Out-branchings with Extremal Number of Leaves

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Outline

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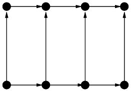
Out-Trees and Out-Branchings

- We say that a subdigraph T of a digraph D is an out-tree if T is an oriented tree with only one vertex s of in-degree zero (its root).
- The vertices of T of out-degree zero are leaves.
- If T is a spanning out-tree, i.e. $V(T) = V(D)$, then T is an out-branching of D.
- A digraph D has an out-branching iff D has only one initial strong component (a strong component C is initial if there are no arcs entering C).

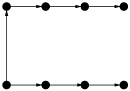
Problems with Extremal Number of Leaves

- Find an out-branching with minimum number of leaves, $\ell_{\min}(D)$.
- Find an out-branching with maximum number of leaves, $\ell_{\max}(D)$.
- If D has no out-branching, let $\ell_{\min}(D) = \ell_{\max}(D) = 0$.

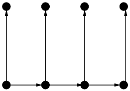
Example



(a) D



(b) Q



(c) R

Figure 1: A digraph D and its out-branchings with minimum and maximum number of leaves (Q and R, respectively).

Aspects to Study

- Classical Complexity (polynomial time or NP-hard).
- Parameterized Complexity (FPT, etc.).
- Bounds on the Number of Leaves.

Fixed Parameter Tractability

Definition

A parameterized problem Π can be considered as a set of pairs (I, k) where I is the problem instance and k (usually an integer) is the parameter.

Definition

Π is called fixed-parameter tractable (FPT) if membership of (I, k) in Π can be decided in time $O(f(k)|I|^c)$, where $|I|$ is the size of I, $f(k)$ is a computable function, and c is a constant independent from k and I.

Outline

Min Leaf in Acyclic Digraphs is in P

- G., Kim and Razgon, 2008
- Algorithm: $D \to B(D) \to M \to M^* \to T$

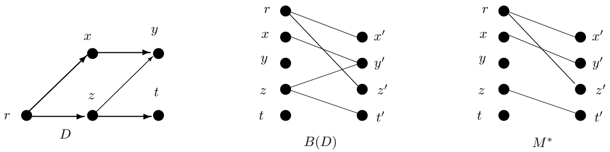


Figure 1: Illustration for the MINLEAF algorithm

Las Vergnas' Theorem

Theorem (Las Vergnas 1976)

Let D be a digraph and let $\alpha(D)$ be the independence number of D. Then $\ell_{\min}(D) \leq \alpha(D)$.

- Tight bound. For transitive acyclic digraphs with unique vertex of in-degree 0, we have $\ell_{\min}(D) = \alpha(D)$.
- Constructive proof yielding a polynomial-time algorithm.

General Case Complexity

Fact

For any fixed k, the problem to decide whether $\ell_{\min}(D) \leq k$ is NP-hard.

Theorem (G., Kim and Razgon, 2008)

Let k be a parameter and $n = |V(D)|$. The problem to decide whether $\ell_{\min}(D) \leq n - k$ is FPT.

A scheme of the proof follows ...

Tree Decompositions

A tree decomposition of an digraph G is a pair (X, T) where T is a tree whose vertices we will call nodes and $X = \{X_i : \ i \in V(T)\}$ is a collection of subsets of $V(G)$ (called bags) such that

1. $\bigcup_{i \in V(T)} X_i = V(G)$,
2. for each arc $vw \in A(G)$, there is an $i \in V(T)$ such that $v, w \in X_i$, and
3. for each $v \in V(G)$ the set of nodes $\{i : \ v \in X_i\}$ forms a subtree of T.

The width of a tree decomposition $(\{X_i : \ i \in V(T)\}, T)$ equals $\max_{i \in V(T)}\{|X_i| - 1\}$. The treewidth $\mathrm{tw}(G)$ of G is the minimum width over all tree decompositions of G.

Parameterized Complexity Result-1

Fact (well-known)

For any digraph D, we have $\alpha(D) + \beta(D) = n$, where $\beta(D)$ is the minimum size of a vertex cover of D.

Fact (mentioned earlier)

For a digraph D, one can check, in polynomial time, whether $\ell_{\text{min}}(D) \leq \alpha(D)$.

Parameterized Complexity Result-2

Fact (simple tree decomposition)

If $\beta(D) \leq k$*, then* $\text{tw}(D) \leq k$*.*

Proof: Let C be a min size vertex cover of D. The bags are $X_x = C$ and $\{X_y = C + y : \; y \in V(D) - C\}$. The tree T is a star with center x and its other vertices are $y \in V(D) - C$.

1. $\bigcup_{z \in V(T)} X_z = V(D)$,
2. for each arc $vw \in A(D)$, there is a $z \in V(T)$ such that $v, w \in X_z$, and
3. for each $v \in V(D)$ the set of nodes $\{z : \; v \in X_z\}$ forms a subtree of T.

Parameterized Complexity Result-3

Fact (dynamic programming)

If $\mathrm{tw}(D) \leq k$*, then we can check whether* $\ell_{\min}(D) \leq n - k$ *in FPT time.*

It follows from the four facts that

Theorem

The problem of checking $\ell_{\min}(D) \leq n - k$ *is FPT.*

In fact, there is an $O(2^{k \log k} + n^3)$-algorithm.

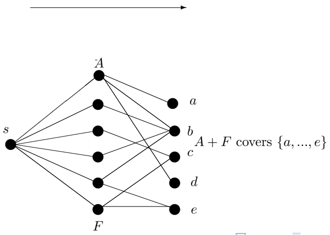
Outline

Acyclic Digraphs

Fact (Alon et al. 2008)

Finding a max leaf out-branching is NP-hard even for acyclic digraphs.

Transformation to the Vertex Cover problem.

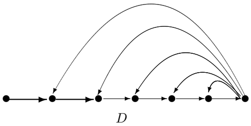


Out-Trees vs Out-Branchings-1

- $\ell^t_{\max}(D)$ max no. leaves in an out-tree
- $\ell_{\max}(D)$ max no. leaves in an out-branching
- Family $\mathcal{L}$: $\ell_{\max}(D) = \ell^t_{\max}(D)$ or 0
- $\mathcal{L}$ includes strong digraphs, acyclic digraphs, semicomplete multipartite digraphs, quasi-transitive digraphs, etc.

Out-Trees vs Out-Branchings-2

There are many digraphs outside $\mathcal{L}$:
$\ell_{\max}(D) = 1$, $\ell^t_{\max}(D) = n - 2$. $D \notin \mathcal{L}$.



Parameterized MaxLeafOT (M. Fellows)

Problem (M. Fellows)

Is it FPT to check whether there is an out-tree with at least k leaves?

Theorem (Alon et al. 2007)

Let $D \in \mathcal{L}$. Then either $\ell_{\max}(D) \geq k$ or D is of pathwidth $\leq 2k^2$.

Fact (Alon et al. 2007)

Let R_v be the set of vertices reachable from v. Then $D[R_v] \in \mathcal{L}$. Also, $\ell^t_{\max}(D) = \max\{\ell_{\max}(D[R_v]) : \ v \in V(D)\}$.

Parameterized MaxLeafOT (cont'd)

Theorem (Alon, Fomin, G., Krivelevich, Saurabh, 2007)

The problem is FPT.

1-Opt OB [Alon et al. 2007]

Parameterized MaxLeafOB (M. Fellows)

Problem (M. Fellows)

Is it FPT to check whether there is an out-branching with at least k leaves?

Theorem (Bonsma and Dorn, 2007)

The problem is FPT.

Similar to Alon et al. but:

- Delete useless arcs from D
- Using an 1-Opt out-branching and some of its backward arcs, try to construct an out-branching with at least k leaves. If no success, then pathwidth$(D) \leq 6k^3$.

Open Problems

- All FPT algorithms mentioned are of complexity $O(2^{k \log k} n^{O(1)})$.
- Find algorithms of complexity $O(2^k n^{O(1)})$ if possible.

Discussion

- Questions?
- Suggestions?
- Comments?