Worst Case Analysis of Greedy, Max-Regret and Other Heuristics for Multidimensional Assignment and Traveling Salesman Problems

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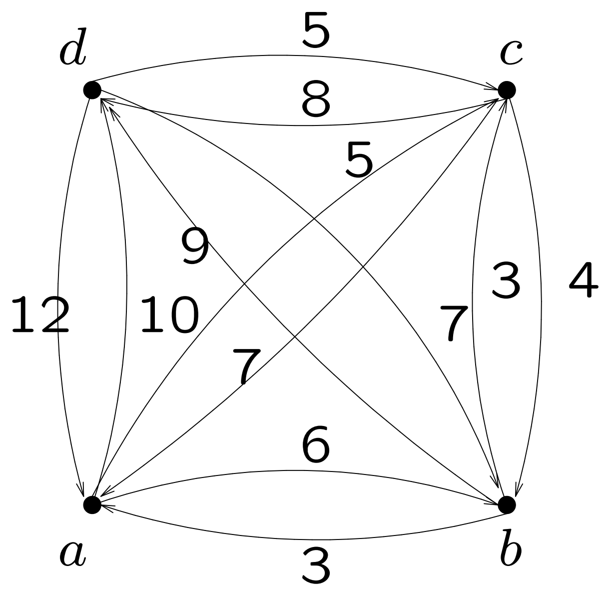
Talk Overview

- ATSP
- Multidimensional Assignment Problem
- General results and results for other problems

ATSP



- A Hamilton cycle is a *tour*; $(n-1)!$ tours
- Many heuristics (how to compare them?)
- Computational experiments
- Approximation and domination analyzes

Domination Number

- Introduced in 1997 by Glover and Punnen
- First results obtained by Rublineckii in 1973 [in Russian]
- For an ATSP heuristic H and ATSP instance I, domn(H, I) is the number of tours in I of weight at least the weight of the tour obtained by H for I
- domn$(H, n) = \min\{\text{domn}(H, I) : |I| = n\}$

Use of Domination Number

- domn is invariant under linear transformations of arc weights

- Worst case comparison:

  domn$(H, n) >$ domn(H', n) for n large enough

  Ben-Arieh, Gutin, Penn, Yeo and Zverovitch (2003) for Generalized ATSP

- Neighborhoods for local search: why some exponential-size neighborhoods perform bad while small-size ones perform well (Orlin et al.)?

Low Dom Number Heuristics

- Greedy and NN are of dom number 1 (may produce the unique worst possible tour), Gutin, Yeo and Zverovitch (2002)

- Any greedy-type heuristic is of dom number 1 (greedy-type introduced by Gutin, Vainshtein and Yeo, 2002, proved by Bendall and Margot, to appear)

- Max-Regret is of dom number 1 (Proc. WAOA'06)

Large Dom Number Heuristics

Any heuristic that always produces a tour of weight not worse that the average weight is of dom number at least $(n-2)!$, $n \neq 6$ (n odd: Sarvanov, 1976, n even: Gutin and Yeo, 2002). Such heuristics are:

(a) Greedy-expectation algorithm (Gutin and Yeo, 2002)

(b) Vertex insertion algorithms (Lifshitz, 1973, Punnen and Kabadi, 2002)

(c) 3-Opt, Lin-Kernighan in polynomial time (Punnen, Margot and Kabadi, 2003)

In computational experiments low dom number algorithms often perform worse than large dom number algorithms

ATSP-Max-Regret

ATSP-Max-Regret-FC: Set $W = T = \emptyset$. While $V \neq W$ do the following: For each $i \in V \setminus W$, compute two lightest arcs (i, j) and (i, k) that are feasible additions to T, and compute the difference $\Delta_i = |w_{ij} - w_{ik}|$. For $i \in V \setminus W$ with maximum Δ_i choose the lightest arc (i, j), which is a feasible addition to T and add (i, j) to M and i to W.

ATSP-Max-Regret: The same with not only outgoing, but also incoming arcs considered.

Question (Ghosh et al., 2007): What is the dom number of ATSP-Max-Regret? (ATSP-Max-Regret often outperforms Greedy)

Dom Number of ATSP-Max-Regret

Theorem *Both* ATSP-Max-Regret-FC *and* ATSP-Max-Regret *are of dom number 1.*

Proof: Consider an instance of ATSP with vertex set $\{1, 2, \ldots, n\}$, $n \geq 2$. The weights: $w_{ik} = \min\{i - k, 0\}$ for each $1 \leq i \neq k \leq n$, $i \neq n$, and $w_{nk} = -k$ for each $1 \leq k \leq n - 1$.

Modify the weights: $w'_{ij} = w_{ij}$ unless $j = i + 1$ modulo n. Set $w'_{i,i+1} = -1 - \frac{1}{n+1}$ for $1 \leq i \leq n$.

ATSP-Max-Regret-FC will use w'.

Proof cont'd (a)

ATSP-Max-Regret-FC constructs the tour

$$T_{MR} = (1, 2, \ldots, n, 1)$$

by choosing the arc $(n-1, n)$, the arc $(n-2, n-1)$, etc. The last two arcs are $(1, 2)$ and $(n, 1)$ (they must be included in the tour).

Indeed, initially $\Delta_{n-1} = \frac{n+2}{n+1} > \Delta_i = 1$ for each $i \neq n-1$. Once $(n-1, n)$ is added to T_{MR}, $\Delta_{n-2} = \frac{n+2}{n+1}$ becomes maximal, etc.

Let T', T'' be tours. Since $\sum_{(i,j)\in K_n^*} |w_{ij} - w'_{ij}| < 1$, $w(T') < w(T'')$ implies $w'(T') < w'(T'')$. Thus, in the rest of the proof we may use w rather than w'.

Proof cont'd (b)

Observe $w(T_{MR}) = -n$.

Let $T = (i_1, i_2, \ldots, i_n, i_1)$ be an arbitrary tour, where $i_1 = 1$. Let $i_s = n$, $P = (i_1, i_2, \ldots, i_s)$.

$w(P) = \sum_{k=1}^{s-1} \min\{0, i_k - i_{k+1}\} \leq$
$\sum_{k=1}^{s-1} (i_k - i_{k+1}) = i_1 - i_s$.

Thus, $w(P) \leq 1 - n$ and $w(P) = 1 - n$ iff $i_1 < i_2 < \cdots < i_s$.

Since $i_s = n$, the weight of the arc (i_s, i_{s+1}) equals $-i_{s+1}$. Thus, $w(T) \leq 1 - n - i_{s+1}$ and $w(T) \geq w(T_{MR})$ iff $i_{s+1} = 1$ and $i_1 < i_2 < \cdots < i_s$. We conclude that $w(T) \geq w(T_{MR})$ iff $T = T_{MR}$. $\square$

Multidimensional Assignment Problem (s-AP)

Let $X = \{1, 2, \ldots, n\}^s$. Each vector $e \in X$ is assigned a real weight $w(e)$.

For a vector e, e_j denotes its jth coordinate. Vectors e, f are *independent* if $e_j \neq f_j$ for each $j = 1, 2, \ldots, s$.

An *assignment* is a set of n independent vectors. The *weight* of an assignment $\{e^1, \ldots, e^n\}$ is $\sum_{i=1}^{n} w(e^i)$.

The aim: to find an assignment of minimum weight.

Applications: In tracking objects, e.g., 5-AP to track elementary particles at CERN

Max-Regret by Balas and Saltzman (1991)

We have a partial assignment A.

Choose $j \in \{1, \ldots, s\}$ and $m \in \{1, \ldots, n\}$.

Choose two lightest vectors e, f with $e_j = f_j = m$ and independent from vectors in A. Find the regret $\Delta_{j,m} = |w(e) - w(f)|$.

Choose j, m with maximum $\Delta_{j,m}$ and the lightest e with $e_j = m$. Add e to A.

Low Dom Number Heuristics

In comput. experiments Balas and Saltzman (1991) show Max-Regret outperforms Greedy. In comput. experiments Robertson (2001) shows the heuristics are of similar performance.

Theorem For s-AP, $s \geq 3$, both Greedy and Max-Regret are of dom number 1.

Theorem For 2-AP, Greedy is of dom number 1.

Theorem For 2-AP, Max-Regret is of dom number at most 2^{n-1}.

Large Dom Number Heuristics

Theorem For s-AP, if a heuristic H always produces an assignment of weight at most the average weight $\bar{w}$ of an assignment, then we have domn$(H, n) \geq ((n-1)!)^{s-1}$.

Proof: Consider an instance $\mathcal{I}$ of s-AP. C denotes the set of all vectors of $\mathcal{I}$ with the first coordinate equal 1; $\mathcal{P} = \{A_f : \ f^1 \in C\}$, where $A_f = \{f^1, f^2, \ldots, f^n\}$ is an assignment with $f_j^i = f_j^1 + i - 1$ (modulo n), $j = 1, 2, \ldots, s$.

Each vector is in exactly one A_f and, thus, $\mathcal{P}$ is a partition of $X = X_1 \times X_2 \times \cdots \times X_s$, $X_i = \{1, 2, \ldots, n\}$, into assignments.

Since $\sum_{f \in C} w(A_f) = w(X)$, $|C| = n^{s-1}$ and $\bar{w} = w(X)/n^{s-1}$, the heaviest assignment A_h in $\mathcal{P}$ is of weight at least $\bar{w}$.

Proof cont'd

Let $S(X_i)$ be the set of all permutations on X_i $(2 \le i \le s)$ and let $\pi_2 \in S(X_2), \ldots, \pi_s \in S(X_s)$. To obtain $\mathcal{P}(\pi_2, \ldots, \pi_s)$ from $\mathcal{P}$, replace f_j^i with $\pi_j(f_j^i)$ for each $j \ge 2$ and $i = 1, 2, \ldots, n$. Thus, we obtain a family

$$\mathcal{F} = \{\mathcal{P}(\pi_2, \ldots, \pi_s) : \pi_2 \in S(X_2), \ldots, \pi_s \in S(X_s)\}$$

of partitions of X into assignments. The family consists of $(n!)^{s-1}$ partitions. We may choose the heaviest assignment in each partition and, thus, obtain a family $\mathcal{A}$ of assignments of weight at least $\bar{w}$.

It's possible to prove that no assignment in $\mathcal{A}$ can be repeated more than n^{s-1} times. Since $\mathcal{A}$ has $(n!)^{s-1}$ assignments with repetitions, we can find (in $\mathcal{A}$) $((n-1)!)^{s-1}$ distinct assignments of weight at least $\bar{w}$. □

ROM Heuristic

Recursive Opt Matching (ROM):

Theorem For 3-AP, 3-Opt (by Balas and Saltzman) is of dom number at least $((n-1)!)^2$.

Compute a new weight $\bar{w}(i,j) = w(X_{ij})/n^{s-2}$, where X_{ij} is the set of all vectors with last two coordinates equal i and j, respectively. Solving the 2-AP with the new weights, find an optimal assignment $M = \{(i, \pi_s(i)) : \ i = 1, 2, \ldots, n\}$, where π_s is a permutation.

Continue for coordinates $s-2$ and $s-1$ with M 'fixing' coordinates $s-1$ and s, etc. Use $X' = \{1, 2, \ldots, n\}^{s-1}$ instead of X.

Theorem

domn$(ROM, n) \geq ((n-1)!)^{s-1}$.

Proof

It suffices to show that the assignment obtained by ROM is of weight at most $\bar{w} = w(X)/n^{s-1}$, the average weight of an assignment. By induction on $s \geq 2$. Clearly the assertion holds for $s = 2$ and consider $s \geq 3$. Observe that

$$\bar{w} = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\bar{w}(i,j) \geq \sum_{i=1}^{n}\bar{w}(i,\pi_s(i)) = \frac{w'(X')}{n^{s-2}}.$$

Let $A = \{(g^1, \pi_s(1)), \ldots, (g^n, \pi_s(n))\}$ be an assignment obtained by ROM, where $g^i \in X'$ such that $g^i_{s-1} = i$ for every $i = 1, \ldots, n$. Let $A' = \{g^1, \ldots, g^n\}$. Then by induction hypothesis, $\bar{w}' = w'(X')/n^{s-2} \geq w'(A') = w(A)$ and we are done. $\square$

Preliminary Computational Experiments (by Gerold Jäger)

100 random examples with n=8, s=6, entries in [0,1000] (sum of all solutions):

1. Greedy: 58310

2. Recursive Opt Matching: 53820

3. Balas-Saltzman: 54637

4. Shifted Recursive Opt Matching: 36878

Results for Other Problems

1. Independence Systems (IS). Sufficient conditions for Greedy to be of domination number 1, Gutin and Yeo, 2002

2. IS. Weights form a finite set. Necessary and sufficient conditions for Greedy to be of domination number 1, Bang-Jensen, Gutin and Yeo, 2004.

3. IS. Sufficient conditions for greedy-type heuristics to be of domination number 1, Bendall and Margot, to appear

4. Dom analysis of various problems: Alon, Gutin and Krivelevich (2004), Berend, Skiena and Twitto (submitted), Gutin, Vainshtein and Yeo 2003, Gutin, Jensen and Yeo (2006), etc.