 1,2,3,4$ with the following exceptions: $c_j(p) = 2$ for each $p \in V(G)$ and $j \neq 2$, and $c_4(z_i) = 1 + M$ for each i not divided by 4.

Consider a homomorphism h of $T(u,v)$ to H defined as follows: $h(z_{3i+1}) = 1$, $h(z_{3i+2}) = 2$, $h(z_{3i+3}) = 3$ for each $i = 0,1,2,3$, $h(u') = h(u) = h(v) = h(y) = 4$ and $h(v') = h(x) = 3$. The cost of h equals M.

Let f be a minimum cost homomorphism of $T(u,v)$ to H. Observe that we must have $f(z_1) = 1$ since $x \rightarrow z_1$, $y \rightarrow z_1$ and $c_4(z_1) = 1 + M$. This and the fact that f is of cost at most M imply that $(f(z_1), f(z_2), \dots, f(z_{12}))$ has to coincide with one of the following two sequences: $(1,2,3,1,2,3,1,2,3,1,2,3)$ or $(1,2,3,4,1,2,3,4,1,2,3,4)$.

If the first sequence is the actual one, then we have $f(z_6) = 3$, $f(u') \in \{1,4\}$, $f(u) \in \{1,2,4\}, f(z_{11}) = 2$, $f(v') = 3$ and $f(v) \in \{1,4\}$. If the second sequence is the actual one, then we have $f(z_6) = 2$, $f(u') = 3$, $f(u) \in \{1,4\}, f(z_{11}) = 3$, $f(v') \in \{1,4\}$ and $f(v) \in \{1,2,4\}$. We can color one of u and v with color 2 and the other with color 1 or 4. However we cannot assign color 2 to both u and v in a homomorphism.

Let g be a minimum cost homomorphism of D to H. Clearly, g must aim at assigning as many vertices of $V(G)$ in D color 2. Notice that if pq is an edge in G, by the arguments above, g cannot assign color 2 to both p and q. However, g can assign color 2 to either p or q (or neither). Thus, g corresponds to a maximum independent set in G and vise versa (the vertices of a maximum independent set are assigned color 2 and all other vertices in $V(G)$ are assigned color 1 or 4). $\diamond$

Theorem 4.8 *Let H be a cyclic k-partite tournament w.p.l., $k \geq 2$. If H is an extension of $\vec{C}_3$ or $\vec{C}_4$, then MinHOM(H) is polynomial time solvable. Otherwise, MinHOM(H) is NP-hard.*

Proof: Assume that MinHOM(H) is polynomial time solvable and H is not an extension of $\vec{C}_3$ and $\vec{C}_4$. Since H is a k-partite tournament, $k \geq 2$, and H has a cycle, there can be two possibilities for the length of a shortest cycle C in H: 3 or 4. Thus, we consider two corresponding cases. Observe that, by Lemma 4.3, no cycle in H has a loop. In particular, C has no loop.

Case 1: We have $C = ijli$. If H has at least four partite sets, then i, j, l together with a vertex from a partite set containing none of i, j, l induce a tournament w.p.l. By Theorem 4.6, MinHOM(H) is NP-hard, a contradiction. So, H has three partite sets.

Let I, J and L be partite sets of H such that $i \in I$, $j \in J$ and $l \in L$. Let s be a vertex outside C and let $s \in I$. If s is dominated by j and l or dominates j and l, then MinHOM(H) is NP-hard by Lemma 4.5, a contradiction. If $j \rightarrow s \rightarrow l$, then MinHOM($H$) is NP-hard by Lemma 4.7, a contradiction. Thus, $l \rightarrow s \rightarrow j$. Similar arguments show that $l \rightarrow I \rightarrow j$. Similarly we can prove that $L \rightarrow I \rightarrow J \rightarrow L$, i.e., H is an extension of $\vec{C}_3$, contradiction.

Case 2: We have $C = i_1 i_2 i_3 i_4 i_1$. Since C is a shortest cycle, i_1, i_3 are belong to the same partite set, say L, and i_2, i_4 belong to the same partite set, say M. Assume first that H is not bipartite. Then there is a vertex q belonging to a partite set different from L and M. Since H has no directed 3-cycle, either q dominates $V(C)$ or $V(C)$ dominates q. In both cases, MinHOM(H) is NP-hard by Lemma 4.5, a contradiction.

Assume that H is bipartite. By Lemma 4.5, the vertices of H can be partitioned into four sets: $I_j = \{p : \ i_{j-1} \to p \to i_{j+1}\}, \ j = 1, 2, 3, 4$, where all indices are taken modulo 4. Suppose that there is pair $q_1 \in I_1, q_2 \in I_2$ of vertices such that $q_2 \to q_1$. Let H' denote the subdigraph of H induced by q_1, i_2, i_3, i_4, q_2. Observe that MinHOM(H') is NP-hard by Lemma 4.5. Similar arguments imply that $I_1 \to I_2 \to I_3 \to I_4 \to I_1$, i.e., H is an extension of $\vec{C}_4$, a contradiction. $\diamond$

5 Further Research

We have managed to obtain dichotomies for tournaments w.p.l. and cyclic multipartite tournaments w.p.l. The reader may ask why we have not obtained dichotomies, for example, for semicomplete digraphs w.p.l. and acyclic multipartite tournaments w.p.l. Our research indicates significant difficulties in obtaining these dichotomies. One of them is a significantly larger number of polynomial time solvable (and NP-hard) cases. In fact, we have recently obtained a dichotomy for semicomplete digraphs w.p.l. [18], but its proof is far too long to be included here. A dichotomy for acyclic multipartite tournaments w.p.l. remains an open problem.

The above indicates that addition of loops may well make dichotomies significantly more complicated. Thus, the problem of obtaining a dichotomy for all semicomplete k-partite digraphs, $k \geq 2$, w.p.l. appears to be a very difficult open problem. We conjecture that there is a dichotomy for the whole class of digraphs w.p.l. and it would be very interesting to verify this conjecture.

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