Polynomial-size Kernels for Problems Parameterized Above Tight Lower Bounds

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Fredericia 2009, 27th November 2009

Outline

1. Introduction
2. Various Parameterizations
3. Strictly Above/Below Expectation Method
4. Linear Ordering Problem AA
5. Exact r-SAT AA
6. Betweenness AA
7. Lin-2 AA

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Fixed-parameter Tractability

- A parameterized problem Π: a set of pairs (I, k) where I is the main part and k (usually an integer) is the parameter.
- Π is fixed-parameter tractable (FPT) if membership of (I, k) in Π can be decided in time $O(f(k)|I|^c)$, where $|I|$ is the size of I, $c = O(1)$ and $f(k)$ is a computable function.
- The idea: for small values of k, $O(f(k)|I|^c)$ is not too large.

Fixed-parameter Tractability

- Π is fixed-parameter tractable (FPT) if membership of (I, k) in Π can be decided in time $O(f(k)|I|^c)$, where $|I|$ is the size of I, $c = O(1)$ and $f(k)$ is a computable function.
- Examples of FPT problems:
  - Does a graph G have a vertex cover of size $\leq k$? An algorithm of runtime $O(1.2852^k + kn)$ (Chen, Kanj and Jia, 2001) instead of an $O(n^k m)$-algorithm.
  - Does a digraph D have a spanning out-tree with $\leq k$ leaves? Algorithms of runtime $4^k n^{O(1)}$ (Kneis, Langer and Rossmanith, 2008) and $3.72^k n^{O(1)}$ (Daligault, Gutin, Kim and Yeo, 2009) instead of an $O(n^k m)$-algorithm.

Kernelization

- A kernelization of Π polynomial-time algorithm that maps an instance $(x, k) \in \Pi$ to an instance $(x', k') \in \Pi$ (the kernel) such that
  - (x, k) is YES iff (x', k') is YES
  - $k' \leq f(k)$ and $|x'| \leq g(k)$ for some functions f and g.
- The function $g(k)$ is called the size of the kernel.
- A parameterized problem is FPT if and only if it is decidable and admits a kernelization.
- Wanted: low degree polynomial-size kernels (for preprocessing).

Parameterized Algorithms Monographs

- R. G. Downey and M. R. Fellows. *Parameterized Complexity*. Springer Verlag, 1999.
- J. Flum and M. Grohe. *Parameterized Complexity Theory*. Springer Verlag, 2006.
- R. Niedermeier. *Invitation to Fixed-Parameter Algorithms*. Oxford University Press, 2006.

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Various Parameterizations

- Above we considered standard parameterizations: the parameter is the size of a set to optimize.
- Parameterizations using structural parameters such as treewidth, cliquewidth, the number of vertices to delete to make G bipartite, etc.
- Parameterizations above and below tight bounds.

Acyclic Subgraphs of Digraphs: Standard Parameterization

- Given a digraph $D = (V, A)$, find an acyclic subgraph $H = (V, B)$ of D with the maximum number of arcs.
- Standard parameterization: $k = |B|$. Namely, does D have an acyclic subgraph with at least k arcs?
- Standard parameterization is FPT as $|B| \geq |A|/2$: if $k \leq |A|/2$ the answer is YES otherwise $|V| \leq |A| + 1 \leq 2k$ and we use a brute-force algorithm of running time $|A|^{O(1)}(2k)!$ to check whether the answer is YES.
- k is supposed to be small (for $|A|^{O(1)}(2k)!$ to be tractable), but $k > |A|/2$ is large when $|A|$ is large.

Acyclic Subgraphs of Digraphs: Parameterization above the Average

- Parameterization Above Tight Lower Bound: Does $D = (V, A)$ have an acyclic subgraph with at least $|A|/2 + k$ arcs? [ACYCLIC AA]
- The bound is tight: For symmetric digraphs, $k = 0$: a digraph D is symmetric if $xy \in A$ implies $yx \in A$.
- Mahajan, Raman and Sikdar (2009): Is ACYCLIC AA fixed-parameter tractable?

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Strictly Above/Below Expectation Method (SABEM): Symmetric Case

- SABEM was recently introduced by GG, Kim, Szeider and Yeo [IWPEC'09].
- Apply some reduction rules to reduce the problem to its special case.
- Introduce a random variable X such that if Prob$(X \geq k) > 0$ then the answer to the problem AA is YES.
- If X is symmetric, then Prob($X \geq \sqrt{\mathbb{E}(X^2)}$) > 0.
- If $k \leq \sqrt{\mathbb{E}(X^2)}$ then YES. Otherwise, $\sqrt{\mathbb{E}(X^2)} < k$ and we can often solve the problem using a brute force algorithm.

Strictly Above/Below Expectation Method: Asymmetric Case

Lemma (Alon, GG, Krivelevich, 2004; Alon, GG, Kim, Szeider, Yeo, SODA'2010)

Let X be a real random variable and suppose that its first, second and forth moments satisfy $\mathbb{E}(X) = 0$, $\mathbb{E}(X^2) = \sigma^2 > 0$ and $\mathbb{E}(X^4) \leq b(\mathbb{E}(X^2))^2$, respectively. Then $\text{Prob}(\ X > \frac{\sigma}{2\sqrt{b}}\ ) > 0$.

Lemma (Hypercontractive Inequality, Bonami, Gross, 1970s)

Let $f = f(x_1, \ldots, x_n)$ be a polynomial of degree r in n variables $x_1, \ldots, x_n$. Define a random variable X by choosing a vector $(\varepsilon_1, \ldots, \varepsilon_n) \in \{-1, 1\}^n$ uniformly at random and setting $X = f(\varepsilon_1, \ldots, \varepsilon_n)$. Then $\mathbb{E}(X^4) \leq 9^r(\mathbb{E}(X^2))^2$.

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Reduction Rule for Linear Ordering Problem AA

- LINEAR ORDERING AA: each arc ij has positive integral weight w_{ij}, does $D = (V, A)$ have an acyclic subgraph of weight at least $W/2 + k$, where $W = \sum_{ij \in A} w_{ij}$?
- Reduction rule: Assume D has a directed 2-cycle iji;
  - if $w_{ij} = w_{ji}$ delete the cycle,
  - if $w_{ij} > w_{ji}$ delete the arc ji and replace w_{ij} by $w_{ij} - w_{ji}$,
  - if $w_{ji} > w_{ij}$ delete the arc ij and replace w_{ji} by $w_{ji} - w_{ij}$.
- Thus, we've reduced LINEAR ORDERING AA to the one on oriented graphs.

SABEM for Linear Ordering AA-1

- Let $D = (V, A)$ be an oriented graph, let $n = |V|$.
- Consider a random bijection: $\alpha : V \rightarrow \{1, \dots, n\}$ and a random variable $X(\alpha) = \frac{1}{2} \sum_{ij \in A} \varepsilon_{ij}(\alpha)$, where $\varepsilon_{ij}(\alpha) = w_{ij}$ if $\alpha(i) < \alpha(j)$ and $\varepsilon_{ij}(\alpha) = -w_{ij}$, otherwise.
- $X(\alpha) = \sum\{w_{ij} : \ ij \in A, \alpha(i) < \alpha(j)\} - W/2$. Thus, the answer is YES iff there is an $\alpha : V \rightarrow \{1, \dots, n\}$ such that $X(\alpha) \geq k$.
- Since $\mathbb{E}(\varepsilon_{ij}) = 0$, we have $\mathbb{E}(X) = 0$.

SABEM for Linear Ordering AA-2

Lemma

$\mathbb{E}(X^2) \geq W^{(2)}/12$, *where* $W^{(2)} = \sum_{ij \in A} w_{ij}^2$.

Since X is symmetric, we have Prob($X \geq \sqrt{W^{(2)}/12}$) > 0. Hence, if $\sqrt{W^{(2)}/12} \geq k$, there is an $\alpha : V \rightarrow \{1, \ldots, n\}$ such that $X(\alpha) \geq k$ and, thus, the answer is YES. Otherwise, $|A| \leq W^{(2)} < 12 \cdot k^2$. Thus, we have:

Theorem (GG, Kim, Szeider, Yeo, IWPEC'09)

LINEAR ORDERING AA *is fixed-parameter tractable.*

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Exact r-SAT

- Exact r-SAT: A CNF formula $\mathcal{F}$ which contains m clauses each with r literals. Is there a truth assignment satisfying all m clauses of $\mathcal{F}$?
- Max Exact r-SAT: Find a truth assignment satisfying the max number of clauses.
- The prob. of a clause to be satisfied: $1 - 2^{-r}$.
- The average number of satisfied clauses: $(1 - 2^{-r})m$. This lower bound is tight.

Exact r-SAT AA-1

- EXACT r-SAT AA: Is there a truth assignment satisfying $\geq (1 - 2^{-r})m + k2^{-r}$ clauses?
- Mahajan, Raman and Sikdar (2009): What is the parameterized complexity of EXACT r-SAT AA for each fixed r?
- Alon, GG, Kim, Szeider and Yeo (SODA 2010): EXACT r-SAT AA is FPT for each fixed r.

Exact r-SAT AA-2

- $-1 =$ true.
- $X = \sum_{C \in \mathcal{F}} [1 - \prod_{x_i \in \text{var}(C)} (1 + \varepsilon_i x_i)]$, where $\varepsilon_i \in \{-1, 1\}$ and $\varepsilon_i = 1$ iff x_i is in C.
- For a truth assignment τ, we have $X = 2^r(\text{sat}(\tau, F) - (1 - 2^{-r})m)$.
- The answer to EXACT r-SAT AA is YES iff $X \geq k$.

Exact r-SAT AA-3

- After algebraic simplification $X = X(x_1, x_2, \ldots, x_n)$ can be written as $X = \sum_{I \in \mathcal{S}} X_I$, where $X_I = c_I \prod_{i \in I} x_i$, each c_I is a nonzero integer and $\mathcal{S}$ is a family of nonempty subsets of $\{1, \ldots, n\}$ each with at most r elements. [This is a Fourier expansion of X.]
- $\mathbb{E}(X) = 0$ [Condition 1 of the Alon et al. inequality]
- $\mathbb{E}(X^2) = \sum_{I \in \mathcal{S}} c_I^2 > 0$ [by Parseval's Theorem]
- By Hypercontractive Inequality, $\mathbb{E}(X^4) \leq 9^r \mathbb{E}(X^2)^2$. [Condition 2 of the Alon et al. inequality]

Exact r-SAT AA-4

- By the Alon et. al. inequality, $\text{Prob}(X \geq \frac{\sqrt{\mathbb{E}(X^2)}}{2 \cdot 3^r}) > 0$.
- $\mathbb{E}(X^2) = \sum_{I \in \mathcal{S}} c_I^2 \geq |\mathcal{S}| > 0$
- If $k \leq \frac{\sqrt{|\mathcal{S}|}}{2 \cdot 3^r}$ then YES.
- Otherwise $n' \leq r|\mathcal{S}| \leq 4r9^r k^2 = O(k^2)$ (n' is the number of variables in the Fourier expansion of X).
- Thus, an $m^{O(1)} 2^{O(k^2)}$-time algorithm.
- More work gives: $O(k^2)$-size kernel.

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Betweenness AA

- Let $V = \{v_1, \ldots, v_n\}$ be a set of variables and let $\mathcal{C}$ be a set of m betweenness constraints of the form $(v_i, \{v_j, v_k\})$.
- Given a bijection $\alpha : V \rightarrow \{1, \ldots, n\}$, we say that a constraint $(v_i, \{v_j, v_k\})$ is satisfied if either $\alpha(v_j) < \alpha(v_i) < \alpha(v_k)$ or $\alpha(v_k) < \alpha(v_i) < \alpha(v_j)$.
- Betweenness: find a bijection α satisfying the max number of constraints in $\mathcal{C}$.
- Tight Lower Bound: $m/3$, the expectation number of satisfied constraints is $m/3$.
- Betweenness AA: Is there α that satisfies $\geq m/3 + \kappa$ constraints? (κ is the parameter)

Difficulties

- Benny Chor's question in Niedermeier's book (2006): What is the parameterized complexity of BETWEENNESS AA?
- Difficult to estimate $\mathbb{E}(X^2)$, practically impossible to do $\mathbb{E}(X^4)$, but we cannot use Hypercontractive Inequality as X is not a polynomial of constant-bounded degree.
- What to do?

Way Around Difficulties-1

- Gutin, Kim, Mnich and Yeo: BETWEENNESS AA is FPT.
- An instance $(V, \mathcal{C})$, where V is the set of variables and $\mathcal{C} = \{C_1, \ldots, C_m\}$ is the set of betweenness constraints.
- A random function $\phi: V \rightarrow \{0, 1, 2, 3\}$.
- ϕ-compatible bijections α: if $\phi(v_i) < \phi(v_j)$ then $\alpha(v_i) < \alpha(v_j)$.

Way Around Difficulties-2

- Let α be a random ϕ-compatible bijection and $\nu_p(\alpha) = 1$ if C_p is satisfied and 0, otherwise.
- Let the *weights* $w(C_p, \phi) = \mathbb{E}(\nu_p(\alpha)) - 1/3$ and $w(\mathcal{C}, \phi) = \sum_{p=1}^{m} w(C_p, \phi)$.

Lemma

If $w(\mathcal{C}, \phi) \geq \kappa$ then $(V, \mathcal{C})$ is a YES*-instance of* BETWEENNESS AA*.*

- Thus, to solve BETWEENNESS AA, it suffices to find ϕ for which $w(\mathcal{C}, \phi) \geq \kappa$.
- We may forget about bijections α !

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Lin-2 AA

- A system of linear equations over $\mathrm{GF}(2)$: $\sum_{i \in I_j} z_i = b_j$, $I_j \subseteq \{1, \ldots, n\}$, $j = 1, \ldots, m$. Equation j has weight $w_j \in \mathbb{Z}_+$.
- The problem Max Lin-2 asks for an assignment of values to the variables that maximizes the total weight of the satisfied equations.

Lin-2 AA

- Let $W = w_1 + \cdots + w_m$. A greedy-type algorithm guarantees a solution of weight $\geq W/2$.
- LIN-2 AA: Does the system have a solution of weight $\geq W/2 + k$?
- Mahajan, Raman and Sikdar (2009): What is the parameterized complexity of LIN-2 AA?

Lin-2 AA

- $X = \sum_{j=1}^{m} X_j$, where $X_j = (-1)^{b_j} w_j \prod_{i \in I_j} x_i$, $x_i \in \{-1, 1\}$.
- Observe that $X_j = w_j$ if equation j is satisfied and $X_j = -w_j$, otherwise.
- The answer to Lin-2 AA is Yes iff $X \geq 2k$.
- Difficulty: in general $|I_j|$ is not bounded from above [we cannot apply Hypercontractive Inequality].

Lin-2 AA

- Not proved to be FPT yet, but proved FPT [GG, Kim, Szeider, Yeo, IWPEC'09] in three cases:
- Case 1: There exists a set U of variables such that each equation of S contains an odd number of variables from U. [Symmetric X]
- Case 2: $\leq O(1)$ variables in each equation. [Alon et al. inequality + Hypercontractive Inequality]
- Case 3: Every variable in $\leq O(1)$ equations. [Alon et al. inequality + $\mathbb{E}(X^4)$ bounded without Hypercontractive Inequality]

Thank you!

- Questions?
- Comments?