Establishing Complexity of Problems Parameterized Above Average

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Outline

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Fixed-parameter Tractability

- A parameterized problem Π: a set of pairs (I, k) where I is the main part and k (usually an integer) is the parameter; I is an instance of the classical sense.
- Π is fixed-parameter tractable (FPT) if membership of (I, k) in Π can be decided in time $O(f(k)|I|^c)$, where $|I|$ is the size of I, $c = O(1)$ and $f(k)$ is a computable function.
- The idea: for small values of k, $O(f(k)|I|^c)$ is not too large.

Fixed-parameter Tractability

Examples of FPT problems:

- Does a graph G have a vertex cover of size $\leq k$? An algorithm of runtime $O(1.2852^k + kn)$ (Chen, Kanj and Jia, 2001) instead of an $O(n^k m)$-algorithm.
- Does a digraph D have a spanning out-tree with $\geq k$ leaves? Algorithms of runtime $4^k n^{O(1)}$ (Kneis, Langer and Rossmanith, 2008) and $3.72^k n^{O(1)}$ (Daligault, Gutin, Kim and Yeo, JCSS 2010) instead of an $O(n^k m)$-algorithm.

Bikernelization-1

- Suggested by Alon, Gutin, Kim, Szeider and Yeo (arXiv'09).
- A bikernelization of Π to Π': a polynomial-time algorithm that maps an instance $(x, k) \in \Pi$ to an instance $(x', k') \in \Pi'$ (the bikernel) such that
  - (x, k) is YES iff (x', k') is YES
  - $k' \leq f(k)$ and $|x'| \leq g(k)$ for some functions f and g.
- The function $g(k)$ is called the size of the bikernel.

Bikernelization-2

- A decidable parameterized problem is FPT iff it is and admits a bikernelization to a parameterized problem.
- Wanted: low degree polynomial-size bikernels to well-studied problems.
- Similar to a theorem in Bodlaender, Thomassé and Yeo on polynomial time and parameter transformations (ESA'09):

Lemma (Alon, Gutin, Kim, Szeider and Yeo)

Let P, P' be a pair of parameterized problems such that P' is in NP and P is NP-complete. If there is a bikernelization from P to P' producing a bikernel of polynomial size, then P has a polynomial-size kernel.

Kernelization

- A kernelization of Π: a polynomial-time algorithm that maps an instance $(x, k) \in \Pi$ to an instance $(x', k') \in \Pi$ (the kernel) such that
  - (x, k) is YES iff (x', k') is YES
  - $k' \leq f(k)$ and $|x'| \leq g(k)$ for some functions f and g.
- The function $g(k)$ is called the size of the kernel.
- A decidable parameterized problem is FPT if and only if it admits a kernelization.
- Wanted: low degree polynomial-size kernels (for preprocessing).
- Does a graph G have a vertex cover of size $\leq k$? Kernel of size $\leq 2k$ (Chen, Kanj and Jia, 2001).

Outline

Acyclic Subgraphs of Digraphs: Parameterization Above Average

- Parameterization Above Average: Does $D = (V, A)$ have an acyclic subgraph with at least $|A|/2 + k$ arcs? [Acyclic AA]
- The bound is tight: For symmetric digraphs, $k = 0$: a digraph D is symmetric if $xy \in A$ implies $yx \in A$.
- Mahajan, Raman and Sikdar (JCSS, 2009): Is Acyclic AA fixed-parameter tractable?

Strictly Above Expectation Method (SAEM): Symmetric Case

- Gutin, Kim, Szeider and Yeo [JCSS, ta]. Problem Π parameterized AA.
- Apply some reduction rules.
- Introduce a random variable X s.t. $\mathbb{E}(X) = 0$ and if $\text{Prob}(X \geq k) > 0$ then the answer to Π is YES.
- If X is symmetric (X and $-X$ have the same distribution), then $\text{Prob}(\ X \geq \sqrt{\mathbb{E}(X^2)}\ ) > 0$.
- If $k \leq \sqrt{\mathbb{E}(X^2)}$ then YES. Otherwise, $\sqrt{\mathbb{E}(X^2)} < k$ and we can often solve the problem using a brute-force algorithm.

Strictly Above Expectation Method (SAEM): Asymmetric Case

Lemma (Alon, Gutin, Krivelevich, 2004; Alon, Gutin, Kim, Szeider, Yeo, SODA'2010)

Let X be a real random variable and suppose that its first, second and forth moments satisfy $\mathbb{E}(X) = 0$, $\mathbb{E}(X^2) = \sigma^2 > 0$ and $\mathbb{E}(X^4) \leq b \cdot (\mathbb{E}(X^2))^2$, respectively. Then $\mathrm{Prob}(\ X > \frac{\sigma}{2\sqrt{b}}\ ) > 0$.

Lemma (Hypercontractive Inequality, Bonami, Gross, 1970s)

Let $f = f(x_1, \ldots, x_n)$ be a polynomial of degree r in n variables $x_1, \ldots, x_n$. Define a random variable X by choosing a vector $(\varepsilon_1, \ldots, \varepsilon_n) \in \{-1, 1\}^n$ uniformly at random and setting $X = f(\varepsilon_1, \ldots, \varepsilon_n)$. Then $\mathbb{E}(X^4) \leq 9^r(\mathbb{E}(X^2))^2$.

Outline

Reduction Rule for Linear Ordering Problem AA

- LINEAR ORDERING AA: Each arc ij has positive integral weight w_{ij}, does $D = (V, A)$ have an acyclic subgraph of weight at least $W/2 + k$, where $W = \sum_{ij \in A} w_{ij}$?
- Reduction rule: Assume D has a directed 2-cycle iji;
  - if $w_{ij} = w_{ji}$ delete the cycle,
  - if $w_{ij} > w_{ji}$ delete the arc ji and replace w_{ij} by $w_{ij} - w_{ji}$,
  - if $w_{ji} > w_{ij}$ delete the arc ij and replace w_{ji} by $w_{ji} - w_{ij}$.
- Thus, we've reduced LINEAR ORDERING AA to the one on oriented graphs.

SAEM for Linear Ordering AA-1

- Let $D = (V, A)$ be an oriented graph, $n = |V|$; a bijection: $\alpha : V \rightarrow \{1, \ldots, n\}$.
- Define $X(\alpha) = \frac{1}{2} \sum_{ij \in A} \varepsilon_{ij}$, where $\varepsilon_{ij} = w_{ij}$ if $\alpha(i) < \alpha(j)$ and $\varepsilon_{ij} = -w_{ij}$, otherwise.
- We have $X(\alpha) = \sum\{w_{ij} : \ ij \in A, \alpha(i) < \alpha(j)\} - W/2$. Thus, the answer is YES iff there is an $\alpha : V \rightarrow \{1, \ldots, n\}$ such that $X(\alpha) \geq k$.
- Consider a random bijection: $\alpha : V \rightarrow \{1, \ldots, n\}$. Then X is a random variable.
- Since $\mathbb{E}(\varepsilon_{ij}) = 0$, we have $\mathbb{E}(X) = 0$.

SABEM for Linear Ordering AA-2

Lemma

$\mathbb{E}(X^2) \geq W^{(2)}/12$, *where* $W^{(2)} = \sum_{ij \in A} w_{ij}^2$.

Since X is symmetric, we have Prob($X \geq \sqrt{W^{(2)}/12}$) > 0. Hence, if $\sqrt{W^{(2)}/12} \geq k$, there is an $\alpha : V \rightarrow \{1, \ldots, n\}$ such that $X(\alpha) \geq k$ and, thus, the answer is YES. Otherwise, $|A| \leq W^{(2)} < 12 \cdot k^2$. Thus, we have:

Theorem (GG, Kim, Szeider, Yeo, JCSS, ta)

LINEAR ORDERING AA *is FPT and has an* $O(k^2)$*-size kernel.*

Outline

Exact r-SAT

- EXACT r-SAT: A CNF formula $\mathcal{F}$ which contains m clauses each with r literals. Is there a truth assignment satisfying all m clauses of $\mathcal{F}$?
- MAX EXACT r-SAT: Find a truth assignment satisfying the max number of clauses.
- The prob. of a clause to be satisfied: $1 - 2^{-r}$.
- The average number of satisfied clauses: $(1 - 2^{-r})m$. This lower bound is tight.

Exact r-SAT AA-1

- Exact r-SAT AA: Is there a truth assignment satisfying $\geq (1-2^{-r})m + k2^{-r}$ clauses?
- Mahajan, Raman and Sikdar (JCSS, 2009): What is the parameterized complexity of Exact r-SAT AA for each fixed r?
- Alon, GG, Kim, Szeider and Yeo (SODA 2010): Exact r-SAT AA is FPT.

Exact r-SAT AA-2

- $-1 =$ true.
- $X = \sum_{C \in \mathcal{F}}[1 - \prod_{x_i \in \mathrm{var}(C)}(1 + \varepsilon_i x_i)]$, where var's $x_i \in \{-1, 1\}$, coef's $\varepsilon_i \in \{-1, 1\}$ and $\varepsilon_i = 1$ iff x_i is in C.
- For a truth assignment τ, we have $X = 2^r(\mathrm{sat}(\tau, F) - (1 - 2^{-r})m)$.
- The answer to EXACT r-SAT AA is YES iff $X \geq k$.

Exact r-SAT AA-3

- After algebraic simplification: $X = \sum_{I \in \mathcal{S}} X_I$, where $X_I = c_I \prod_{i \in I} x_i$, where each c_I is a nonzero integer and $\mathcal{S}$ is a family of nonempty subsets of $[n]$ each with at most r elements.
- This is a Fourier expansion of X over orthogonal basis $\prod_{i \in I} x_i, \ I \subseteq [n]$.

Exact r-SAT AA-4

- Choose x_i randomly. Then X is random.
- $\mathbb{E}(X) = 0$ [Condition 1 of the Alon et al. inequality]
- $\mathbb{E}(X^2) = \sum_{I \in \mathcal{S}} c_I^2 > 0$ [by Parseval's Theorem]
- By Hypercontractive Inequality, $\mathbb{E}(X^4) \leq 9^r \mathbb{E}(X^2)^2$. [Condition 2 of the Alon et al. inequality]

Exact r-SAT AA-5

- By the Alon et. al. inequality, $\mathrm{Prob}(X \geq \frac{\sqrt{\mathbb{E}(X^2)}}{2\cdot 3^r}) > 0$.
- $\mathbb{E}(X^2) = \sum_{I\in\mathcal{S}} c_I^2 \geq |\mathcal{S}| > 0$
- If $k \leq \frac{\sqrt{|\mathcal{S}|}}{2\cdot 3^r}$ then YES.
- Otherwise $r|\mathcal{S}| \leq 4r9^r k^2 = O(k^2)$.

Exact r-SAT AA-6

- MAX r-LIN AA: $X = \sum_{I \in \mathcal{S}} X_I$, where $X_I = c_I \prod_{i \in I} x_i$; is $\max X \geq k$?
- Thus, an $O(k^2)$-size bikernel from Exact r-SAT AA to Max r-Lin AA.
- More work gives: $O(k^2)$-size kernel (Alon, GG, Kim, Szeider and Yeo, arXiv'09):

Outline

Boolean CSP AA-1

- Let r be a fixed positive integer.
- Let Φ be a set of Boolean functions, each with at most r var's out of n var's $x_1, \ldots, x_n$.
- MAX-r-CSP: $\mathcal{F} = \{f_1, f_2, \ldots, f_m\}$, $f_i \in \Phi$; satisfy the max number of f_i's.

Boolean CSP AA-2

- Alon, GG, Kim, Szeider and Yeo, arXiv'09.
- MAX-r-CSP AA: Is there a truth assignment satisfying $\geq \mathbb{E}(\mathrm{sat}(F)) + k$ formulas?
- If Φ is closed under replacing each x_i by $\overline{x}_i$, then $\mathbb{E}(\mathrm{sat}(F))$ is a tight LB.
- There is an $O(k^2)$-size bikernel from MAX-r-CSP AA to MAX r-LIN AA.
- Bikernel Lemma implies polynomial-size kernel for MAX-r-CSP AA.

Outline

Fourier Expansions of Pseudo-boolean Functions

- Pseudo-boolean function: an arbitrary $f: \{-1,+1\}^n \rightarrow \mathbb{R}$.
- It can be uniquely written as $f(x) = \sum_{S \subseteq [n]} c_S \prod_{i \in S} x_i$ (Fourier expansion of f).
- c_S are the Fourier coefficients of f; $c_S = \hat{f}(S)$.
- $\prod_{i \in S} x_i$ form an orthogonal basis.

Pseudo-boolean Functions and Linear Equations

- $f(x) = \hat{f}(\emptyset) + \sum_{S \in \mathcal{F}} c_S \prod_{i \in S} x_i$, where $\mathcal{F} = \{\emptyset \neq S \subseteq [n] : \ c_S \neq 0\}$.
- A weighted system $Az = b$ of linear equations on $\mathbb{F}_2^n$: for each $S \in \mathcal{F}$, we have an equation $\sum_{i \in S} z_i = b_S$ with weight $|c_S|$, where $b_S = 0$ if $c_S > 0$ and $b_S = 1$, otherwise.
- The max excess of $Az = b$ is max of the total weight of satisfied equations minus the total weight of falsified equations.
- $\max_{x \in \{-1,+1\}^n} f(x) - \hat{f}(\emptyset) =$ max excess of $Az = b$.
- Lower bounds on $\max f$ via max excess.

Max Lin

- MAX LIN: Given a weighted system (all weights are positive) of linear equations over $\mathbb{F}_2^n$, maximize the max excess of $Az = b$.
- MAX r-LIN: Each equation has at most r variables.
- Håstad (2001): unless P=NP for each $\epsilon > 0$, there is no polynomial algorithm for distinguishing instances of MAX 3-LIN in which at least $(1-\epsilon)m$ equations can be simultaneously satisfied from instances in which less than $(1/2+\epsilon)m$ equations can be simultaneously satisfied.

Max Lin AA

- Max Lin AA: Given a weighted system (all weights are positive integers) of m linear equations over $\mathbb{F}_2^n$, is the max excess of $Az = b$ at least k?
- It can be solved in time $O(m^{k+O(1)})$ (Crowston, GG, Jones, Kim, Ruzsa, SWAT 2010)
- We believe that Max Lin AA is not FPT (too 'general').

FPT Special Cases of Max Lin AA: 'Symmetric'

- Assumption: $\mathrm{rank} A = n$, the number of variables.
- Using Symmetric SAEM (GG, Kim, Szeider, Yeo, JCSS 2010):

Theorem

If $\exists$ a set U of vars s.t. each equation has odd number of vars from U, then Max Lin AA *is FPT and has a quadratic kernel.*

FPT Special Cases of Max Lin AA: ‘Small’ Systems

Assumptions: (i) Equations in $Az = b$ are distinct, (2) $\mathrm{rank} A = n$.

Theorem (Excess Theorem; Crowston, GG, Jones, Kim, Ruzsa, SWAT 2010)

Let $k \geq 2$. If $k \leq m \leq 2^{n/(k-1)} - 2$, then the maximum excess of $Az = b$ is at least k, i.e., $Az = b$ constitutes a YES*-instance. Moreover, we can find an assignment that achieves an excess of at least k in time $m^{O(1)}$.*

Using the Excess Theorem:

Theorem (Crowston, GG, Jones, Kim, Ruzsa, SWAT 2010)

Let $p(n)$ be an arbitrary function s.t. $p(n) = o(n)$. If $m \leq 2^{p(n)}$ then Max Lin AA *is FPT.*

Smaller Kernels

Using the Excess Theorem:

Theorem (Crowston, GG, Jones, Kim, Ruzsa, SWAT 2010)

For each fixed integral $r \geq 2$ Max r-Lin AA *admits a kernel on* $O(k \log k)$ *variables.*

Theorem (Crowston, GG, Jones, Kim, Ruzsa, SWAT 2010)

For each fixed integral $r \geq 2$ Max Exact r-SAT AA *admits a kernel on* $O(k \log k)$ *variables.*

Outline

Betweenness AA

- Let $V = \{v_1, \ldots, v_n\}$ be a set of variables and let $\mathcal{C}$ be a set of m betweenness constraints of the form $(v_i, \{v_j, v_k\})$.
- Given a bijection $\alpha : V \rightarrow \{1, \ldots, n\}$, we say that a constraint $(v_i, \{v_j, v_k\})$ is satisfied if either $\alpha(v_j) < \alpha(v_i) < \alpha(v_k)$ or $\alpha(v_k) < \alpha(v_i) < \alpha(v_j)$.
- BETWEENNESS: find a bijection α satisfying the max number of constraints in $\mathcal{C}$.
- Tight Lower Bound: $m/3$, the expectation number of satisfied constraints is $m/3$.

Difficulties

- BETWEENNESS AA: Is there α that satisfies $\geq m/3+\kappa$ constraints? (κ is the parameter)
- Benny Chor's question in Niedermeier's book (2006): What is the parameterized complexity of BETWEENNESS AA?
- Difficult to estimate $\mathbb{E}(X^2)$, practically impossible to do $\mathbb{E}(X^4)$, but we cannot use Hypercontractive Inequality as X is not a polynomial of constant-bounded degree.
- What to do?

Way Around Difficulties-1

- Gutin, Kim, Mnich and Yeo [JCSS, ta]: BETWEENNESS AA has an $O(\kappa^2)$-kernel.
- An instance $(V, \mathcal{C})$, where V is the set of variables and $\mathcal{C} = \{C_1, \ldots, C_m\}$ is the set of betweenness constraints.
- A random function $\phi: \ V \rightarrow \{0, 1, 2, 3\}$.
- ϕ-compatible bijections α: if $\phi(v_i) < \phi(v_j)$ then $\alpha(v_i) < \alpha(v_j)$.

Way Around Difficulties-2

- Let α be a random ϕ-compatible bijection and $\nu_p(\alpha) = 1$ if C_p is satisfied and 0, otherwise.
- Let the *weights* $w(C_p, \phi) = \mathbb{E}(\nu_p(\alpha)) - 1/3$ and $w(\mathcal{C}, \phi) = \sum_{p=1}^{m} w(C_p, \phi)$.

Lemma

If $w(\mathcal{C}, \phi) \geq \kappa$ then $(V, \mathcal{C})$ is a YES*-instance of* BETWEENNESS AA*.*

- Thus, to solve BETWEENNESS AA, it suffices to find ϕ for which $w(\mathcal{C}, \phi) \geq \kappa$.
- We may forget about bijections α !

Thank you!

- Questions?
- Comments?